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Strata of Smooth Space Curves Having Unstable Normal Bundle.

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Sunto. – *Per $d \gg g$, vengono trovate curve liscie in $\mathbb{P}^3$ di grado d e genere g aventi fibrato normale instabile con grado di instabilità σ, per ogni $1 \leqslant \sigma \leqslant d-4$. Inoltre per $4g-2 \leqslant \sigma \leqslant d-4$, viene trovata una famiglia di curve in $\mathbb{P}^3$ di grado d e genere g avente fibrato normale instabile con grado di instabilità σ e formante uno strato dello schema di Hilbert della giusta dimensione che è $4d-g+1-2\sigma$.*

Introduction.

Several authors studied the normal bundle of spaces curves, but even today our knowledge of this subject is not satisfactory.

The normal bundle of a smooth rational space curve is well-known. Eisenbud and Van de Ven in [6] and [7] gave a complete geometric description of the strata associated to the splitting type of the normal bundle of smooth rational space curves.

For $g \geqslant 1$, one can stratify the Hilbert scheme of degree d and genus g smooth spaces curves C by the following integer $s(N_C)$ associated to the normal bundle N_C:

$$s(N_C) = \frac{1}{2}\deg N_C - \deg L_{\max}, \text{ where } L_{\max} \text{ is a maximal line subbundle of } N_C.$$

If $s(N_C) = s > 0$, we say that N_C is stable with stability degree s. If $s(N_C) = s < 0$, we say that N_C is unstable with instability degree $\sigma = -s$. If $s(N_C) = 0$, N_C is semi-stable non-stable.

Some natural questions arise.

For every integer s such that $-(d+g-4) \leqslant s \leqslant [g/2]$ does exist a smooth space curve C with $s(N_C) = s$?

Let $N_{d,g}(s)$ be the stratum parametrizing the smooth space curves C of degree d and genus g with $s(N_C) = s$. Does $N_{d,g}(s)$ have an irreducible component of the «right» dimension?

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For $g \geqslant 2$ and a large d the general degree d genus g curve C has a superstable normal bundle N_C, i.e. N_C is stable with stability degree $[g/2]$ ([10]).

For $g = 1$, by using the geometric construction of Eisenbud and Van de Ven, Hulek and Sacchiero in [20] proved that the degree d genus 1 general space curve has a semi-stable normal bundle and they found all the instability degrees σ that normal bundles of elliptic curves can admit.

In this paper we find a lot of smooth space curves having an unstable normal bundle. For example, if either $g \geqslant 3$ and $d \geqslant 4g + 2$ or $g = 2$ and $d \geqslant 12$, we find a degree d genus g smooth curve C in $\mathbb{P}^3$ having an unstable normal bundle N_C with instability degree σ, where $1 \leqslant \sigma \leqslant d - 4$ (Theorem 4.3, Proposition 4.5, Theorem 4.6).

Moreover for $4g - 2 \leqslant \sigma \leqslant d - 4$, we find an irreducible component of the stratum $N_{d,g}(-\sigma)$ consisting of degree d genus g smooth space curves having an unstable normal bundle with instability degree σ of the right dimension, that is $4d - g + 1 - 2\sigma$ (Theorem 4.3).

Also for $g = 1$, $d \geqslant 7$ and $3 \leqslant \sigma \leqslant d - 4$, we can find a good irreducible component of the stratum $N_{d,1}(-\sigma)$ of the right dimension, that is $4d - 2\sigma$. Thus we can calculate the dimension of some strata that Hulek and Sacchiero found in [20] (Theorem 4.8).

We use the geometric construction that Eisenbud and Van de Ven gave in [6] and [7], i.e. we use a developable ruled surface S_L containing C to describe a line subbundle L of the normal bundle N_C. S_L is called the characteristic surface of L and it is the dual surface of the curve Λ in $\mathbb{P}^{3\vee}$ image of the Gauss morphism induced by L.

We calculate the dimension of our strata by showing that normal sheaves N_f of morphisms f from a curve C to $\mathbb{P}^3$ form a flat algebraic family of sheaves (Proposition 1.3) and by describing a natural $\mathcal{H}$-morphism (see Theorem 1.6)

$$\gamma\colon \mathring{\mathrm{Quot}}^{1,b}(N_\phi(-1),\, \mathcal{C} \times_S \mathcal{H},\, \mathcal{H}) \to \mathcal{H}om^b_{\mathcal{H}}(\mathcal{C} \times_S \mathcal{H},\, \mathbb{P}^{3\vee}_{\mathcal{H}})$$

where S is the fine moduli space of genus g smooth curves with level m structure, $\mathcal{C}$ is the universal curve over S, $\mathcal{H}$ is the scheme of degree d S-morphisms of $\mathcal{C}$ in $\mathbb{P}^3_S$, ϕ is the universal degree d S-morphism and the first scheme denotes the quasi-projective scheme of rank 1 degree b quotient sheaves of $N_\phi(-1)$ that are flat over $\mathcal{H}$ and locally free over $\mathcal{C} \times_S \mathcal{H}$.

The above morphism γ gives in a natural way the morphism G that Eisenbud and Van de Venn described in [7] Theorem 5.1.

All schemes that we consider are separated and of finite type over an algebraically closed field k of characteristic 0.

1. – A family of normal sheaves.

Let $\mathcal{C}$ be a smooth curve of genus g over a scheme S and let d be a positive integer.

We consider the functor $\underline{\mathrm{Hom}}_S^d(\mathcal{C}, \mathbb{P}_S^3)$ from the category of S-schemes to the category of sets associating to an S-scheme T the set $\mathrm{Hom}_T^d(\mathcal{C} \times_S T, \mathbb{P}_T^3)$ of T-morphisms f from $\mathcal{C} \times_S T$ to $\mathbb{P}_T^3$ of degree d.

This functor is representable (see [14]). We denote by $\mathcal{H} = \mathcal{H}\mathrm{om}_S^d(\mathcal{C}, \mathbb{P}_S^3)$ the S-scheme representing it and by $\phi\colon \mathcal{C} \times_S \mathcal{H} \to \mathbb{P}_{\mathcal{H}}^3$ the universal $\mathcal{H}$-morphism. The bijective map from $\mathrm{Hom}_S(T, \mathcal{H})$ to $\mathrm{Hom}_T^d(\mathcal{C} \times_S T, \mathbb{P}_T^3)$ associates to an S-morphism $\varphi\colon T \to \mathcal{H}$ the T-morphism induced by the S-morphism $p \circ \phi \circ (1_{\mathcal{C}} \times \varphi)$, where p is the projection from $\mathbb{P}_{\mathcal{H}}^3$ to $\mathbb{P}_S^3$.

We note that a k-point of $\mathcal{H}\mathrm{om}_S^d(\mathcal{C}, \mathbb{P}_S^3)$ is a pair (C, f), where C is a genus g smooth curve which is the fibre of $\mathcal{C}$ at a k-point of S and f is a degree d morphism from C to $\mathbb{P}_k^3$.

