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C^* Algebras Associated with von Neumann Algebras.

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Sunto. – *Ad un'algebra di von Neumann separabile M, in forma standard su di uno spazio di Hilbert H, si associa la C* algebra $\mathcal{O}_M$ definita come la C* algebra $\mathcal{O}_{\mathcal{U}(M)}$ costituita dai punti fissi dell' algebra di Cuntz generalizzata $\mathcal{O}_H$ mediante l'azione canonica del gruppo $\mathcal{U}(M)$ degli unitari di M. Si dà una caratterizzazione di $\mathcal{O}_M$ nel caso in cui M è un fattore iniettivo. In seguito, come applicazione della teoria dei sistemi asintoticamente abeliani, si mostra che, se ω è uno stato vettoriale normale e fedele di M, la restrizione ad $\mathcal{O}_M$ dello stato prodotto tensoriale infinito $\bigoplus_{n=1}^{\infty} \omega$ di $\mathcal{O}_H$ è uno stato puro.*

1. – Introduction.

Let H be a separable Hilbert space. We set $H^0 = \boldsymbol{C}$; for $r > 0$ we denote by $H^r = H \otimes H \otimes \ldots \otimes H$ the Hilbert space r-fold tensor power of H and, for r, $s \geq 0$, we denote by (H^r, H^s) the set of all bounded linear mappings from H^r into H^s, so that, in particular, $(H^1, H^0) = (H, \boldsymbol{C}) = H^*$ coincides with the topological dual of H.

Thus, denoting by $I: H \to H$ the identical map, we have, for all r, $s \geq 0$, the inclusions

$$\begin{array}{ccc} (H^r, H^s) & \to & (H^{r+1}, H^{s+1}) = (H^r \otimes H, H^s \otimes H), \\ T & \mapsto & T \otimes I . \end{array}$$

We then denote, for all $k \in \boldsymbol{Z}$, by

$${}^0\mathcal{O}_H^k = \lim_{r \to +\infty} (H^r, H^{r+k})$$

the direct limit of the (H^r, H^{r+k})'s and by

$${}^0\mathcal{O}_H = \bigoplus_{k \in Z} {}^0\mathcal{O}_H^k$$

the algebraic direct sum of the $\mathcal{O}_H^k$' s.

${}^0\mathcal{O}_H$ carries, in a natural way, a structure of $\boldsymbol{Z}$-graded $*$-algebra over $\boldsymbol{C}$. Indeed, denoting by $\widetilde{T} \in {}^0\mathcal{O}_H^k$ (respectively by $\widetilde{S} \in {}^0\mathcal{O}_H^h$) the class of an element $T \in (H^r, H^{r+k})$ (resp. $S \in (H^s, H^{s+h})$), we can find $p, q \in \boldsymbol{N}$ such that $s + h + q =$

$r+p$; we then set

$$\widetilde{T}\widetilde{S} = [(T \otimes I^{\otimes p}) \circ (S \otimes I^{\otimes q})]^{\sim} \in {}^{0}\mathcal{O}_{H}^{k+h}$$

and

$$(\widetilde{T})^{*} = (T^{*})^{\sim} \in {}^{0}\mathcal{O}_{H}^{-k}$$

which are well-defined as one checks immediately.

In [CPDR] it is shown that ${}^{0}\mathcal{O}_{H}$ is endowed with a unique C^{*}-norm $\|\cdot\|$, namely the C^{*}-maximal norm, so that its completion

$$\mathcal{O}_{H} = ({}^{0}\mathcal{O}_{H})^{\sim\|\cdot\|},$$

called the generalized Cuntz algebra [CPDR], is a simple C^{*}-algebra.

A concrete realisation of such a C^{*}-algebra can be given as follows.

Recall ([R]) that a Hilbert space in a (separable) von Neumann algebra $\mathcal{M}$ is a norm-closed vector space $H \leqslant \mathcal{M}$ such that $\forall \phi, \psi \in H$ one has $\phi^{*}\psi \in \boldsymbol{C}I$; the relation

$$\phi^{*}\psi = (\phi|\psi)\, I$$

defines an inner product $(\,|\,)$ wich endowes H with a Hilbert space structure (note that H is norm-closed and that the norm arising from the inner product coincides in fact with the norm $\|\cdot\|_{\mathcal{M}}$ of $\mathcal{M}$; thus H is $(\,|\,)$-complete). The support of H is the projection in $\mathcal{M}$ defined by

$$p_{H} = \sum_{i \in J}^{s} \psi_{i}\psi_{i}^{*}$$

$\{\psi_{i}\}_{i\in J}$ being any orthonormal basis of the Hilbert space H, and the series being convergent with respect to the strong-operator topology in $\mathcal{M}$.

