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On Nonhomogeneous Reinforcements of Varying Shape and Different Exponents.

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Sunto. – *Studiamo un problema ellittico quasilineare concernente un dominio circondato da un rinforzo sottile di spessore variabile, in cui il coefficiente dell'equazione è (localmente) non costante. Esso concerne due diversi esponenti, uno nel dominio e l'altro nel rinforzo, una condizione di Dirichlelet sulla frontiera esterna e una condizione di trasmissione. Prediciamo il comportamento asintotico della soluzione quando lo spessore, insieme con il coefficiente nel rinforzo, tende a zero perché essi siano convenientemente riferiti fra di loro.*

1. – Introduction.

Let Ω be a bounded domain of $\mathbb{R}^N$ and let Γ be the union of certain connected components of $\partial\Omega$. We assume that Ω is surrounded along Γ by a thin reinforcement

$$\Sigma^\varepsilon = \{\sigma + t\nu(\sigma),\ \sigma \in \Gamma,\ 0 < t < h^\varepsilon(\sigma)(\leqslant \varepsilon)\}$$

where $\nu(\sigma)$ denotes the outer normal to Ω at the point $\sigma \in \Gamma$. Let $\Omega^\varepsilon = \overline{\Omega} \cup \Sigma^\varepsilon$ be the reinforced domain. We study the limit behaviour (when ε tends to zero) of some quasilinear problems with two exponents $p,\ q \in (1, \infty)$ of the type

$$(1)^\varepsilon \qquad \begin{cases} -\operatorname{div}(|\nabla u^\varepsilon|^{p-2}\nabla u^\varepsilon) + |u^\varepsilon|^{p-2}u^\varepsilon = f^\varepsilon \ \text{ in } \ \Omega\,, \\ -\operatorname{div}(\mu^\varepsilon |\nabla u^\varepsilon|^{q-2}\nabla u^\varepsilon) = g^\varepsilon \ \text{ in } \ \Sigma^\varepsilon\,, \\ u^\varepsilon = 0 \ \text{ on } \ \partial\Omega^\varepsilon\,, \\ (+\text{transmission conditions on } \Gamma). \end{cases}$$

For $\Gamma = \partial\Omega$, $p = q$, μ^ε constant in Σ^ε and all the Σ^ε having the same shape (i.e. $h^\varepsilon(\sigma) = \varepsilon h(\sigma)$), such problems have been studied by E. Acerbi and G. Buttazzo [1], who generalized the results obtained by H. Brezis, L. A. Caffarelli and A. Friedman [7] for the linear case (see also the inspiring works of E. Sanchez-Palencia [13], [14]). In [1] it is proved that the limit problem depends on the limit of the sequence of numbers $\mu^\varepsilon/\varepsilon^{q-1}$.

The present paper is inspired by [1]; however the shape of the reinforcement may depend on ε, that is we consider «general» functions h^ε and the reinforcement material may be inhomogeneous along Γ, but μ^ε is constant along each normal to Γ: in other words $\mu^\varepsilon(x) = \mu^\varepsilon(\sigma(x))$ where $\sigma(x)$ denotes the projection of $x \in \Sigma^\varepsilon$ on Γ. We define $a^\varepsilon : \Gamma \to \mathbb{R}$ by $a^\varepsilon = \mu^\varepsilon (h^\varepsilon)^{1-q}$. We assume essentially that $a^\varepsilon \in L^\infty(\Gamma)$ with $1/a^\varepsilon$ bounded in $L^\infty(\Gamma)$ and that Γ is divided into two parts:

– Γ_1 such that $1/a^\varepsilon$ tends to zero in $L^{q'-1}(\Gamma_1)$, $(1/q + 1/q' = 1)$,

– Γ_2 such that a^ε tends to a in *weak* $*$-$L^\infty(\Gamma_2)$ and such that h^ε does not oscillate too much on Γ_2 (if N=2 and Γ_2 is a closed curve, we can allow as in [8], $h^\varepsilon(\sigma) = n^{-r} H(ny(\sigma))$ for $\sigma \in \Gamma_2$, $\varepsilon = n^{-r}$, $y(\sigma) =$ arc length (σ), H periodic of period Y where Y is the length of Γ_2, $0 < H \leqslant 1$, $H \in \mathcal{C}^1(\mathbb{R}^+)$ and $r \geqslant q'$).

We prove that the limit problem has the form

$$
(1) \qquad \begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) + |u|^{p-2}u = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega \backslash \Gamma_2, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + a|u|^{q-2}u = 0 \text{ on } \Gamma_2. \end{cases}
$$

As E. Acerbi and G. Buttazzo did in [1], we use the Γ-convergence theory introduced by E. De Giorgi [11] (see also H. Attouch [2] and G. Dal Maso [10]) and we actually are able to predict the explicit limit in more general minimization problems than those associated to $(1)^\varepsilon$: in the energy functional, the term

$$
\frac{1}{p}\int_\Omega |v|^p\,dx + \frac{1}{p}\int_\Omega |\nabla v|^p\,dx + \frac{1}{q}\int_{\Sigma^\varepsilon} \mu^\varepsilon \circ \sigma |\nabla v|^q\,dx
$$

can be generalized to

$$
F(v_{|\Omega}) + \int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma}\, G(h^\varepsilon \circ \sigma |\nabla v|)\,dx
$$

where (e.g.) $(\mu^\varepsilon \circ \sigma)(x) = \mu^\varepsilon(\sigma(x))$. The present results were announced in [3] and extensions will be considered in [4], [5] and [12].

This introduction would not be complete without quoting the very general paper of G. Buttazzo, G. Dal Maso and U. Mosco [9], where the limit problem involves a measure which is given in terms of suitable asymptotic capacities associated with Σ^ε. Here the aim is to get a simple explicit limit problem but it seems to us that our hypothesis could be weakened and that connected problems could be considered. In particular, the torsional rigidity problem is treated in [4], where the term $|\nabla v|$ is replaced by an anisotropic one.

2. – Statement of the problem and of the result.

Consider a bounded domain Ω in $\mathbb{R}^N$ with boundary $\partial\Omega$. Let us denote by Γ the union of certain connected components of $\partial\Omega$. (Of course $\Gamma = \partial\Omega$ is allowed.) Let $\Omega' \supset \Omega$ be the domain having boundary $(\partial\Omega \backslash \Gamma) \cup \Gamma'$ where Γ' is at given distance t' to Γ. We assume that t' is small enough so that $\Sigma' = \Omega' \backslash \overline{\Omega}$ is $\mathcal{C}^1$-diffeomorphic to $\Gamma \times (0, t')$ by the mapping

$$x \in \Sigma' \to (\sigma(x),\ t(x)) \in \Gamma \times (0, t'),$$

$$t(x) = \min\{|x - \sigma|,\ \sigma \in \Gamma\},$$

$$\sigma(x) = \arg\min\{|x - \sigma|,\ \sigma \in \Gamma\}$$

($t(x)$ is the distance from $x \in \Sigma'$ to Γ and $\sigma(x)$ the projection of x on Γ). One has $\Omega' = \overline{\Omega} \cup \Sigma'$,