DEFINITION 1.1. – We consider the canonical morphism $d\phi\colon T_{\mathcal{C} \times_S \mathcal{H}/\mathcal{H}} \to \phi^* T_{\mathbb{P}_{\mathcal{H}}^3/\mathcal{H}}$. The quotient sheaf $\mathrm{coker}(d\phi)$ is denoted by N_ϕ and is called the *normal sheaf of the universal morphism* ϕ (*universal normal sheaf* for short).

We will prove that N_ϕ gives the family of normal sheaves of morphisms from $\mathcal{C}$ to $\mathbb{P}_S^3$.

LEMMA 1.2. – *The following exact sequence*

$$0 \to T_{\mathcal{C} \times_S \mathcal{H}/\mathcal{H}} \to \phi^* T_{\mathbb{P}_{\mathcal{H}}^3/\mathcal{H}} \to N_\phi \to 0$$

remains exact after any base change.

PROOF. – We prove that the sequence remains exact after every S-morphism from T to $\mathcal{H}$ and after every k-morphism from S' to S. Let $\varphi\colon T \to \mathcal{H}$ be an S-morphism and let us consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathcal{C} \times_S T & \xrightarrow{\phi_T} & \mathbb{P}_T^3 & \longrightarrow & T \\
{\scriptstyle 1 \times \varphi} \downarrow & & \downarrow {\scriptstyle 1 \times \varphi} & & \downarrow {\scriptstyle \varphi} \\
\mathcal{C} \times_S \mathcal{H} & \xrightarrow{\phi} & \mathbb{P}_{\mathcal{H}}^3 & \longrightarrow & \mathcal{H}
\end{array}$$

From [15] 20.5.4.1 and 20.5.7.3, we obtain the following commutative diagram:

$$\begin{array}{ccccccc}
\phi_T^* \Omega_{\mathbb{P}^3_T/T} & \longrightarrow & \Omega_{\mathcal{C}\times_S T/T} & \longrightarrow & \Omega_{\mathcal{C}\times_S T/\mathbb{P}^3_T} & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \\
(1\times\varphi)^* \phi^* \Omega_{\mathbb{P}^3_{\mathcal{H}}/\mathcal{H}} & \longrightarrow & (1\times\varphi)^* \Omega_{\mathcal{C}\times_S \mathcal{H}/\mathcal{H}} & \longrightarrow & (1\times\varphi)^* \Omega_{\mathcal{C}\times_S \mathcal{H}/\mathbb{P}^3_{\mathcal{H}}} & \longrightarrow & 0
\end{array}$$

Since $\Omega_{X\times_S Y/Y} \cong \Omega_X \otimes \mathcal{O}_Y$, the two first vertical arrows are isomorphisms. Moreover $\Omega_{\mathcal{C}\times_S T/\mathbb{P}^3_T}$ is a torsion sheaf supported at a closed scheme of $\mathcal{C}\times_S T$ and then its dual sheaf is zero.

$\mathcal{C}$ and $\mathbb{P}^3_S$ are two smooth S-schemes, then dualizing the above diagram gives us the following commutative diagram (see [16] Proposition 16.5.11):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_{\mathcal{C}\times_S T/T} & \longrightarrow & \phi^* T_{\mathbb{P}^3_T/T} & \longrightarrow & N_{\phi_T} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & (1\times\varphi)^* T_{\mathcal{C}\times_S \mathcal{H}/\mathcal{H}} & \longrightarrow & (1\times\varphi)^* \phi^* T_{\mathbb{P}^3_{\mathcal{H}}/\mathcal{H}} & \longrightarrow & (1\times\varphi)^* N_\phi & \longrightarrow & 0
\end{array}$$

Since the vertical arrows are isomorphisms, the first row is an exact sequence.

Now let $S'\to S$ be a base extension. We write $\mathcal{C}'=\mathcal{C}\times_S S'$ and $\mathcal{H}'=\mathcal{H}\times_S S'$; we note that $\mathcal{H}'$ is the scheme representing the functor $\underline{\mathrm{Hom}}^d_{S'}(\mathcal{C}', \mathbb{P}^3_{S'})$. We have the following commutative diagram:

$$\begin{array}{ccccc}
\mathcal{C}'\times_{S'} \mathcal{H}' & \xrightarrow{\phi'} & \mathbb{P}^3_{\mathcal{H}'} & \longrightarrow & \mathcal{H}' \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{C}\times_S \mathcal{H} & \xrightarrow{\phi} & \mathbb{P}^3_{\mathcal{H}} & \longrightarrow & \mathcal{H}
\end{array}$$

and we conclude as above.

PROPOSITION 1.3. – *The universal normal sheaf* N_ϕ *is flat over* $\mathcal{H}$.

PROOF. – The sequence of Lemma 1.2 remains exact after any base change and $\phi^* T_{\mathbb{P}^3_{\mathcal{H}}/\mathcal{H}}$ is a flat sheaf over $\mathcal{H}$. Thus the quotient sheaf N_ϕ is flat over $\mathcal{H}$ (see [4] ch. I § 2 n. 5). ■

The Euler exact sequence $0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3} \otimes k^4 \to T_{\mathbb{P}^3}(-1) \to 0$ gives the following exact sequences:

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & T_{\mathcal{C} \times_S \mathcal{H}/\mathcal{H}}(-1) & & \\ & & & & & & \downarrow & & \\ 0 & \to & \mathcal{O}_{(\mathcal{C} \times_S \mathcal{H})}(-1) & \to & \mathcal{O}_{(\mathcal{C} \times_S \mathcal{H})} \otimes k^4 & \to & \phi^* T_{\mathbb{P}^3_{\mathcal{H}}/\mathcal{H}}(-1) & \to & 0 \\ & & & & & & \downarrow & & \\ & & & & & & N_\phi(-1) & & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

We have $\mathbb{P}(N_\phi(-1)) \subset \mathbb{P}(\phi^* T_{\mathbb{P}^3_{\mathcal{H}}/\mathcal{H}}(-1)) \subset \mathcal{C} \times_S \mathbb{P}^{3\vee}_{\mathcal{H}}$.

Proposition 1.4. – *The scheme $\mathbb{P}(N_\phi)$ is flat over $\mathcal{H}$.*

Proof. – $\mathbb{P}(N_\phi)$ as an $\mathcal{H}$-subscheme of $\mathbb{P}(\phi^* T_{\mathbb{P}^3_{\mathcal{H}}/\mathcal{H}})$ is locally defined by a unique equation.

Thus $\mathbb{P}(N_\phi)$ is a flat $\mathcal{H}$-scheme if and only if it is a Cartier $\mathcal{H}$-divisor of $\mathbb{P}(\phi^* T_{\mathbb{P}^3_{\mathcal{H}}/\mathcal{H}})$.

Write $L = T_{\mathcal{C} \times_S \mathcal{H}/\mathcal{H}}(-1)$, $T = \phi^* T_{\mathbb{P}^3_{\mathcal{H}}/\mathcal{H}}(-1)$ and $N = N_\phi(-1)$.