Suppose we are given a family $\{\psi_{i}\}_{i\in J} \subset \mathcal{B}(\mathcal{H})$ of isometries on a Hilbert space $\mathcal{H}$ satisfying the relation $\sum_{i\in J}^{s} \psi_{i}\psi_{i}^{*} = I$; this implies in particular that $\psi_{i}^{*}\psi_{j} = \delta_{i,j}I$, $\forall i, j \in I$ so that $H = \overline{\operatorname{span}\{\psi_{i} \colon i \in J\}}^{\|\cdot\|_{\mathcal{B}(\mathcal{H})}}$ is a Hilbert space in $\mathcal{B}(\mathcal{H})$ with support the identity: $p_{H} = I$.

We then pose $H^{0} = \boldsymbol{C}I$, for $r > 0$

$$H^{r} = \overline{\operatorname{span}\{\psi_{i_1}\psi_{i_2}\ldots\psi_{i_r} \,|\, i_l \in J\}}^{\|\cdot\|_{\mathcal{B}(\mathcal{H})}}$$

which is a Hilbert space as well (and can in fact be identified with the r-fold tensor power of H) and, for $r, s \geqslant 0$,

$$(H^{r}, H^{s}) = \overline{\operatorname{span}\{\psi_{i_1}\psi_{i_2}\ldots\psi_{i_s}\psi_{j_1}^{*}\psi_{j_2}^{*}\ldots\psi_{j_r}^{*} \,|\, i_l, j_t \in J\}}^{s}.$$

Since $p_{H} = I$, we have the injection

$$(H^{r}, H^{s}) \to (H^{r+1}, H^{s+1})$$

given by

$$\psi_{i_1}\psi_{i_2}\cdots\psi_{i_s}\psi^*_{i_1}\psi^*_{i_2}\cdots\psi^*_{i_r}\mapsto \sum_{l\in J}^{s}\psi_{i_1}\psi_{i_2}\cdots\psi_{i_s}\psi_l\psi^*_l\psi^*_{j_1}\psi^*_{j_2}\cdots\psi^*_{j_r}.$$

The sets

$$^0\mathcal{O}^k_H=\bigcup_{r,\,r+k>0}(H^r,H^{r+k}),\qquad k\in\boldsymbol{Z}$$

are independent subspaces of $\mathcal{B}(\mathcal{H})$ so that if $^0\mathcal{O}_H=\bigcup_{k\in Z}{}^0\mathcal{O}^k_H$ we have

$$^0\mathcal{O}_H=\bigoplus_{k\in Z}{}^0\mathcal{O}^k_H .$$

$^0\mathcal{O}_H$, with the product inherited as a subspace of $\mathcal{B}(\mathcal{H})$, becomes a $\boldsymbol{Z}$-graded $*$-algebra. We then have that $\overline{^0\mathcal{O}_H}^{\|\cdot\|_{\mathcal{B}(\mathcal{H})}}$ as a sub C^*-algebra of $\mathcal{B}(\mathcal{H})$ is isomorphic to the generalized Cuntz algebra $\mathcal{O}_H$ as defined before, which is therefore simple and has the following universality property ([CPDR]):

Suppose that $\{\psi'_i\}_{i\in J}\subset\mathcal{B}(\mathcal{H}')$ is another family of isometries satisfying $\sum_{i\in J}^s\psi'_i\psi'^*_i=I$; then if H' denotes the Hilbert space they generate and $\mathcal{O}_{H'}\leqslant\mathcal{B}(\mathcal{H}')$ is the corresponding generalized Cuntz algebra, there exists a unique $*$-isomorphism $\alpha\colon\mathcal{O}_H\to\mathcal{O}_{H'}$ of $\mathcal{O}_H$ onto $\mathcal{O}_{H'}$ such that $\alpha(\psi_i)=\psi'_i$, $\forall i\in J$.

We remark that in case H is a finite dimensional Hilbert space, say $\dim(H)=d$, then $\mathcal{O}_H$ is isomorphic to the Cuntz algebra $\mathcal{O}_d$ ([Cu]).