$$\Sigma' = \{\sigma + t\nu(\sigma),\ \sigma \in \Gamma,\ 0 < t < t'\}.$$

Let $\varepsilon < t'$ be a small parameter (hereafter ε will represent a sequence of positive numbers tending to zero) and let $h^\varepsilon : \Gamma \to \mathbb{R}^+ \backslash \{0\}$ be a positive $\mathcal{C}^1$-function such that

$$\forall \sigma \in \Gamma, \qquad h^\varepsilon(\sigma) \leqslant \varepsilon; \tag{2.1}$$

h^ε defines the reinforcement Σ^ε of Ω:

$$\Sigma^\varepsilon = \{\sigma + t\nu(\sigma),\ \sigma \in \Gamma,\ 0 < t < h^\varepsilon(\sigma)\}$$

and consequently the reinforced domain $\Omega^\varepsilon = \overline{\Omega} \cup \Sigma^\varepsilon$. We define $\Gamma^\varepsilon = \partial\Sigma^\varepsilon \backslash \Gamma$. Note that $\Omega \subset \Omega^\varepsilon \subset \Omega'$.

With the above geometrical data and given $p, q \in (1, \infty)$, we consider the functional space

$$V^\varepsilon = \{v : \Omega^\varepsilon \to \mathbb{R},\ v_{|\Omega} \in W^{1,p}(\Omega),\ v_{|\Sigma^\varepsilon} \in W^{1,q}(\Sigma^\varepsilon),\ v_{|\partial\Omega^\varepsilon} = 0, (v_{|\Omega})_{|\Gamma} = (v_{|\Sigma^\varepsilon})_{|\Gamma}\}.$$

We are given data f^ε, g^ε, F, G, a^ε such that

– $f^\varepsilon \in L^{p'}(\Omega)$, $g^\varepsilon \in L^{q'}(\Sigma^\varepsilon)$, $1/p + 1/p' = 1/q + 1/q' = 1$,

– $F : W^{1,p}(\Omega) \to \mathbb{R}^+$ is a lower semi-continuous strictly convex functional such that

$$\exists \lambda > 0, \qquad \forall v \in W^{1,p}(\Omega), \qquad F(v) \geqslant \lambda \|v\|^p_{W^{1,p}(\Omega)}, \tag{2.2}$$

– $G : \mathbb{R}^+ \to \mathbb{R}^+$ is a monotone nondecreasing, continuous, strictly convex function and

$$\exists \mu_1,\ \mu_2 > 0, \qquad \forall \xi \in \mathbb{R}^+, \qquad \mu_1 \xi^q \leqslant G(\xi) \leqslant \mu_2 \xi^q, \tag{2.3}$$

– $a^\varepsilon \in L^\infty(\Gamma)$, $a^\varepsilon > 0$ a.e. and $1/a^\varepsilon \in L^\infty(\Gamma)$.

Now we are able to define $J^\varepsilon \colon V^\varepsilon \to \mathbb{R}^+$ by

$$J^\varepsilon(v) = F(v_{|\Omega}) + \int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G(h^\varepsilon \circ \sigma |\nabla v|)\, dx - \int_\Omega f^\varepsilon v\, dx - \int_{\Sigma^\varepsilon} g^\varepsilon v\, dx$$

where (e.g.) $(a^\varepsilon \circ \sigma)(x) = a^\varepsilon(\sigma(x))$ and where the integral over Σ^ε is meaningful since by (2.1), (2.3) the nonnegative integrand is bounded by $\mu_2 |\nabla v|^q \|a^\varepsilon\|_{L^\infty(\Gamma)} \varepsilon^{q-1}$.

Our aim is to study the limit, as ε tends to zero, of the sequence of minimization problems

$$(\mathscr{P}^\varepsilon) \qquad \operatorname{Inf}\{J^\varepsilon(v),\ v \in V^\varepsilon\}.$$

For fixed ε, we have

PROPOSITION 1. – $(\mathscr{P}^\varepsilon)$ *has a unique solution* $u^\varepsilon \in V^\varepsilon$.

PROOF. – In this proof, as well as in the whole paper, c denotes various constants. The functional J^ε is strictly convex. Moreover $v \to G(h^\varepsilon \circ \sigma |\nabla v|)$ is continuous from $W^{1,q}(\Sigma^\varepsilon)$ into $L^1(\Sigma^\varepsilon)$ (use the Theorem IV. 9, p. 58 in [6], (2.3) and the Lebesgue dominated convergence Theorem), so that the integral on Σ^ε is a continuous function on $W^{1,q}(\Sigma^\varepsilon)$. It is easy to check (using Poincaré inequality on Σ^ε) that

$$\|v\|_{V^\varepsilon} = \|v\|_{W^{1,p}(\Omega)} + \|\nabla v\|_{L^q(\Sigma^\varepsilon)}$$

is a norm on V^ε which is equivalent to the usual one induced by $W^{1,p}(\Omega) \times W^{1,q}(\Sigma^\varepsilon)$ and that J^ε is a lower semi-continuous strictly convex function on V^ε. Moreover J^ε is coercive since by (2.2), (2.3), Hölder inequality and Poincaré inequality on Σ^ε, one has with α^ε such that $\alpha^\varepsilon \leqslant a^\varepsilon(\sigma) h^\varepsilon(\sigma)^{q-1}$ a.e. $\sigma \in \Gamma$

$$J^\varepsilon(v) \geqslant \lambda \|v\|^p_{W^{1,p}(\Omega)} + \mu_1 \alpha^\varepsilon \int_{\Sigma^\varepsilon} |\nabla v|^q\, dx - \|f^\varepsilon\|_{L^{p'}(\Omega)} \|v\|_{W^{1,p}(\Omega)} - \|g^\varepsilon\|_{L^{q'}(\Sigma^\varepsilon)} \|v\|_{L^q(\Sigma^\varepsilon)} \geqslant$$

$$(\lambda \|v\|^p_{W^{1,p}(\Omega)} - \|f^\varepsilon\|_{L^{p'}(\Omega)} \|v\|_{W^{1,p}(\Omega)}) + (\mu_1 \alpha^\varepsilon \|\nabla v\|^q_{L^q(\Sigma^\varepsilon)} - C^\varepsilon \|g^\varepsilon\|_{L^{q'}(\Sigma^\varepsilon)} \|\nabla v\|_{L^q(\Sigma^\varepsilon)})$$

and since when $\|v\|_{V^\varepsilon} \to +\infty$ at least one of $\|v\|_{W^{1,p}(\Omega)}$ or $\|\nabla v\|_{L^q(\Sigma^\varepsilon)}$ tends to infinity.