We have the following exact sequence: $0 \to L \to T \to N \to 0$. Consider the following projection q:

$$\mathbb{P}(N) \subset \mathbb{P}(T) \subset \mathcal{C} \times_S \mathcal{H} \times_S \mathbb{P}^{3\vee}_S \xrightarrow{q} \mathcal{C} \times_S \mathcal{H}.$$

We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & q^* L & \to & q^* T & \to & q^* N & \to & 0 \\ & & {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\tau} & & \downarrow & & \\ 0 & \to & \mathfrak{I}_{\mathbb{P}(N)} \otimes \mathcal{O}_{\mathbb{P}(T)}(1) & \to & \mathcal{O}_{\mathbb{P}(T)}(1) & \to & \mathcal{O}_{\mathbb{P}(N)}(1) & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

(α is induced by τ).

Since morphisms f are not constant, locally the intersection of $q^* L$ and $\ker \tau$ in $q^* T$ is zero. Then $q^* T$ is locally a direct sum of $q^* L$ and $\ker \tau$. So we conclude that α is an isomorphism and $\mathfrak{I}_{\mathbb{P}(N)}$ is isomorphic to the line bundle $\mathcal{O}_{\mathbb{P}(T)}(-1) \otimes q^* L$.

DEFINITION 1.5. – The natural morphism $\pi \colon \mathbb{P}(N_\phi(-1)) \hookrightarrow \mathcal{C} \times_S \mathbb{P}^{3\vee}_{\mathcal{H}} \to \mathbb{P}^{3\vee}_{\mathcal{H}}$ is called the *universal morphism giving the family of algebraic dual surfaces of degree d morphisms from* $\mathcal{C}$ *to* $\mathbb{P}^3_S$.

Note that we have $\deg N_\phi(-1) = \deg \pi = 2d - 2 + 2g$.

Let b be a positive integer, we denote by $\overset{\circ}{\mathrm{Quot}}{}^{1,\,b}(N_\phi(-1),\, \mathcal{C} \times_S \mathcal{H},\, \mathcal{H})$ the quasi-projective scheme of rank 1 degree b quotient sheaves of $N_\phi(-1)$, that are flat over $\mathcal{H}$ and locally free over $\mathcal{C} \times_S \mathcal{H}$.

THEOREM 1.6. – *We have a natural* $\mathcal{H}$*-morphism*

$$\gamma \colon \overset{\circ}{\mathrm{Quot}}{}^{1,\,b}(N_\phi(-1),\, \mathcal{C} \times_S \mathcal{H},\, \mathcal{H}) \to \mathcal{H}\mathrm{om}^b_{\mathcal{H}}(\mathcal{C} \times_S \mathcal{H},\, \mathbb{P}^{3\vee}_{\mathcal{H}})$$

where $\mathcal{H}\mathrm{om}^b_{\mathcal{H}}(\mathcal{C} \times_S \mathcal{H},\, \mathbb{P}^{3\vee}_{\mathcal{H}})$ *denotes the scheme of degree b* $\mathcal{H}$*-morphisms from* $\mathcal{C} \times_S \mathcal{H}$ *to* $\mathbb{P}^{3\vee}_{\mathcal{H}}$.

PROOF. – Both $\mathcal{H}$-schemes represent contravariant functors from the category of $\mathcal{H}$-schemes to the category of sets. We denote these functors by F and G respectively.

We construct a functor map $g \colon F \to G$ in order to define the morphism γ.

Let T be an $\mathcal{H}$-scheme. The Euler exact sequence gives the following exact sequences:

$$\begin{array}{ccccc} \mathcal{O}_{(\mathcal{C} \times_S \mathcal{H}) \times_{\mathcal{H}} T} \otimes k^4 & \to & \phi^* T_{\mathbb{P}^3_{\mathcal{H}}/\mathcal{H}}(-1) \otimes_{\mathcal{O}_{\mathcal{H}}} \mathcal{O}_T & \to & 0 \\ & & \downarrow & & \\ & & N_\phi(-1) \otimes_{\mathcal{O}_{\mathcal{H}}} \mathcal{O}_T & & \\ & & \downarrow & & \\ & & 0 & & \end{array}$$

If Q is an element of $F(T)$, then Q is also a rank 1 locally free quotient of $\mathcal{O}_{(\mathcal{C} \times_S \mathcal{H}) \times_{\mathcal{H}} T} \otimes k^4$ and so Q induces a canonical $\mathcal{H}$-morphism λ_Q from $(\mathcal{C} \times_S \mathcal{H}) \times_{\mathcal{H}} T$ to $\mathbb{P}^{3\vee}_{\mathcal{H}}$, such that $Q = \lambda_Q^* \mathcal{U}$, where $\mathcal{U}$ denotes the universal quotient of $\mathbb{P}^{3\vee}_{\mathcal{H}}$. Thus λ_Q has degree b. So we have defined a map $g_T \colon F(T) \to G(T)$, $g_T(Q) = \lambda_Q$.

Let $h \colon T \to T'$ be an $\mathcal{H}$-morphism of $\mathcal{H}$-schemes. We have $g_T \circ F(h) = G(h) \circ g_{T'}$ because $g_T \circ F(h)(Q) = \lambda_{h^* Q}$, with $(\lambda_{h^*Q})^* \mathcal{U} = h^* Q$, and $G(h) \circ g_{T'}(Q) = \lambda_Q \circ (1 \times h)$, with $(\lambda_Q \circ (1 \times h))^* \mathcal{U} = (1 \times h)^* (\lambda_Q^* \mathcal{U}) = (1 \times h)^* Q = h^* Q$. ∎

We write $\mathcal{Q}_b = \mathring{\mathrm{Quot}}^{1,b}(N_\phi(-1), \mathcal{C}\times_S \mathcal{H}, \mathcal{H})$ and *we omit* the index b when there can be no ambiguity.

DEFINITION 1.7. – The morphism γ of Theorem 1.6 gives a natural degree b $\mathcal{H}$-morphism from $\mathcal{C}\times_S \mathcal{H}\times_{\mathcal{H}}\mathcal{Q}$ to $\mathbb{P}^{3\vee}_{\mathcal{H}}$ and thus a degree b $\mathcal{Q}$-morphism $\lambda\colon \mathcal{C}\times_S\mathcal{Q}\to\mathbb{P}^{3\vee}_{\mathcal{Q}}$.

The morphism λ is called the *degree b Gauss morphism associated to the universal normal sheaf* N_ϕ.

Now we consider the canonical morphism $d\lambda\colon T_{\mathcal{C}\times_S\mathcal{Q}/\mathcal{Q}}\to\lambda^* T_{\mathbb{P}^{3\vee}_{\mathcal{Q}}/\mathcal{Q}}$ induced by the Gauss morphism λ, the normal sheaf $N_\lambda = \operatorname{coker} d\lambda$ of λ and the morphism $\sigma\colon \mathbb{P}(N_\lambda(-1))\subset\mathcal{C}\times_S\mathbb{P}^3_{\mathcal{Q}}\to\mathbb{P}^3_{\mathcal{Q}}$. We have, as above, that the sheaf N_λ is flat over $\mathcal{Q}$ and the scheme $\mathbb{P}(N_\lambda)$ is flat over $\mathcal{Q}$. Note that the morphism σ has degree $2b-2+2g$.