As a consequence of the universality property, if $u\in\mathcal{U}(H)$ is a unitary of the Hilbert space H, there exists a unique automorphism $\alpha_u\in\mathrm{Aut}(\mathcal{O}_H)$ such that $\alpha_u(\psi)=u(\psi)$ for all $\psi\in H$ and the application

$$\begin{array}{ccc}\mathcal{U}(H)&\to&\mathrm{Aut}(\mathcal{O}_H)\\ u&\mapsto&\alpha_u\end{array}$$

is a homomorphism.

If $G\leqslant\mathcal{U}(H)$ is any subgroup of unitaries of H we denote by $\mathcal{O}_G$ the fixed point subalgebra $\mathcal{O}_G=\mathcal{O}^G_H=\{x\in\mathcal{O}_H\colon \alpha_g(x)=x,\ \forall g\in G\}$ of $\mathcal{O}_H$ under the automorphisms α_g, $g\in G$ and we call the corresponding action by automorphisms a *canonical action*.

In particular for $G=\boldsymbol{T}$, the subgroup of the homoteties of H, we denote by $\mathcal{O}^0_H$ the subalgebra $\mathcal{O}_{\boldsymbol{T}}$, called the 0-th grade of $\mathcal{O}_H$; then the map

$$\begin{array}{ccc}m\colon\mathcal{O}_H&\to&\mathcal{O}^0_H,\\ X&\mapsto&\displaystyle\int_{\boldsymbol{T}}\alpha_\tau(X)\,d\tau,\end{array}$$

is a conditional expectation; observe that in fact one has $\mathcal{O}^0_H=({}^0\mathcal{O}^0_H)^{-\|\cdot\|}$.

Let now M be an injective factor [C] acting in standard form [Ha] on a separable Hilbert space H. One can consider $H=L^2(M,\theta)$ as the Hilbert space coming from the GNS-construction relative to the couple (M, θ), where $\theta \in M_*$ is a faithful normal state on M.

DEFINITION. – *We define $\mathcal{O}_M$ to be the subalgebra $\mathcal{O}_{\mathcal{U}(M)}$ of $\mathcal{O}_H$ consisting of the fixed points in $\mathcal{O}_H$ under the canonical action of the unitary group $\mathcal{U}(M)$ of M.*

Since for any factor M one has $\boldsymbol{T}I \subset \mathcal{U}(M)$, the inclusion $\mathcal{O}_M \subset \mathcal{O}_H^0$ holds. The following theorem characterizes $\mathcal{O}_M$ as a C^* algebra.

THEOREM 1. – *The C^*-algebra $\mathcal{O}_M$ is isomorphic to the C^*-inductive limit of the w^*-crossed products of $M' \otimes M' \otimes \ldots \otimes M'$, n factors, with the action of the symmetric group $S(n)$ by permutating the factors*:

$$\mathcal{O}_M \cong \lim_{n \to \infty} {}^{C^*}(M' \otimes M' \otimes \ldots \otimes M') \times S(n).$$

Let now $\omega = \omega_\psi$ denote the faithful normal state induced on M by a vector $\psi_\infty \in H$ and denote by $\tilde{\omega}$ the restriction to $\mathcal{O}_M$ of the infinite tensor state [vN] $\bigotimes_{n=1} \omega$ on $\mathcal{O}_H^0$.

THEOREM 2. – *State $\tilde{\omega}$ is pure.*

The following two sections are devoted to the proofs of the theorems stated above; in particular in section 3, where the second theorem is proved, some notions relative to the theory of asymptotically abelian systems are recalled from [DKKR] and [DKS]. Some problems that naturally arise in this framework shall follow in the last section.

2. – Proof of Theorem 1.

Let M_n, $n = 1, 2, \ldots$ be any increasing sequence of finite type I factors with strongly dense union in M.

By Kaplansky' s density theorem ([T] Thm 4.8), the unitary group $\mathcal{U}(M)$ is the strong closure of the amenable group $G = \bigcup_{n=1}^{\infty} \mathcal{U}(M_n)$, the union of the unitary groups of the M_n's. Thus, denoting $\mathcal{O}_G \cap (H^n, H^n)$ by $(H^n, H^n)_G$ we have ([CDPR]):

$$\mathcal{O}_M = \mathcal{O}_G = \lim_{n \to \infty} (H^n, H^n)_G.$$

Now Theorem 1 follows immediately from the next lemma which is of some interest in itself.