We study the limit behaviour of $(\mathscr{P}^\varepsilon)$ under the following additional assumptions on $a^\varepsilon, h^\varepsilon, f^\varepsilon, g^\varepsilon$ valid when ε tends to zero. First $\{1/a^\varepsilon\}_\varepsilon$ is bounded in $L^\infty(\Gamma)$:

$$(2.4) \qquad \exists \alpha > 0, \quad \text{a.e. } \sigma \in \Gamma, \quad \forall \varepsilon,\ a^\varepsilon(\sigma) \geqslant \alpha.$$

Moreover, up to a set of $(N-1)$ dimensional measure zero, there exists a partition of Γ into two open regular subsets Γ_1 and Γ_2 independent of ε, (one of the Γ_i being possibly empty, none of them being necessarily connected) such that

$$\Gamma_1 = \emptyset \ \text{ or } \ \frac{1}{a^{\varepsilon}_{|\Gamma_1}} \to 0 \ \text{ in } \ L^{q'-1}(\Gamma_1), \tag{2.5}$$

and

$$\Gamma_2 = \emptyset \ \text{ or } \ \exists a \in L^{\infty}(\Gamma_2), \qquad a^{\varepsilon}_{|\Gamma_2} \rightharpoonup a \ \text{ in weak } *\text{-}L^{\infty}(\Gamma_2), \tag{2.6}$$

$$\left\{ \frac{1}{h^{\varepsilon}} |\nabla h^{\varepsilon}|^q \right\}_{\varepsilon} \ \text{ is bounded in } \ L^{\infty}(\Gamma_2). \tag{2.7}$$

Finally we assume that

$$\{\|g^{\varepsilon}\|_{L^{q'}(\Sigma^{\varepsilon})}\}_{\varepsilon} \ \text{ is bounded and } \ \exists f \in L^{p'}(\Omega), \qquad f^{\varepsilon} \rightharpoonup f \ \text{ in weak-} L^{p'}(\Omega). \tag{2.8}$$

Let us comment (2.7). It means that h^{ε} «does not oscillate too much» on Γ_2. Of course it holds true if (e.g.) $h^{\varepsilon}(\sigma) \equiv \varepsilon h(\sigma)$ with $h \in \mathcal{C}^1(\partial\Omega)$, $h > 0$, but also if (e.g.) $N = 2$, Γ_2 is a closed curve, $\varepsilon = n^{-r}$, $h^{\varepsilon}(\sigma) = n^{-r} H(ny(\sigma))$ for $\sigma \in \Gamma_2$, $y(\sigma) = \text{arc length}(\sigma)$, H periodic of period Y where Y is the length of Γ_2, $0 < H \leqslant 1$, $H \in \mathcal{C}^1(\mathbb{R}^+)$ and $r \geqslant q'$.

Under the above assumptions we have

THEOREM 1. – *Let u^{ε} be the solution of*

$$(\mathcal{P}^{\varepsilon}) \quad \begin{cases} \operatorname{Inf} \{J^{\varepsilon}(v),\ v \in V^{\varepsilon}\}, \\ V^{\varepsilon} = \{v \colon \Omega^{\varepsilon} \to \mathbb{R},\ v_{|\Omega} \in W^{1,p}(\Omega),\ v_{|\Sigma^{\varepsilon}} \in W^{1,q}(\Sigma^{\varepsilon}), \\ \qquad\qquad v_{|\partial\Omega^{\varepsilon}} = 0, (v_{|\Omega})_{|\Gamma} = (v_{|\Sigma^{\varepsilon}})_{|\Gamma}\}, \\ J^{\varepsilon}(v) = F(v_{|\Omega}) + \displaystyle\int_{\Sigma^{\varepsilon}} \frac{a^{\varepsilon} \circ \sigma}{h^{\varepsilon} \circ \sigma} G(h^{\varepsilon} \circ \sigma |\nabla v|)\, dx - \int_{\Omega} f^{\varepsilon} v\, dx - \int_{\Sigma^{\varepsilon}} g^{\varepsilon} v\, dx. \end{cases}$$

Let us define $(\mathcal{P})$ by

$$(\mathcal{P}) \quad \begin{cases} \operatorname{Inf} \{J(v),\ v \in V\}, \\ V = \{v \in W^{1,p}(\Omega),\ v_{|\partial\Omega \setminus \Gamma_2} = 0,\ v_{|\Gamma_2} \in L^q(\Gamma_2)\}, \end{cases}$$

(the restriction $v_{|\partial\Omega\setminus\Gamma_2} = 0$ being effective only if $\Gamma_2 \neq \partial\Omega$ and $v_{|\Gamma_2} \in L^q(\Gamma_2)$ only if $\Gamma_2 \neq \emptyset$ and $q > p$),

$$J(v) = F(v) + \int_{\Gamma_2} a\, G(|v|)\, d\sigma - \int_{\Omega} fv\, dx.$$

Then $(\mathcal{P})$ *has a unique solution* $u \in V$. *Moreover when* $\varepsilon \to 0$,

1) $u^{\varepsilon}_{|\Omega}$ *tends to* u *in weak-*$W^{1,p}(\Omega)$ *and in* $L^p(\Omega)$;

2) *the function* $\tilde{u}^{\varepsilon}: \Sigma' \to \mathbb{R}$ *given by* $\tilde{u}^{\varepsilon} = u^{\varepsilon}$ *in* Σ^{ε} *and* 0 *in* $\Sigma' \setminus \Sigma^{\varepsilon}$ *tends to zero in* $L^q(\Sigma')$;

3) $u^{\varepsilon}_{|\partial\Omega} \to u_{|\partial\Omega}$ *in* $L^p(\partial\Omega)$, $u^{\varepsilon}_{|\Gamma} \rightharpoonup u_{|\Gamma}$ *in weak-*$L^q(\Gamma)$ (*the weak convergence is of interest only if* $q > p$);

4) $J^{\varepsilon}(u^{\varepsilon}) \to J(u)$.

Except for an example given in the last section, the rest of the paper is devoted to the proof of this theorem.

3. – Existence and uniqueness of the solution u.

This is very classical if $\Gamma_2 = \emptyset$ or if $q \leq p$: V is a closed subspace of $W^{1,p}(\Omega)$, J is strictly convex and lower semi-continuous on V equipped with the topology induced by $W^{1,p}(\Omega)$, J is coercive since by (2.2)

$$J(v) \geq \lambda \|v\|^p_{W^{1,p}(\Omega)} - \|f\|_{L^{p'}(\Omega)} \|v\|_{W^{1,p}(\Omega)} .$$

If $\Gamma_2 \neq \emptyset$ and if $q > p$, V is a Banach space for

$$\|v\|_V = \|v\|_{W^{1,p}(\Omega)} + \|v_{|\Gamma_2}\|_{L^q(\Gamma_2)} ,$$

again J is coercive since by (2.2), (2.3), (2.4), (2.6)

$$J(v) \geq [\lambda \|v\|^p_{W^{1,p}(\Omega)} - \|f\|_{L^{p'}(\Omega)} \|v\|_{W^{1,p}(\Omega)}] + \alpha\mu_1 \|v_{|\Gamma_2}\|^q_{L^q(\Gamma_2)}$$

and since, when $\|v\|_V \to +\infty$, either the bracket or the last term tends to infinity.

4. – A priori estimates and consequences for u^{ε}.

The a priori estimates are given in

LEMMA 1.