DEFINITION 1.8. – The morphism σ from $\mathbb{P}(N_\lambda(-1))$ to $\mathbb{P}^3_{\mathcal{Q}}$ is called the *universal morphism giving the family of degree* $2b-2+2g$ *algebraic characteristic surfaces of degree d morphisms from* $\mathcal{C}$ *to* $\mathbb{P}^3_S$.

We consider as an example $\mathcal{C}=\mathbb{P}^1_k$. We denote by $q_b\colon\mathcal{Q}_b\to\mathcal{H}$ the structural morphism and by $q_b(\mathcal{Q}_b)$ the scheme image of q_b.

We denote by $N(2d-2-b, b)$ the scheme $(q_b(\mathcal{Q}_b)-q_{b-1}(\mathcal{Q}_{b-1}))\cap\mathcal{H}'$, where $\mathcal{H}'$ denotes the open set of $\mathcal{H}$ consisting of embeddings. A k-point of $N(2d-2-b, b)$ is an embedding $f\colon\mathbb{P}^1_k\to\mathbb{P}^3_k$ of degree d such that the vector bundle $N_f(-1)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1_k}(2d-2-b)\oplus\mathcal{O}_{\mathbb{P}^1_k}(b)$. We have the following:

COROLLARY 1.9 (see [7]). – *If* $b<d-1$, *then there is a natural morphism* $G\colon N(2d-2-b, b)\to\mathcal{H}\mathrm{om}^b(\mathbb{P}^1_k, \mathbb{P}^{3\vee}_k)$.

PROOF. – Let $\mathcal{Q}'$ the inverse image of $\mathcal{H}'$ by q_b. If $b<d-1$, then q_b is an isomorphism between $\mathcal{Q}'$ and $N(2d-2-b, b)$. So the $\mathcal{H}$-morphism $\gamma\colon\mathcal{Q}\to\mathcal{H}\mathrm{om}^b_{\mathcal{H}}(\mathbb{P}^1_k\times\mathcal{H}, \mathbb{P}^{3\vee}_{\mathcal{H}})$ of Theorem 1.6 gives the morphism G. ■

In Corollary 1.9 we have obtained in a *natural way* the morphism G that Eisenbud and Van de Ven described in [7] Theorem 5.1. The morphism G has been the basis for studying the stratification of the scheme $\mathcal{H}'$ of embeddings f from $\mathbb{P}^1_k$ to $\mathbb{P}^3_k$ by the splitting type of $N_f(-1)$ (see [6] and [7]).

2. – Dual and characteristic surfaces of curves.

Now we want to describe fibres of the morphism γ of Theorem 3 by means of curves on developable ruled surfaces. We use the geometric construction that Eisenbud and Van de Venn gave in [6] and [7].

We consider a smooth curve C of genus g over k and a morphism $f\colon C \to \mathbb{P}^3_k$ of degree $d \geq 2$. The fibre $(N_f)_x$ of the normal sheaf N_f of f is a free $\mathcal{O}_{C,x}$-module if and only if x is not a ramification point of f and $f(x)$ is not a cusp. Moreover, if f is an embedding, the normal sheaf N_f is isomorphic to the normal bundle N_X of $X = f(C)$ in $\mathbb{P}^3_k$.

DEFINITION 2.1. – The quotient N'_f of the normal sheaf N_f by its torsion subsheaf R_f is called the *normal bundle of the morphism f.*

The number $\kappa = \deg R_f$ is called *the number of points x of C such that either x is a ramification point or f(x) is a cusp* (see [25] §3).

The image of the canonical morphism $\pi\colon \mathbb{P}(N'_f(-1)) \to \mathbb{P}^{3\vee}_k$ is called the *dual surface of f.*

Let $\wp^m(f^*\mathcal{O}_{\mathbb{P}^3_k}(1))$ be the bundle of principal parts of order m ($m = 1, 2$), it has rank $(m+1)$ and we have a canonical morphism $a^m\colon (k^4)^\vee \otimes \mathcal{O}_C \to \wp^m(f^*(\mathcal{O}_{\mathbb{P}^3_k}(1))$.

DEFINITION 2.2 (see [25]). – The bundle $\wp^m_f = \operatorname{Im}(a^m)$ is called the *osculating bundle of order m of f.*

The rank 2 bundle $\wp^1_f$ gives a natural morphism τ^1_f from C to the Grassmannian $G(1,3)$ of lines in $\mathbb{P}^3_k$, describing tangents lines of $f(C)$. The bundle $\wp^2_f$ gives a morphism τ^2_f from C to $\mathbb{P}^{3\vee}_k$ describing osculating planes of $f(C)$.

Assume that f is a non-degenerate morphism. The image of the canonical morphism $p\colon \mathbb{P}(\wp^1_f) \hookrightarrow C \times \mathbb{P}^3_k \to \mathbb{P}^3_k$ is the tangent developable surface T_f of f. The image of τ^2_f in $\mathbb{P}^{3\vee}_k$ is the dual curve of $f(C)$ and τ^2_f is denoted by $f^\vee$.

If f is a degenerate morphism, the image of τ^1_f is the (plane) dual curve of $f(C)$ and τ^1_f is denoted by $f^\vee$.

LEMMA 2.3. – *We have the following exact sequence of bundles over C*:

$$0 \to (\wp^1_f)^\vee \to k^4 \otimes \mathcal{O}_C \to N'_f(-1) \to 0\,.$$

PROOF. – From properties satisfied by osculating bundles and described in [25], we deduce the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{O}_C(-1) & = & \mathcal{O}_C(-1) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \wp^1(f^*\mathcal{O}_{\mathbb{P}^3_k}(1))^\vee & \overset{a^{1\vee}}{\to} & k^4\otimes\mathcal{O}_C & \to & N_f(-1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & T_C(-1) & \to & f^*T_{\mathbb{P}^3_k}(-1) & \to & N_f(-1) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Thus we find the following exact sequence:

$$0\to N_f(-1)^\vee\to k^{4\vee}\otimes\mathcal{O}_C\overset{a^1}{\to}\wp^1(f^*\mathcal{O}_{\mathbb{P}^3_k}(1))$$

and we obtain the thesis by dualizing it.

PROPOSITION 2.4. – *If f is a non-degenerate morphism from C to $\mathbb{P}^3_k$, then the dual surface S_f of f is the tangent developable surface of the morphism $f^\vee$ giving the dual curve of $f(C)$.*

If f is a degenerate morphism, then we have $N_f'(-1)\cong\mathcal{O}_C\oplus(f^\vee)^*\mathcal{O}_{\mathbb{P}^{2\vee}}(1)$ and the dual surface S_f of f is the cone over the (plane) dual curve $f^\vee(C)$ with vertex the point corresponding to the plane containing $f(C)$.