LEMMA. – *The commutant of* $(H^n, H^n)_G$, *i.e. the von Neumann algebra generated by the tensor products* $u \otimes u \otimes \ldots \otimes u$ (*n times*) *as* u *varies in* $\mathcal{U}(M)$, *coincides with the fixed point subalgebra of* $M \otimes M \otimes \ldots \otimes M$ *under the action of* $S(n)$ *by permutating the factors in the tensor product.*

PROOF. – By the joint continuity of the product with respect to the strong topology on bounded subsets, we have that $(H^n, H^n)'_G$ is the von Neumann algebra generated by the subalgebras

$$R_{n,m} = \{u \otimes u \otimes \ldots \otimes u : n \text{ times}, \, u \in \mathcal{U}(M_m)\}''$$

as m and n range over the positive integers.

By the theorem of Weyl we have that $R_{n,m}$ consists of the fixed points $(M_m \otimes M_m \otimes \ldots \otimes M_m)_{S(n)}$ under the action of $S(n)$.

Denote by μ_n the average over this action on $\mathcal{B}(H) \otimes \mathcal{B}(H) \otimes \ldots \otimes \mathcal{B}(H) \cong \mathcal{B}(H^{\otimes^n})$; we then have

$$R_{n,m} = \mu_n (M_m \otimes M_m \otimes \ldots \otimes M_m).$$

Since μ_n is normal we obtain

$$\overline{\bigcup_{m=1}^{\infty} R_{n,m}} = \mu_n \left(\overline{\bigcup_{m=1}^{\infty} M_m \otimes M_m \otimes \ldots \otimes M_m} \right)$$

i.e.

$$(H^n, H^n)'_G = \mu_n (M \otimes M \otimes \ldots \otimes M),$$

as desired. ■

END OF THE PROOF OF THEOREM 1. – By the lemma we have

$$(H^n, H^n)_G = \{(M \otimes M \ldots \otimes M) \cap \mathcal{U}(S(n))'\}' =$$

$$(M \otimes M \ldots \otimes M)' \vee \mathcal{U}(S(n))'' = (M' \otimes M' \ldots \otimes M') \times^{w^*} S(n),$$

where the commutant theorem has been used and the w^*-product refers to the action of $S(n)$. ■

3. – Proof of Theorem 2.

We begin this section by recalling some definitions and results from [DKKR] and [DKS].

Let $\mathcal{A}$ be a C^* algebra, G a locally compact non compact group acting by automorphisms on $\mathcal{A}$. Then $\{\mathcal{A}, G, \alpha\}$ is an *asymptotically abelian system* if

$\forall \varepsilon > 0$, $\forall A, B \in \mathcal{A}$ and any state $\Phi \in \mathcal{S}(\mathcal{A})$ of $\mathcal{A}$, there exists a compact subset $K \subset G$ such that $g \notin K$ implies

$$|\Phi(A\alpha_g(B) - \alpha_g(B)A)| < \varepsilon .$$

If $\phi \in \mathcal{S}(\mathcal{A})$ is a G-invariant state, i.e. $\phi(\alpha_g(A)) = \phi(A)$ for all $g \in G$ and $A \in \mathcal{A}$, then the GNS construction relative to $(\mathcal{A}, G, \alpha, \phi)$ yields a representation π_ϕ of the C^* algebra $\mathcal{A}$ on a Hilbert space H_ϕ with a cyclic vector Ω_ϕ and a unitary representation $U_\phi \colon G \ni g \mapsto U_\phi(g) \in \mathcal{U}(H_\phi)$ such that

$$(\Omega_\phi \,|\, \pi_\phi(A)\Omega_\phi) = \phi(A),$$

$$\pi_\phi(\alpha_g(A)) = U_\phi(g)\, \pi_\phi(A)\, U_\phi(g)^* ,$$

$$U_\phi(g)\, \Omega_\phi = \Omega_\phi ,$$

for all $g \in G$ and $A \in \mathcal{A}$.

Let now $\mathcal{R}$ denote the von Neumann algebra generated by the set $\pi_\phi(\mathcal{A}) \cup U_\phi(G) \subset \mathcal{B}(H_\phi)$ and let E_0 denote the orthogonal projection in H_ϕ onto the subspace $\{x \in H_\phi \colon U_\phi(g)x = x, \forall g \in G\}$ of the G-invariant vectors.