(1) $\exists c > 0, \ \forall \varepsilon, \ \forall v \in W^{1,q}(\Sigma^{\varepsilon}), \ v_{|\Gamma^{\varepsilon}} = 0 \Rightarrow \int_{\Sigma^{\varepsilon}} |v|^q \, dx \leq c \int_{\Sigma^{\varepsilon}} (h^{\varepsilon} \circ \sigma)^q \, |\nabla v|^q \, dx$, $\int_{\Gamma} |v|^q \, d\sigma \leq c \int_{\Sigma^{\varepsilon}} (h^{\varepsilon} \circ \sigma)^{q-1} \, |\nabla v|^q \, dx$ (*which applies to* u^{ε}),

(2) *$u^\varepsilon_{|\Omega}$ is bounded in $W^{1,p}(\Omega)$,*

(3) *$F(u^\varepsilon_{|\Omega})$ and $\displaystyle\int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G(h^\varepsilon \circ \sigma |\nabla u^\varepsilon|)\, dx$ are bounded,*

(4) *$u^\varepsilon_{|\Gamma}$ is bounded in $L^q(\Gamma)$,*

(5) *$\displaystyle\int_{\Sigma^\varepsilon} |u^\varepsilon|^q\, dx \to 0$.*

PROOF OF (1). – Let $v \in \mathcal{C}^1(\overline{\Sigma}^\varepsilon)$, $v_{|\Gamma^\varepsilon} = 0$. One has

$$v(\sigma + t\nu(\sigma)) = -\int_t^{h^\varepsilon(\sigma)} \nabla v(\sigma + \theta\nu(\sigma))\cdot\nu(\sigma)\, d\theta\,,$$

so that by Hölder inequality

$$|v(\sigma + t\nu(\sigma))|^q \leqslant (h^\varepsilon(\sigma) - t)^{q-1} \int_0^{h^\varepsilon(\sigma)} |\nabla v(\sigma + \theta\nu(\sigma))|^q\, d\theta\,,$$

$$\int_\Gamma \int_0^{h^\varepsilon(\sigma)} |v(\sigma + t\nu(\sigma))|^q\, dt\, d\sigma \leqslant \frac{1}{q} \int_\Gamma \int_0^{h^\varepsilon(\sigma)} (h^\varepsilon(\sigma))^q |\nabla v(\sigma + \theta\nu(\sigma))|^q\, d\theta\, d\sigma$$

and using the $\mathcal{C}^1$-diffeomorphism of Σ' onto $\Gamma \times (0, t')$ one gets (since $\{(\sigma, t), \sigma \in \Gamma, 0 < t < h^\varepsilon(\sigma)\}$ is diffeomorphic to Σ^ε, the diffeomorphism being independent of ε)

$$\int_{\Sigma^\varepsilon} |v|^q\, dx \leqslant c \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^q |\nabla v|^q\, dx$$

for any $v \in \mathcal{C}^1(\overline{\Sigma}^\varepsilon)$, $v_{|\Gamma^\varepsilon} = 0$. Then the first result in (1) follows by continuity (with respect to v and for the $W^{1,q}(\Sigma^\varepsilon)$-topology) of the two members of the last inequality and by density of $\{v \in \mathcal{C}^1(\overline{\Sigma}^\varepsilon), v_{|\Gamma^\varepsilon} = 0\}$ in $\{v \in W^{1,q}(\Sigma^\varepsilon), v_{|\Gamma^\varepsilon} = 0\}$.

As for the second inequality in (1), the same proof gives

$$\int_\Gamma |v(\sigma)|^q\, d\sigma \leqslant \int_\Gamma \int_0^{h^\varepsilon(\sigma)} (h^\varepsilon(\sigma))^{q-1} |\nabla v(\sigma + \theta\nu(\sigma))|^q\, d\theta\, d\sigma \leqslant c \int_{\Sigma^\varepsilon} (h^\varepsilon \circ \sigma)^{q-1} |\nabla v|^q\, dx\,,$$

first for $v \in \mathcal{C}^1(\overline{\Sigma}^\varepsilon)$, $v_{|\Gamma^\varepsilon} = 0$ and then for $v \in W^{1,q}(\Sigma^\varepsilon)$, $v_{|\Gamma^\varepsilon} = 0$, since $v \to v_{|\Gamma}$ is continuous from $W^{1,q}(\Sigma^\varepsilon)$ into $L^q(\Gamma)$.

PROOF OF (2). – Let α be as in (2.4). For small $\varepsilon(\varepsilon \leqslant 1)$ one has by (2.1)

$$(4.1)\qquad \alpha\int_{\Sigma^\varepsilon}(h^\varepsilon\circ\sigma)^q\,|\nabla u^\varepsilon|^q\,dx \leqslant \alpha\int_{\Sigma^\varepsilon}(h^\varepsilon\circ\sigma)^{q-1}\,|\nabla u^\varepsilon|^q\,dx \leqslant$$

$$\int_{\Sigma^\varepsilon}(a^\varepsilon\circ\sigma)(h^\varepsilon\circ\sigma)^{q-1}\,|\nabla u^\varepsilon|^q\,dx \leqslant \frac{1}{\mu_1}\int_{\Sigma^\varepsilon}\frac{a^\varepsilon\circ\sigma}{h^\varepsilon\circ\sigma}\,G(h^\varepsilon\circ\sigma|\nabla u^\varepsilon|)\,dx \quad \text{(by (2.3))},$$

so that by (2.2)

$$(4.2)\qquad \lambda\|u^\varepsilon\|^p_{W^{1,p}(\Omega)} + \mu_1\alpha\int_{\Sigma^\varepsilon}(h^\varepsilon\circ\sigma)^q\,|\nabla u^\varepsilon|^q\,dx \leqslant$$

$$F(u^\varepsilon_{|\Omega}) + \int_{\Sigma^\varepsilon}\frac{a^\varepsilon\circ\sigma}{h^\varepsilon\circ\sigma}\,G(h^\varepsilon\circ\sigma|\nabla u^\varepsilon|)\,dx \leqslant F(0) + \int_\Omega f^\varepsilon u^\varepsilon\,dx + \int_{\Sigma^\varepsilon} g^\varepsilon u^\varepsilon\,dx$$

(as u^ε solves $(\mathcal{P}^\varepsilon)$ and as $G(0)=0$)

$$\leqslant F(0) + c\|u^\varepsilon\|_{W^{1,p}(\Omega)} + c\|u^\varepsilon\|_{L^q(\Sigma^\varepsilon)} \quad \text{(by (2.8))}.$$

Using (1) it follows that

$$[\lambda\|u^\varepsilon\|^p_{W^{1,p}(\Omega)} - c\|u^\varepsilon\|_{W^{1,p}(\Omega)}] + \left[\frac{\mu_1\alpha}{c}\,\|u^\varepsilon\|^q_{L^q(\Sigma^\varepsilon)} - c\|u^\varepsilon\|_{L^q(\Sigma^\varepsilon)}\right] \leqslant F(0).$$

As the first (resp. second) bracket tends to infinity when $\|u^\varepsilon\|_{W^{1,p}(\Omega)}$ (resp. $\|u^\varepsilon\|_{L^q(\Sigma^\varepsilon)}$) tends to infinity, it follows that $u^\varepsilon_{|\Omega}$ is bounded in $W^{1,p}(\Omega)$ and that $\|u^\varepsilon\|_{L^q(\Sigma^\varepsilon)}$ is bounded.