PROOF. – If f is non-degenerate then, from the above Lemma and from Lemma 5.2 of [25], we obtain canonical isomorphisms of the following exact sequences:

$$\begin{array}{ccccccccc}
0 & \to & (N'_{f^\vee}(-1))^\vee & \to & k^4\otimes\mathcal{O}_C & \to & \wp^1_{f^\vee} & \to & 0 \\
 & & \uparrow\wr & & \| & & \uparrow\wr & & \\
0 & \to & (\wp^1_f)^\vee & \to & k^4\otimes\mathcal{O}_C & \to & N_f'(-1) & \to & 0
\end{array}$$

If f is degenerate, we have $N_f'(-1)\cong\mathcal{O}_C\oplus N'_{f,H}(-1)$, where H is the plane containing $f(C)$ and $N'_{f,H}$ denotes the normal bundle of f in H. From Lemma 2.3 $N'_{f,H}(-1)$ is isomorphic to $(f^\vee)^*\mathcal{O}_{\mathbb{P}^{2\vee}}(1)$ and we have the assertion. ■

Let L be a line subbundle of $N_f'(-1)$. It gives an exact sequence of vector bundles $0 \to L \to N_f'(-1) \to Q \to 0$. The injection $\mathbb{P}(Q) \hookrightarrow \mathbb{P}(N_f'(-1))$ gives a section of the projective bundle $\mathbb{P}(N_f'(-1))$.

DEFINITION 2.5. – The restriction of the morphism π to $\mathbb{P}(Q)$ (see Definition 2.1) gives a morphism $\lambda\colon C \to \mathbb{P}_k^{3\vee}$ called the *Gauss morphism associated to the line subbundle L of $N_f'(-1)$* (see [7]). We have $\lambda^*(\mathcal{O}_{\mathbb{P}^{3\vee}}(1)) \cong Q$.

If $\deg \lambda > 0$, let x be a point of C, the dual line in $\mathbb{P}_k^3$ of the tangent line of $\lambda(C)$ at $\lambda(x)$ is called the *characteristic line of L at x*.

If $\deg \lambda > 1$, the set of characteristic lines of L forms a ruled surface S_L called the *characteristic surface of L*.

PROPOSITION 2.6. – *The characteristic line of L at $x \in C$ contains the point $f(x)$.*

PROOF. – We denote by $k[x_0, x_1, x_2, x_3]$ and $k[X_0, X_1, X_2, X_3]$ the homogeneous coordinate ring of $\mathbb{P}^3$ and $\mathbb{P}^{3\vee}$ respectively. We assume that $f(x)$ and $\lambda(x)$ are contained in the affine open sets $x_0 \neq 0$ and $X_0 \neq 0$ respectively.

Morphisms f and λ are locally defined at x by equations of the following type respectively:

$$\begin{cases} x_1 = a_1(t) \\ x_2 = a_2(t) \\ x_3 = a_3(t) \end{cases} \quad \text{and} \quad \begin{cases} X_1 = A_1(t) \\ X_2 = A_2(t) \\ X_3 = A_3(t). \end{cases}$$

Suppose that for both morphisms f and λ the point x gives neither a ramification point nor a cusp.

The characteristic line of L at x has equations

$$x_0 + \sum_{i=1}^{3} A_i(0)\, x_i = \sum_{i=1}^{3} A_i'(0)\, x_i = 0 .$$

By using the duality between tangent developable surfaces and dual ones, we have that the plane $x_0 + \sum_{i=1}^{3} A_i(t)\, x_i = 0$ contains the tangent line of $f(C)$ at $f(t)$, then we have $1 + \sum_{i=1}^{3} A_i(t)\, a_i(t) = \sum_{i=1}^{3} A_i(t)\, a_i'(t) = 0$. The derivate of the first equation gives $\sum_{i=1}^{3} A_i'(t)\, a_i(t) = 0$ and then the characteristic line of L at x contains $f(x)$.

For the other points x of C the proof is similar.

REMARK 2.7. – Let $f\colon C \to \mathbb{P}^3$ be a degree d morphism, with $d \geqslant 2$, and $0 \to L \to N_f'(-1) \to Q \to 0$ an exact sequence of vector bundles, with $\deg Q \geqslant 2$. The line subbundle L of $N_f'(-1)$ induces a Gauss morphism

$\lambda: C \to \mathbb{P}^{3\vee}$, with $\lambda^*(\mathcal{O}_{\mathbb{P}^{3\vee}}(1)) \cong Q$, and the curve $\lambda(C)$ lies on the dual surface S_f of f.

The above Proposition prove that the characteristic surface S_L of L (i.e. the dual surface of λ) contains the curve $f(C)$ and we have a quotient $N'_\lambda(-1) \to f^*\mathcal{O}_{\mathbb{P}^3}(1) \to 0$. We can say that f is the Gauss morphism associated to a line subbundle of $N'_\lambda(-1)$.

Note that $f(C)$ is the projection of a unisecant section of $\mathbb{P}(N'_\lambda(-1))$. If λ is birational, $\mathbb{P}(N'_\lambda(-1))$ is isomorphic to the desingularisation $\widetilde{S}_\lambda$ of the dual surface S_λ of λ. If λ is a r: 1 morphism ($r > 1$), the desingularisation $\widetilde{S}_\lambda$ of S_λ is isomorphic to a rank 2 projective bundle $\mathbb{P}(E)$ on the desingularisation of the curve $\lambda(C)$. In this case $f(C)$ is the projection of a r-secant section of $\mathbb{P}(E) \cong \widetilde{S}_\lambda$.

We can conclude that the fibre of the morphism γ of Theorem 1.6 at the point (C, λ) is given by the pairs (f, Q), where f is the morphism associated to a degree d line bundle quotient of $N'_\lambda(-1)$ and Q is $\lambda^*\mathcal{O}_{\mathbb{P}^{3\vee}}(1)$.

3. – Stratifications.

Let E be a rank 2 vector bundle over a genus g smooth curve C. To E associate the integer $\tilde{s}(E) = \deg E - 2 \max\{\deg L\}$, where the maximum is taken over all rank 1 subsheaves L of E (see [21]).

A rank 1 subsheaf L of E of maximal degree is a line subbundle of E and it is called a *maximal subbundle* of E.

Note that $\tilde{s}(E) \equiv \deg E (\mathrm{mod}\, 2)$ and that E is called *stable* (resp. *semistable*) if and only if $\tilde{s}(E) > 0$ (resp. $\tilde{s}(E) \geqslant 0$). If $\tilde{s}(E) < 0$ E is called *unstable*.

Let C_0 be a section of the projective bundle $\mathbb{P}(E)$ of minimal self-intersection. C_0 is given by a maximal subbundle of E and we have $C_0^2 = \tilde{s}(E)$. Nagata proved in [22] that $\tilde{s}(E) \leqslant g$.