Then the commutator $\mathcal{R}'$ and the compression $E_0 \mathcal{R} E_0$ are abelian von Neumann algebras and the mapping

$$\mathcal{R}' \ni T \mapsto TE_0 \in E_0 \mathcal{R} E_0$$

is a surjective *-isomorphism. In particular for each $A \in \mathcal{A}$ there exists a unique $M_\phi(A) \in \mathcal{R}'$ such that

$$M_\phi(A)\, E_0 = E_0 \pi_\phi(A)\, E_0$$

because the mapping M_ϕ is linear and positive; in particular $\{M_\phi(A) \colon A \in \mathcal{A}\}$ is weakly-operator dense in $\mathcal{R}'$.

If in addition the group G is amenable one has $\mathcal{R}' \subset \pi_\phi(\mathcal{A})''$ and the following conditions are equivalent:

(i) $(\Omega_\phi \,|\, \pi_\phi(A)\, M_\phi(B)\, \Omega_\phi) = \phi(A)\, \phi(B)$, $\forall A, B \in \mathcal{A}$.

(ii) Ω_ϕ is the only G-invariant unit-vector of H_ϕ, i.e. $E_0 = E_{\Omega_\phi}$, where E_{Ω_ϕ} denotes the orthogonal projection of H_ϕ onto the subspace $\boldsymbol{C}\Omega_\phi$.

(iii) $M_\phi(A) \in \boldsymbol{C}I$, $\forall A \in \mathcal{A}$.

(iv) $\mathcal{R} = \mathcal{B}(H_\phi)$.

(v) $\mathcal{R}$ is a factor.

(vi) ϕ is an extremal element of the convex set of all G-invariant states over $\mathcal{A}$.

In our setting, if $\widetilde{M}_\omega$ denotes the von Neumann algebra generated by $\pi_{\tilde{\omega}}(\mathcal{O}_M)$,where $(\pi_{\tilde{\omega}}, H_{\tilde{\omega}}, \xi_{\tilde{\omega}})$ denotes the GNS construction relative to the couple $(\mathcal{O}_M, \tilde{\omega})$, then the proof of Theorem 1 yields that $\widetilde{M}_\omega$ is isomorphic to the w^*-crossed product of $\bigotimes_{n=1}^{\infty} {}^{\psi_i = \psi} M'$ with the action of the infinite permutation group $S(\infty) = \bigcup_{n=1}^{\infty} S(n)$.

But if $\mathcal{A} = \bigotimes_{n=1}^{\infty} {}^{\psi_n = \psi} M'$, $G = S(\infty)$, and α denotes the action of the amenable group G by permutation of the factors, then $(\mathcal{A}, G, \alpha)$ is an asymptotic abelian system since, given any $\varepsilon > 0$ and $A, B \in \mathcal{A}$, one can find an integer n_ε such that if $g \in S(\infty) \backslash S(n_\varepsilon)$ then $\| A\alpha_g(B) - \alpha_g(B) A \| < \varepsilon$. Moreover the faithful normal state $\omega = \omega_\psi$ is $S(\infty)$-invariant and $\Omega_\omega = \bigotimes_{n=1}^{\infty} \psi_n$ is the unique $S(\infty)$-invariant vector so that $\mathcal{R} = \widetilde{M}_\omega$ equals the whole of $\mathcal{B}(H_\omega)$, equivalently $\pi_{\tilde{\omega}}$ is irreducible, i.e. $\tilde{\omega}$ is pure.

4. – Problems.

Problem 1.. – Let $N \subset M$ be an inclusion of type II_1 factors with finite Jones' index [J], $[M : N] < \infty$; then we have the controvariant inclusion $\mathcal{O}_M \subset \mathcal{O}_N$. Can we associate with this inclusion of C^* algebras some invariants which are computable? For instance what is the relation between the Watatani [W] index for $\mathcal{O}_M \subset \mathcal{O}_N$ and Jones' index $[M : N]$?

Problem 2. – With ϱ the identity representation of the unitary group $\mathcal{U}(M)$ of a factor M: $\varrho(u) = u$, $u \in \mathcal{U}(M)$, investigate the inclusion ([CDPR])

$$\mathcal{O}_\varrho \subset \mathcal{O}_M .$$

Is it proper if M is not hyperfinite? What about for $M = \mathcal{L}(\boldsymbol{F}_n)$ the von Neumann algebra of the free group $\boldsymbol{F}_n$ on n generators?

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