PROOF OF (3). – From the lines following (4.2)

$$F(u^\varepsilon_{|\Omega}) + \int_{\Sigma^\varepsilon}\frac{a^\varepsilon\circ\sigma}{h^\varepsilon\circ\sigma}\,G(h^\varepsilon\circ\sigma|\nabla u^\varepsilon|)\,dx \qquad (\leqslant F(0) + c\|u^\varepsilon\|_{W^{1,p}(\Omega)} + c\|u^\varepsilon\|_{L^q(\Sigma^\varepsilon)})$$

is bounded since, as just proved, $\|u^\varepsilon\|_{W^{1,p}(\Omega)}$ and $\|u^\varepsilon\|_{L^q(\Sigma^\varepsilon)}$ are bounded.

PROOF OF (4). – From (1), (3) and the lines following (4.1)

$$\int_\Gamma |u^\varepsilon|^q\,d\sigma \leqslant c\int_{\Sigma^\varepsilon}(h^\varepsilon\circ\sigma)^{q-1}\,|\nabla u^\varepsilon|^q\,dx \leqslant \frac{c}{\mu_1\alpha}\int_{\Sigma^\varepsilon}\frac{a^\varepsilon\circ\sigma}{h^\varepsilon\circ\sigma}\,G(h^\varepsilon\circ\sigma|\nabla u^\varepsilon|)\,dx \leqslant c\,.$$

PROOF OF (5). – From (1) and (2.3)

$$\int_{\Sigma^\varepsilon}|u^\varepsilon|^q\,d\sigma \leqslant c\int_{\Sigma^\varepsilon}(h^\varepsilon\circ\sigma)^q\,|\nabla u^\varepsilon|^q\,dx \leqslant \frac{c}{\lambda_1}\int_{\Sigma^\varepsilon}G(h^\varepsilon\circ\sigma|\nabla u^\varepsilon|)\,dx =$$

$$\frac{c}{\lambda_1}\int_{\Sigma^\varepsilon}\frac{a^\varepsilon\circ\sigma}{h^\varepsilon\circ\sigma}\,\frac{h^\varepsilon\circ\sigma}{a^\varepsilon\circ\sigma}\,G(h^\varepsilon\circ\sigma|\nabla u^\varepsilon|)\,dx \leqslant \frac{c\varepsilon}{\lambda_1\alpha}\int_{\Sigma^\varepsilon}\frac{a^\varepsilon\circ\sigma}{h^\varepsilon\circ\sigma}\,G(h^\varepsilon\circ\sigma|\nabla u^\varepsilon|)\,dx$$

(using (2.1) and (2.4))

$$\leq c\varepsilon \text{ by } (3).$$

From the a-priori estimates we are going to deduce.

LEMMA 2. – *There exists a subsequence ε' of ε and an element u of V such that*

$$u^{\varepsilon'}_{|\Omega} \rightharpoonup u \text{ in weak-}W^{1,p}(\Omega),$$

$$u^{\varepsilon'}_{|\Omega} \to u \text{ in } L^p(\Omega),$$

$$u^{\varepsilon'}_{|\partial\Omega} \to u_{|\partial\Omega} \text{ in } L^p(\partial\Omega) \text{ (hence in } L^q(\partial\Omega) \text{ if } q \leq p),$$

$$u^{\varepsilon'}_{|\Gamma} \rightharpoonup u_{|\Gamma} \text{ in weak-}L^q(\Gamma)$$

(and $\tilde{u}^\varepsilon = (u^\varepsilon$ in Σ^ε, 0 in $\Sigma' \setminus \Sigma^\varepsilon) \to 0$ in $L^q(\Sigma'))$. Moreover

$$\liminf \int_{\Sigma^{\varepsilon'}} \frac{a^{\varepsilon'} \circ \sigma}{h^{\varepsilon'} \circ \sigma} G(h^{\varepsilon'} \circ \sigma |\nabla u^{\varepsilon'}|)\, dx \geq \int_{\Gamma_2} a\, G(|u|)\, d\sigma. \tag{4.3}$$

PROOF. – We remember that the injection mapping: $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact and that the trace mapping $v \in W^{1,p}(\Omega) \to v_{|\partial\Omega} \in L^p(\partial\Omega)$ also is compact. Hence all the convergences follow from Lemma 1 and from compactness and we know that $u \in W^{1,p}(\Omega)$, $u = 0$ on $\partial\Omega \setminus \Gamma$ and $u_{|\Gamma} \in L^q(\Gamma)$. It just remains to prove (4.3) and to prove that $u_{|\Gamma_1} = 0$ if $\Gamma_1 \neq \emptyset$.

– For any $v \in \mathcal{C}^1(\overline{\Sigma^\varepsilon})$, $v_{|\Gamma^\varepsilon} = 0$ we have, using the diffeomorphism of Σ' on $\partial\Omega \times (0, t')$, and refining an argument already used above

$$\int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G(h^\varepsilon \circ \sigma |\nabla v|)\, dx = \int_\Gamma \int_0^{h^\varepsilon(\sigma)} \frac{a^\varepsilon(\sigma)}{h^\varepsilon(\sigma)} G(h^\varepsilon(\sigma) |\nabla v(\sigma + t\nu(\sigma))|)\, \psi(\sigma, t)\, dt\, d\sigma$$

where $\psi(\sigma, t) = \psi(\sigma, 0) + \dfrac{\partial \psi}{\partial t}(\sigma, \theta(\sigma, t))\, t = 1 + \Phi(\sigma, t)\, t$ with Φ bounded $(|\Phi| \leq M)$. Hence

$$\int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G(h^\varepsilon \circ \sigma |\nabla v|)\, dx = \tag{4.4}$$

$$\int_\Gamma \int_0^{h^\varepsilon(\sigma)} \frac{a^\varepsilon(\sigma)}{h^\varepsilon(\sigma)} G(h^\varepsilon(\sigma) |\nabla v(\sigma + t\nu(\sigma))|)\, dt\, d\sigma + A^\varepsilon$$

with

$$|A^\varepsilon| \leqslant \int_\Gamma \int_0^{h^\varepsilon(\sigma)} \frac{a^\varepsilon(\sigma)}{h^\varepsilon(\sigma)}\, G\big(h^\varepsilon(\sigma)\,|\nabla v(\sigma + t\nu(\sigma))|\big)\,|\Phi(\sigma,t)|\, t\, dt\, d\sigma \leqslant$$

$$\varepsilon M \int_\Gamma \int_0^{h^\varepsilon(\sigma)} \frac{a^\varepsilon(\sigma)}{h^\varepsilon(\sigma)}\, G\big(h^\varepsilon(\sigma)\,|\nabla v(\sigma + t\nu(\sigma))|\big)\, d\sigma\, dt$$