The normal bundle N of a smooth curve in $\mathbb{P}^3$ has an even degree. We give the following definition:

DEFINITION 3.1. – If N is a rank 2 vector bundle of even degree, we put $s(N) = (1/2)\, \tilde{s}(N)$. If $s(N) > 0$, the bundle N is stable and we say that $s(N)$ is the *stability degree of* N; if $s(N) < 0$, N is unstable, we write $\sigma(N) = -s(N)$ and we say that $\sigma(N)$ is the *instability degree of* N.

The definition of stability degree of the normal bundle of a space curve used by Ellingsrud and Hirschowitz in [10] is the one given above.

We want to study curves in $\mathbb{P}^3$ having an unstable normal bundle.

Note that we have the following fact:

LEMMA 3.2. – *An unstable rank 2 vector bundle N has a unique maximal subbundle.*

PROOF. – Let us suppose that there exist two maximal subbundles L_1 and L_2 of N. We have two exact sequences of vector bundles over C: $0 \to L_1 \to N \to Q_1 \to 0$ and $0 \to L_2 \to N \to Q_2 \to 0$. Then we have a natural morphism $\varphi: N \to Q_1 \oplus Q_2$. The morphism φ is non-null, then we have $\operatorname{rank}(\ker\varphi) \leq 1$. L_1 and L_2 are two different line subbundles of N, then $\ker\varphi$ is of rank 0 and so φ is an injective morphism. Since N is unstable, we have $\deg N > \deg Q_1 + \deg Q_2$, absurd.

Now we define the stratification of the Hilbert scheme of smooth curves in $\mathbb{P}^3$ with unstable normal bundle by the instability degree of the normal bundle and we estimate the «right» dimension of these strata.

NOTATION 3.3. – Let $\mathcal{C} \to M_{g,m}$ be the universal curve over the fine moduli space of genus g smooth curves with level m structure, for $g \geq 2$.

We *denote by* $\mathcal{H}_{d,g,m}$ the open set of $\mathcal{H}om^d_{M_{g,m}}(\mathcal{C}, \mathbb{P}^3_{M_{g,m}})$ consisting of embeddings and by $\phi: \mathcal{C} \times_{M_{g,m}} \mathcal{H}_{d,g,m} \to \mathbb{P}^3_{M_{g,m}}$ the universal morphism. In the étale topology, we can consider the relative Picard scheme $\operatorname{Pic}^{\mu}_{\mathcal{H}_{d,g,m}}(\mathcal{C} \times_{M_{g,m}} \mathcal{H}_{d,g,m})$ and its closed subschemes (see [18])

$$\mathcal{P}_{d,g,m}(\mu) = \{(C, f, L) \in \operatorname{Pic}^{\mu}_{\mathcal{H}_{d,g,m}}(\mathcal{C} \times_{M_{g,m}} \mathcal{H}_{d,g,m}) / h^0(C, N_f \otimes L^{\vee}) \geq 1\}.$$

(Notice that L is a subsheaf of N_f if and only if $h^0(C, N_f \otimes L^{\vee}) \geq 1$.)

If $\mathcal{P}_{d,g,m}(\mu)$ is non-empty, then we have $\operatorname{codim} \mathcal{P}_{\mathrm{d,g,m}}(\mu) \leq |\chi| + 1$, where $\chi = \chi(N_f \otimes L^{\vee}) = 4d - 2\mu$ (see [1] and [18]).

Let $\mathcal{P}^0_{d,g,m}(\mu)$ be the scheme image of the natural projection of $\mathcal{P}_{d,g,m}(\mu)$ in $\mathcal{H}_{d,g,m}$.

Note that if L is a rank 1 subsheaf of N_f, then for every point p of C the line bundle $L(-p)$ is, in a natural way, a rank 1 subsheaf of N_f. So in $\mathcal{H}_{d,g,m}$ we have $\mathcal{P}^0_{d,g,m}(\mu+1) \subset \mathcal{P}^0_{d,g,m}(\mu)$. Let us *denote by* $\mathcal{N}_{d,g,m}(2d-1+g-\mu)$ the locally closed scheme $\mathcal{P}^0_{d,g,m}(\mu) - \mathcal{P}^0_{d,g,m}(\mu+1)$. Let $\sigma = \mu - 2d + 1 - g > 0$. Then $\mathcal{N}_{d,g,m}(-\sigma)$ parametrizes all the degree d embeddings f of genus g smooth curves with level m structure in $\mathbb{P}^3$ having an unstable normal bundle N_f with degree of instability σ.

For $g = 1$, we consider the fine moduli space $M_{1,m}$ of polarized smooth elliptic curves with level m structure and we can do as above.

REMARK 3.4. – Let σ be a positive integer. If ξ is a k-point of $\mathcal{N}_{d,g,m}(-\sigma)$ giving an embedding $f: C \to \mathbb{P}^3$ with $h^1(f^* T_{\mathbb{P}^3}) = 0$, then for each irreducible component W of $\mathcal{N}_{d,g,m}(-\sigma)$ containing ξ we have $\dim W \geq 4d - g + 1 - 2\sigma$ if $g \geq 2$ and $\dim W \geq 4d + 1 - 2\sigma$ if $g = 1$.

Indeed, let $\mu = 2d - 1 + g + \sigma$. We have $\operatorname{codim} \mathcal{P}_{d,g,m}(\mu) \leq |\chi| + 1$, where $\chi = \chi(N_f \otimes L^\vee) = -2(g + \sigma - 1)$ (see [1] and [18]). From Lemma 3.2 we have $\dim \mathcal{N}_{d,g,m}(-\sigma) = \dim \mathcal{P}_{d,g,m}(\mu) \geq \dim \mathcal{H}_{d,g,m} + g - 2(g + \sigma - 1) - 1$.

We have that ξ is a smooth point in $\mathcal{H}_{d,g,m}$ and $h^0(f^* T_{\mathbb{P}^3}) = 4d + 3 - 3g$. Thus every irreducible component of $\mathcal{H}_{d,g,m}$ containing ξ has dimension $\geq 4d$ if $g \geq 2$ and $\geq 4d + 1$ if $g = 1$.

In a similar way, we can stratify the Hilbert scheme $I_{d,g,m}$ of degree d genus g smooth curves C in $\mathbb{P}^3$ with level m structure and also the Hilbert scheme $I_{d,g}$ of degree d genus g smooth curves C in $\mathbb{P}^3$ by the normal bundle N_C.

NOTATION 3.5. – If $\sigma > 0$, we *denote* by $N_{d,g,m}(-\sigma)$ and $N_{d,g}(-\sigma)$ the strata in $I_{d,g,m}$ and $I_{d,g}$ respectively parametrizing curves C having unstable normal bundle N_C with instability degree σ.

If $h^1(N_C) = 0$, C is a smooth point in the Hilbert scheme and every irreducible component of $N_{d,g,m}(-\sigma)$ and of $N_{d,g}(-\sigma)$ containing C has dimension $\geq 4d - g + 1 - 2\sigma$.