(as $|\Phi| \leqslant M$ and $0 \leqslant t \leqslant h^\varepsilon(\sigma) \leqslant \varepsilon$)

$$\leqslant c\varepsilon M \int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma}\, G(h^\varepsilon \circ \sigma |\nabla v|)\, dx :$$

we get from (4.4)

$$\int_\Gamma \int_0^{h^\varepsilon(\sigma)} \frac{a^\varepsilon(\sigma)}{h^\varepsilon(\sigma)}\, G\big(h^\varepsilon(\sigma)\,|\nabla v(\sigma + t\nu(\sigma))|\big)\, d\sigma\, dt \leqslant$$
$$(1 + c\varepsilon M) \int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma}\, G(h^\varepsilon \circ \sigma |\nabla v|)\, dx\,. \tag{4.5}$$

As already used

$$v(\sigma) = -\int_0^{h^\varepsilon(\sigma)} \nabla v(\sigma + t\nu(\sigma)) \cdot \nu(\sigma)\, dt$$

and as G is nondecreasing

$$G(|v(\sigma)|) \leqslant G\left(\int_0^{h^\varepsilon(\sigma)} |\nabla v(\sigma + t\nu(\sigma))|\, dt \right) =$$

$$G\left(\frac{1}{h^\varepsilon(\sigma)} \int_0^{h^\varepsilon(\sigma)} h^\varepsilon(\sigma)\, |\nabla v(\sigma + t\nu(\sigma))|\, dt \right) \leqslant \frac{1}{h^\varepsilon(\sigma)} \int_0^{h^\varepsilon(\sigma)} G\big(h^\varepsilon(\sigma)\,|\nabla v(\sigma + t\nu(\sigma))|\big)\, dt$$

(by Jensen inequality). It follows

$$\int_\Gamma a^\varepsilon(\sigma)\, G(|v(\sigma)|)\, d\sigma = \int_\Gamma \frac{a^\varepsilon(\sigma)}{h^\varepsilon(\sigma)}\, h^\varepsilon(\sigma)\, G(|v(\sigma)|)\, d\sigma \leqslant$$

$$\int_\Gamma \frac{a^\varepsilon(\sigma)}{h^\varepsilon(\sigma)} \int_0^{h^\varepsilon(\sigma)} G\big(h^\varepsilon(\sigma)\,|\nabla v(\sigma + t\nu(\sigma))|\big)\, dt\, d\sigma \leqslant$$

$$(1 + c\varepsilon M) \int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma}\, G(h^\varepsilon \circ \sigma |\nabla v|)\, dx \qquad \text{(by (4.5))}\,.$$

This is also true by density for any $v \in W^{1,q}(\Sigma^\varepsilon)$ such that $v_{|\Gamma^\varepsilon} = 0$ and in particular for u^ε:

$$(4.6)\quad \int_\Gamma a^\varepsilon(\sigma)\, G(|u^\varepsilon|)\, d\sigma \leqslant (1+c\varepsilon M) \int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma}\, G(h^\varepsilon \circ \sigma |\nabla u^\varepsilon|)\, dx \leqslant c(1+c\varepsilon M) \leqslant c$$

(by Lemma 1)

which gives from (2.3), (2.5)

$$\int_{\Gamma_1} |u^\varepsilon(\sigma)|\, d\sigma = \int_{\Gamma_1} (a^\varepsilon(\sigma))^{-1/q} (a^\varepsilon(\sigma))^{1/q} |u^\varepsilon(\sigma)|\, d\sigma \leqslant$$

$$\left(\int_{\Gamma_1} (a^\varepsilon(\sigma))^{-q'/q} d\sigma\right)^{1/q'} \left(\int_{\Gamma_1} a^\varepsilon(\sigma) |u^\varepsilon(\sigma)|^q\, d\sigma\right)^{1/q} \leqslant$$

$$\left(\int_{\Gamma_1} [(a^\varepsilon(\sigma))^{-1}]^{q'-1}\, d\sigma\right)^{1/q'} (\mu_1)^{-1/q} \left(\int_{\Gamma_1} a^\varepsilon(\sigma)\, G(|u^\varepsilon(\sigma)|)\, d\sigma\right)^{1/q} \leqslant$$

$$c\left(\int_{\Gamma_1} [(a^\varepsilon(\sigma))^{-1}]^{q'-1}\, d\sigma\right)^{1/q'} \to 0\,.$$

As $u^{\varepsilon'}_{|\Gamma_1} \to u_{|\Gamma_1}$ in (strong) $L^p(\Gamma_1)$, we get $u_{|\Gamma_1} = 0$.

– Now we prove (4.3). This is trivial if Γ_2 is empty and is easy to prove from (4.6) if $q \leqslant p$ since then $u^{\varepsilon'}_{|\Gamma} \to u_{|\Gamma}$ in (strong-) $L^q(\Gamma)$, which implies that $G(|u^{\varepsilon'}_{|\Gamma}|) \to G(|u_{|\Gamma}|)$ in (strong-) $L^1(\Gamma)$, so that as $a^\varepsilon \rightharpoonup a$ in weak*-$L^\infty(\Gamma_2)$

$$\int_{\Gamma_2} a\, G(|u|)\, d\sigma = \lim \int_{\Gamma_2} a^{\varepsilon'} G(|u^{\varepsilon'}|)\, d\sigma \leqslant$$

$$\liminf \int_{\Sigma^{\varepsilon'}} \frac{a^{\varepsilon'} \circ \sigma}{h^{\varepsilon'} \circ \sigma}\, G(h^{\varepsilon'} \circ \sigma |\nabla u^{\varepsilon'}|)\, dx \qquad \text{by (4.6)}\,.$$

If Γ_2 is not empty and $q > p$, the proof (also valid otherwise) goes as follows. By convexity and monotonicity

$$G(|u^\varepsilon|) \geqslant G(|u|) + D(|u^\varepsilon| - |u|) \geqslant G(|u|) - D|u^\varepsilon - u|$$

for any D in $\partial G(|u|)$ (the subdifferential of G at the point $|u|$). We notice that (2.3) implies

$$\exists \mu > 0\,, \qquad \forall \xi \in \mathbb{R}^+\,, \qquad \forall d \in \partial G(\xi)\,, \qquad 0 \leqslant d \leqslant \mu \xi^{q-1}$$

and hence $D \in L^{q'}(\Gamma)$,

$$\int_{\Gamma} a^{\varepsilon'} G(|u^{\varepsilon'}|)\, d\sigma \geqslant \int_{\Gamma_2} a^{\varepsilon'} G(|u^{\varepsilon'}|)\, d\sigma \geqslant \int_{\Gamma_2} a^{\varepsilon'} G(|u|)\, d\sigma - \int_{\Gamma_2} a^{\varepsilon'} D|u^{\varepsilon'} - u|\, d\sigma\,.$$

Now

$$\int_{\Gamma_2} a^{\varepsilon'} G(|u|)\, d\sigma \to \int_{\Gamma_2} a\, G(|u|)\, d\sigma\,.$$

Moreover

$$\int_{\Gamma_2} a^{\varepsilon'} D|u^{\varepsilon'} - u|\, d\sigma \to 0$$

since D belongs to $L^{q'}(\Gamma_2)$ and $a^{\varepsilon'}|u^{\varepsilon'} - u| \rightharpoonup 0$ in weak-$L^q(\Gamma_2)$ (because $a^{\varepsilon'}|u^{\varepsilon'} - u|$ is bounded in $L^q(\Gamma_2)$ and $a^{\varepsilon'}|u^{\varepsilon'} - u| \to 0$ in (strong) $L^p(\Gamma_2)$).

Therefore by (4.6)

$$\liminf \int_{\Sigma^{\varepsilon'}} \frac{a^{\varepsilon'} \circ \sigma}{h^{\varepsilon'} \circ \sigma}\, G(h^{\varepsilon'} \circ \sigma |\nabla u^{\varepsilon'}|)\, dx \geqslant \liminf \int_{\Gamma} a^{\varepsilon'} G(|u^{\varepsilon'}|)\, d\sigma \geqslant \int_{\Gamma_2} aG(|u|)\, d\sigma\,.$$

5. – Proof of the convergence of $(\mathcal{P}^\varepsilon)$ to $(\mathcal{P})$.

We first prove it from Lemma 3, whose proof is postponed.

LEMMA 3. – *For any $v \in \mathcal{C}^1(\overline{\Omega})$ such that $v_{|\Gamma_1} = 0$, there exists a sequence of elements $v^\varepsilon \in V^\varepsilon$ such that*

$$v^\varepsilon_{|\Omega} \to v \ \text{ in } L^p(\Omega)\,, \qquad \|v^\varepsilon\|_{L^q(\Sigma^\varepsilon)} \to 0\,,$$

$$\limsup \left\{ F(v^\varepsilon_{|\Omega}) + \int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma}\, G(h^\varepsilon \circ \sigma |\nabla v^\varepsilon|)\, dx \right\} \leqslant F(v) + \int_{\Gamma_2} a\, G(|v|)\, d\sigma\,.$$

PROOF OF THEOREM 1 ASSUMING LEMMA 3. – Let u^ε be the solution of $(\mathcal{P}^\varepsilon)$. From Lemma 1,

$$\int_{\Sigma'} |\tilde{u}^\varepsilon|^q\, dx = \int_{\Sigma^\varepsilon} |u^\varepsilon|^q\, dx \to 0\,,$$

so that 2) is proved. Let $u \in V$ and ε' be as in Lemma 2. As the solution of $(\mathcal{P})$ is unique it just remains to prove that u solves $(\mathcal{P})$ and that $J^{\varepsilon'}(u^{\varepsilon'}) \to J(u)$. (Then a classical argument gives $\varepsilon' \equiv \varepsilon$: one has the convergences for the whole sequence ε.) Let $v \in \mathcal{C}^1(\overline{\Omega})$ such that $v_{|\Gamma_1} = 0$ and let v^ε be as in Lemma 3.

A) $\liminf J^{\varepsilon'}(u^{\varepsilon'}) \leqslant \limsup J^{\varepsilon'}(u^{\varepsilon'}) \leqslant \limsup J^{\varepsilon'}(v^{\varepsilon'}) =$

$$\limsup \left\{ F(v^{\varepsilon'}_{|\Omega}) + \int_{\Sigma^{\varepsilon'}} \frac{a^{\varepsilon'} \circ \sigma}{h^{\varepsilon'} \circ \sigma} G(h^{\varepsilon'} \circ \sigma |\nabla v^{\varepsilon'}|)\, dx - \int_\Omega f^{\varepsilon'} v^{\varepsilon'}\, dx - \int_{\Sigma^{\varepsilon'}} g^{\varepsilon'} v^{\varepsilon'}\, dx \right\}.$$

But $\int_\Omega f^{\varepsilon'} v^{\varepsilon'}\, dx \to \int_\Omega fv\, dx$ since $f^{\varepsilon'} \rightharpoonup f$ in weak-$L^{p'}(\Omega)$ and $v^{\varepsilon'}_{|\Omega} \to v$ in $L^p(\Omega)$. Moreover

$$\left| \int_{\Sigma^{\varepsilon'}} g^{\varepsilon'} v^{\varepsilon'}\, dx \right| \leqslant \|g^{\varepsilon'}\|_{L^{q'}(\Sigma^{\varepsilon'})} \|v^{\varepsilon'}\|_{L^q(\Sigma^{\varepsilon'})} \leqslant c \|v^{\varepsilon'}\|_{L^q(\Sigma^{\varepsilon'})} \qquad \text{(by (2.8))} \to 0\,,$$

so that

$$\liminf J^{\varepsilon'}(u^{\varepsilon'}) \leqslant \limsup J^{\varepsilon'}(u^{\varepsilon'}) \leqslant$$

$$\limsup \left\{ F(v^{\varepsilon'}_{|\Omega}) + \int_{\Sigma^{\varepsilon'}} \frac{a^{\varepsilon'} \circ \sigma}{h^{\varepsilon'} \circ \sigma} G(h^{\varepsilon'} \circ \sigma |\nabla v^{\varepsilon'}|)\, dx \right\} - \int_\Omega fv\, dx \leqslant$$

$$F(v) + \int_{\Gamma_2} a\, G(|v|)\, d\sigma - \int_\Omega fv\, dx \qquad \text{(by Lemma 3)} = J(v)\,.$$

B) $$J^{\varepsilon'}(u^{\varepsilon'}) = F(u^{\varepsilon'}_{|\Omega}) + \int_{\Sigma^{\varepsilon'}} \frac{a^{\varepsilon'} \circ \sigma}{h^{\varepsilon'} \circ \sigma} G(h^{\varepsilon'} \circ \sigma |\nabla u^{\varepsilon'}|)\, dx - \int_\Omega f^{\varepsilon'} u^{\varepsilon'}\, dx - \int_{\Sigma^{\varepsilon'}} g^{\varepsilon'} u^{\varepsilon'}\, dx\,.$$

From Lemma 2 and (2.8) we get

$$\liminf J^{\varepsilon'}(u^{\varepsilon'}) \geqslant \liminf \left\{ F(u^{\varepsilon'}_{|\Omega}) + \int_{\Sigma^{\varepsilon'}} \frac{a^{\varepsilon'} \circ \sigma}{h^{\varepsilon'} \circ \sigma} G(h^{\varepsilon'} \circ \sigma |\nabla u^{\varepsilon c}|)\, dx \right\} - \int_\Omega fu\, dx \geqslant$$

$$\liminf F(u^{\varepsilon'}_{|\Omega}) + \liminf \int_{\Sigma^{\varepsilon'}} \frac{a^{\varepsilon'} \circ \sigma}{h^{\varepsilon'} \circ \sigma} G(h^{\varepsilon'} \circ \sigma |\nabla u^{\varepsilon'}|)\, dx - \int_\Omega fu\, dx \geqslant$$

$$F(u) + \int_{\Gamma_2} a\, G(|u|)\, d\sigma - \int_\Omega fu\, dx = J(u)$$

(by using the lower semi-continuity of F and the convergence $u^{\varepsilon'}_{|\Omega} \rightharpoonup u$ in weak-$W^{1,p}(\Omega)$ and by using (4.3)).

C) From A) and B): For any $v \in \mathcal{C}^1(\overline{\Omega})$ such that $v_{|\Gamma_1} = 0$

$$J(u) \leqslant \liminf J^{\varepsilon'}(u^{\varepsilon'}) \leqslant \limsup J^{\varepsilon'}(u^{\varepsilon'}) \leqslant J(v)\,.$$

By density and by continuity this is also true for any $v \in V$, that is u solves $(\mathcal{P})$ and $J^{\varepsilon'}(u^{\varepsilon'})$ tends to $J(u)$. This completes the proof of Theorem 1, except that we have to prove Lemma 3.