There exist natural morphisms $\mathcal{H}_{d,g,m} \xrightarrow{\alpha} I_{d,g,m} \xrightarrow{\alpha'} I_{d,g}$ and then also [4] $\mathcal{N}_{d,g,m}(-\sigma) \xrightarrow{\alpha} N_{d,g,m}(-\sigma) \xrightarrow{\alpha'} N_{d,g}(-\sigma)$.

We note that fibres of α' are finite, while fibres of α are finite if $g \geq 2$ and of dimension 1 if $g = 1$.

4. – Unstable normal bundles and some strata of the right dimension.

NOTATION 4.1. – We denote by $D_S(g)$ the minimum integer d such that there exists a degree d genus g smooth curve C in $\mathbb{P}^3$ whose normal bundle N_C is stable and by $D_S^0(g)$ (resp. $D_{SS}^0(g)$) the minimum integer d such that there exists a degree d genus g smooth curve C in $\mathbb{P}^3$ whose normal bundle N_C is stable (resp. semi-stable) and satisfies the condition $h^1(N_C) = 0$.

If $g \geq 3$ we have $D_S^0(g) \leq g + 3$ and for $g = 2$ we have $D_S^0(2) = 6$ (see [10]). For $g = 0, 1$ the normal bundle is not stable.

LEMMA 4.2. – *For every triple of integers* (d, g, σ) *such that* $d \geq 3g + D_S(g) - 1$ *and* $4g - 2 \leq \sigma \leq d - 1 + g - D_S(g)$, *there exists a degree d genus g smooth curve in* $\mathbb{P}^3$ *whose normal bundle is unstable with instability degree* σ.

PROOF. – We consider a smooth curve C of genus g, an integer b such that $D_S(g) \leq b \leq d - 3g + 1$ and an embedding $\lambda: C \to \mathbb{P}^{3\vee}$ of degree b whose normal bundle N_λ is stable. Let s be the stability degree of N_λ $(1 \leq s \leq [g/2])$.

A maximal line subbundle F_0 of $N_\lambda(-1)$ gives an exact sequence $0 \to F_0 \to N_\lambda(-1) \to \mathcal{O}_C(D_0) \to 0$ and an unisecant section C_0 of $\mathbb{P}(N_\lambda)$ of minimal self-intersection $C_0^2 = -e = 2s$.

If D_1 is a divisor on C of degree $\geqslant -2s + 2g + 1$, then the divisor $\widetilde{C} = C_0 + D_1 f$ is very ample (see [3]) and then a general curve of the linear system $|\widetilde{C}|$ is projected by $p\colon \mathbb{P}(N_\lambda(-1) \to \mathbb{P}^3$ into a smooth curve. So we have an embedding $f\colon C \to \mathbb{P}^3$ such that $f^*(\mathcal{O}_{\mathbb{P}^3}(1)) = \mathcal{O}_C(D_1 + D_0)$ (see Remark 2.7).

If we pick up a divisor D_1 of degree $d - b + 1 - g - s$, we obtain an embedding f of degree d. The bundle $N_f(-1)$ has $\lambda^* \mathcal{O}_{\mathbb{P}^{3\vee}}(1)$ as a quotient (Remark 2.7) and then it has a line subbundle L of degree $2d - 2 + 2g - b$. Thus $N_f(-1)$ is unstable with instability degree $\sigma = d - 1 + g - b$.

THEOREM 4.3. – *For* $g \geqslant 2$, $d \geqslant 3g + D_S^0(g) - 1$ *and* $4g - 2 \leqslant \sigma \leqslant d - 1 + g - D_S^0(g)$, *the stratum* $N_{d,g}(-\sigma)$ *of the Hilbert scheme* $I_{d,g}$ *parametrizing degree d genus g smooth curves having an unstable normal bundle with instability degree* σ *is non-empty and it has an irreducible component of the right dimension* $4d - g + 1 - 2\sigma$.

PROOF. – Let $b = d - 1 + g - \sigma$, b is an integer such that $D_S^0(g) \leqslant b \leqslant d - 3g + 1$.

We consider the universal curve $\mathcal{C} \to M_{g,m}$, the scheme $\mathcal{H} = \mathcal{H}\mathrm{om}^d_{M_{g,m}}(\mathcal{C}, \mathbb{P}^3_{M_{g,m}})$ and the $\mathcal{H}$-morphism γ of Theorem 1.6.

We consider the open subscheme $\mathcal{H}_{d,g,m}$ of $\mathcal{H}$ consisting of embeddings and the stratum $\mathcal{N}_{d,g,m}(b - d + 1 - g)$ (see Notation 3.3).

Let (C, f) be a k-point of $\mathcal{N}_{d,g,m}(b - d + 1 - g)$, since $b < d - 1 + g$, the normal bundle N_f is unstable and it has a unique maximal subbundle. Thus $N_f(-1)$ has a unique rank 1 quotient of degree b. We have that the $M_{g,m}$-scheme $\mathcal{N}_{d,g,m}(b - d + 1 - g)$ is isomorphic to an open subscheme of $\mathring{\mathrm{Quot}}^{1,b}(N_\phi(-1), \mathcal{C} \times_{M_{g,m}} \mathcal{H}, \mathcal{H})$ and γ is also a k-morphism

$$\gamma\colon \mathcal{N}_{d,g,m}(b - d + 1 - g) \to \mathcal{H}\mathrm{om}^b_{M_{g,m}}(\mathcal{C}, \mathbb{P}^{3\vee}_{M_{g,m}}).$$

We consider also the natural projections

$$\begin{array}{ccc} \mathcal{N}_{d,g,m}(b - d + 1 - g) & & \mathcal{H}_{b,g,m} \\ \alpha \downarrow & \text{and} & \beta \downarrow \\ N_{d,g,m}(b - d + 1 - g) & & I_{b,g,m} \end{array}$$

where $\mathcal{H}_{b,g,m}$ denotes the open subscheme of $\mathcal{H}\mathrm{om}^b_{M_{g,m}}(\mathcal{C}, \mathbb{P}^{3\vee}_{M_{g,m}})$ consisting of embeddings and $I_{b,g,m}$ denotes the open subscheme of the Hilbert scheme $\mathrm{Hilb}_{b,g,m}(\mathbb{P}^{3\vee})$ consisting of smooth curves.

Let Λ be a degree b genus g smooth curve of $\mathbb{P}^{3\vee}$ having a stable normal bundle N_Λ with $h^1(N_\Lambda) = 0$. We denote by $H_{b,g,m}$ the irreducible component of dimension $4b$ of $I_{d,g,m}$ defined by Λ with a level m structure.

Λ is given by a degree b embedding $\lambda\colon C \to \mathbb{P}^{3\vee}$ of $\mathcal{H}_{b,g,m}$.

The fibre of γ at (C, λ) is an (irreducible) open subscheme of the Hilbert scheme of curves in $\mathbb{P}(N_\lambda(-1))$ of class $\widetilde{C} = C_0 + (d - b + 1 - g - s)f$ in the Neron-Severi group, where s denotes the stability degree of N_λ.

By the Kodaira Vanishing Theorem, we have $h^1(\mathcal{O}(\widetilde{C})) = 0$ (see [3] §1) and then, by the Riemann-Roch Theorem, we have $\dim|\widetilde{C}| = \widetilde{C}^2 + 1 - 2g$. Thus we have found a non-empty irreducible component of $\mathcal{N}_{d,g,m}(b - d + 1 - g)$ of dimension $4b + \widetilde{C}^2 + 1 - g = 4d - g + 1 - 2\sigma$.

We conclude by considering the (finite) projections into $N_{d,g,m}(b - d + 1 - g)$ and $N_{d,g}(b - d + 1 - g) = N_{d,g}(-\sigma)$. ■

When there exist degree b curves having stable normal bundle with maximum stability degree $s = [g/2]$, we can amplify the range of σ in the above Theorem. We have:

PROPOSITION 4.4. – *For $g \geq 4$, $d \geq 6g - 2$ and $4g - 1 - [g/2] \leq \sigma \leq 4g - 3$, the stratum $N_{d,g}(-\sigma)$ of the Hilbert scheme $I_{d,g}$ parametrizing degree d genus g smooth curves having an unstable normal bundle with instability degree σ is non-empty and it has an irreducible component of the right dimension $4d - g + 1 - 2\sigma$.*

PROOF. – For $b \geq 3g$ the general curve Λ of $I_{b,g}$ has a stable normal bundle N_Λ with stability degree $s = [g/2]$ and it satisfies the condition $h^1(N_\Lambda) = 0$ (see [10]). At this point we can proceed as in the above proof, by using general curves of $I_{b,g}$ with $3g \leq b \leq d - 3g + [g/2]$. ■

For $g = 2$ we obtain also the following

PROPOSITION 4.5. – *For $g=2$, $d \geq 12$ and $\sigma = d-4$, the stratum $N_{d,2}(4-d)$ of the Hilbert scheme $I_{d,2}$ parametrizing degree d genus 2 smooth curves having an unstable normal bundle with instability degree $d-4$ is non-empty and it has an irreducible component of the right dimension $2d+7$.*

PROOF. – For $b = 5$ and $g = 2$ the general curve Λ of $I_{5,2}$ lies on a smooth quadric. Thus its normal bundle is unstable with instability degree 1. Moreover we have the condition $h^1(N_\Lambda) = 0$, so we can proceed as in the proof of Theorem 4.3. ■

Since $D_S^0(g) \leq g + 3$ for $g \geq 3$ and $D_S^0(2) = 6$ (see [10]), if either $g \geq 3$ and $d \geq 4g + 2$ or $g = 2$ and $d \geq 12$, we found an irreducible component of the stratum $N_{d,g}(-\sigma)$ of the right dimension for every stability degree σ such that $4g-2 \leqslant \sigma \leqslant d-4$.

For $1 \leqslant \sigma \leqslant 4g-3$, we can prove that the stratum $N_{d,g}(-\sigma)$ is non-empty.

THEOREM 4.6. – *Assume $d \geqslant 1+2g+\sqrt{1+8g}$, then there exist degree d genus g smooth curves in $\mathbb{P}^3$ having an unstable normal bundle with instability degree σ, for every $1 \leqslant \sigma \leqslant (1/2)\,d+2g-2$.*

PROOF. – There exists a birational morphism $\lambda\colon C \to \mathbb{P}^{3\vee}$ of degree $d+g-1-\sigma$ such that the curve image Λ is plane with $\kappa = d+4g-4-2\sigma$ cusps. In fact its (plane) dual curve $\lambda^\vee\colon C \to \mathbb{P}^3$ is of degree d and has $\kappa_0 = d+g-1+\sigma$ cusps, it satisfies conditions (17′) of [30] p. 221 and its existence is proved in [30] p. 222.

We consider $\pi\colon \mathbb{P}(N'_\lambda(-1)) \to \mathbb{P}^3$, the surface image is the cone S over $\lambda^\vee(C)$ with vertex the point corresponding to the plane containing $\lambda(C)$.

We recall that $N'_\lambda(-1) = \mathcal{O}_C \oplus \mathcal{O}_C(D)$, where D is the divisor of C giving the morphism $\lambda^\vee$. $\mathcal{O}_C(D)$ as a line subbundle of $N'_\lambda(-1)$ gives the unisecant section C_0 of $\mathbb{P}(N'_\lambda(-1))$ of minimal self-intersection $C_0^2 = -d$. Furthermore $\mathcal{O}_C(D)$ as a quotient of $N'_\lambda(-1)$ gives the unisecant section $\widetilde{C} = C_0 + Df$.

The complete linear system $|D|$ gives a degree d embedding $F\colon C \to \mathbb{P}^{d-g}$. $\lambda^\vee(C)$ is the projection of $F(C)$ from a linear space l of dimension $d-g-3$. The secant variety of $F(C)$ is of dimension 3 and so it intersects l at a finite set of points. Then the general hyperplane l_0 of l does not meet the secant variety of $F(C)$. The projection from l_0 maps $F(C)$ onto a smooth curve contained in a cone over $\lambda^\vee(C)$.

The general curve of the linear system $|\widetilde{C}|$ is projected by π onto a smooth curve of degree d having an unstable normal bundle with instability degree σ.

COROLLARY 4.7. – *For $g=2$ and $d \geqslant 10$, $g=3$ and $d \geqslant 12$, $g \geqslant 4$ and $d \geqslant 4g-2$ there exist smooth curves C of degree d and genus g having unstable normal bundle with instability degree σ, for every $1 \leqslant \sigma \leqslant 4g-3$.*

PROOF. – For $g \geqslant 2$, we have $4g-3 \leqslant (1/2)\,d+2g-2$ if and only if $d \geqslant 4g-2$, then we apply the above Theorem for $d \geqslant \max\{1+2g+\sqrt{1+8g}, 4g-2\}$. ■

For $g=1$ and $d \geqslant 6$ Hulek and Sacchiero found degree d elliptic smooth curves in $\mathbb{P}^3$ having unstable normal bundle with instability degree σ, for every $1 \leqslant \sigma \leqslant d-4$.

Now we can deduce from the following Theorem that for $d \geqslant 7$ and $3 \leqslant \sigma \leqslant d-4$ the stratum $N_{d,1}(-\sigma)$ has an irreducible component of the right dimension.

THEOREM 4.8. – *For* $g \geqslant 1$, $d \geqslant 3g + D^0_{SS}(g)$ *and* $4g - 1 \leqslant \sigma \leqslant d - 1 + g - D^0_{SS}(g)$, *the stratum* $N_{d,g}(-\sigma)$ *of the Hilbert scheme* $I_{d,g}$ *parametrizing degree d genus g smooth curves having an unstable normal bundle with instability degree* σ *is non-empty and it has an irreducible component of the right dimension* $4d - g + 1 - 2\sigma$.

PROOF. – We consider the integers b such that $D^0_{SS}(g) \leqslant b \leqslant d - 3g$ and we proceed as in the proof of Theorem 4.3. ■

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