PROOF OF LEMMA 3. – Let $v \in \mathcal{C}^1(\overline{\Omega})$ such that $v_{|\Gamma_1} = 0$. We are going to define v^ε on Ω^ε having the desired properties.

Let us introduce

$$\Sigma_i' = \{\sigma + t\nu(\sigma),\ \sigma \in \Gamma_i,\ 0 < t < t'\}, \qquad i = 1, 2$$

and let $w \in W^{1,q}(\Sigma_2')$ be such that $w_{|\Gamma_2} = v_{|\Gamma_2}$ and $w_{|\partial\Sigma_2' \setminus \Gamma_2} = 0$. Now we define $\tilde{v}\colon \Omega' \to \mathbb{R}$ by

$$\tilde{v} = \begin{cases} v & \text{in } \Omega\,, \\ 0 & \text{in } \Sigma_1'\,, \\ w & \text{in } \Sigma_2'\,. \end{cases}$$

It is easy to check that $\tilde{v} \in V'$ where

$$V' = \{v\colon \Omega' \to \mathbb{R},\ v_{|\Omega} \in W^{1,p}(\Omega),\ v_{|\Sigma'} \in W^{1,q}(\Sigma'),\ v_{|\partial\Omega'} = 0, (v_{|\Omega})_{|\Gamma} = (v_{|\Sigma'})_{|\Gamma}\}\,.$$

Finaly let $v^\varepsilon = (\tilde{v}\varphi^\varepsilon)_{|\Omega^\varepsilon}$ where φ^ε is the continuous function defined on $\overline{\Omega'}$ by

$$\varphi^\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \overline{\Omega}\,, \\ 0 & \text{if } x \in \overline{\Omega'} \setminus \Omega^\varepsilon\,, \\ 1 - \dfrac{t(x)}{(h^\varepsilon \circ \sigma)(x)} & \text{if } x \in \overline{\Sigma}^\varepsilon\,. \end{cases}$$

We notice that $\varphi^\varepsilon_{|\overline{\Sigma}^\varepsilon} \in \mathcal{C}^1(\overline{\Sigma}^\varepsilon)$, $\varphi^\varepsilon_{|\Gamma} = 1$, $\varphi^\varepsilon_{|\Gamma^\varepsilon} = 0$ and $0 \leqslant \varphi^\varepsilon \leqslant 1$. One has

$$v^\varepsilon_{|\Omega} = v \in W^{1,p}(\Omega)\,, \qquad v^\varepsilon_{|\Sigma^\varepsilon} = \tilde{v}_{|\Sigma^\varepsilon}\varphi^\varepsilon_{|\Sigma^\varepsilon} \in W^{1,q}(\Sigma^\varepsilon)$$

since $\tilde{v}_{|\Sigma^\varepsilon} \in W^{1,q}(\Sigma^\varepsilon)$ (as $\tilde{v} \in V'$) and $\varphi^\varepsilon_{|\overline{\Sigma}^\varepsilon} \in \mathcal{C}^1(\overline{\Sigma}^\varepsilon)$. Clearly $v^\varepsilon_{|\partial\Omega^\varepsilon} = 0$ and $(v^\varepsilon_{|\Omega})_{|\Gamma} = (\tilde{v}_{|\Omega})_{|\Gamma} = (\tilde{v}_{|\Sigma'})_{|\Gamma}$ (since $\tilde{v} \in V'$) $= (\tilde{v}_{|\Sigma^\varepsilon})_{|\Gamma}\ (\varphi^\varepsilon_{|\Sigma^\varepsilon})_{|\Gamma} = ((\tilde{v}\varphi^\varepsilon)_{|\Sigma^\varepsilon})_{|\Gamma} = (v^\varepsilon_{|\Sigma^\varepsilon})_{|\Gamma}$. Hence $v^\varepsilon \in V^\varepsilon$. The convergence of $v^\varepsilon_{|\Omega}$ to v in $L^p(\Omega)$ is trivial since $v^\varepsilon_{|\Omega} \equiv v$. Moreover

$$\|v^\varepsilon\|^q_{L^q(\Sigma^\varepsilon)} = \int_{\Sigma_2^\varepsilon} |w|^q (\varphi^\varepsilon)^q\, dx \leqslant \int_{\Sigma_2^\varepsilon} |w|^q\, dx \to 0$$

since $|w|^q \in L^1(\Sigma_2')$ and since

$$|\Sigma_2^\varepsilon| = \int_{\Sigma_2^\varepsilon} dx \leq c \int_{\Gamma_2} \int_0^{h^\varepsilon(\sigma)} dt\, d\sigma \leq c\varepsilon \int_{\Gamma_2} d\sigma \to 0 .$$

It just remains to prove the last inequality in Lemma 3. One has

$$F(v^\varepsilon_{|\Omega}) = F(v) \tag{5.1}$$

and one writes for simplicity

$$H^\varepsilon(x,\, \xi) = \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma}(x)\, G((h^\varepsilon \circ \sigma)(x)\, \xi) ,$$

so that

$$\int_{\Sigma^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G(h^\varepsilon \circ \sigma |\nabla v^\varepsilon|)\, dx = \int_{\Sigma_2^\varepsilon} H^\varepsilon(x,\, |\nabla v^\varepsilon|)\, dx , \tag{5.2}$$

where H^ε is a nondecreasing convex function of ξ, which implies that $H^\varepsilon(x, |\cdot|)$ is convex. Using a classical convexity argument valid for any $\theta \in (0, 1)$,

$$\int_{\Sigma_2^\varepsilon} H^\varepsilon(x,\, |\nabla v^\varepsilon|)\, dx = \int_{\Sigma_2^\varepsilon} H^\varepsilon\left(x,\, \left| \theta\varphi^\varepsilon \frac{\nabla w}{\theta} + (1-\theta)\, w \frac{\nabla \varphi^\varepsilon}{1-\theta} \right|\right) dx \leq$$

$$\theta \int_{\Sigma_2^\varepsilon} H^\varepsilon\left(x,\, \left| \varphi^\varepsilon \frac{\nabla w}{\theta} \right|\right) dx + (1-\theta) \int_{\Sigma_2^\varepsilon} H^\varepsilon\left(x,\, \left| w \frac{\nabla \varphi^\varepsilon}{1-\theta} \right|\right) dx .$$

By definition of H^ε and by (2.1), (2.3), (2.6)

$$\theta \int_{\Sigma_2^\varepsilon} H^\varepsilon\left(x,\, \left| \varphi^\varepsilon \frac{\nabla w}{\theta} \right|\right) dx = \theta \int_{\Sigma_2^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G\left(h^\varepsilon \circ \sigma \frac{\varphi^\varepsilon |\nabla w|}{\theta}\right) dx \leq$$

$$\frac{\mu_2}{\theta^{q-1}} \int_{\Sigma_2^\varepsilon} (a^\varepsilon \circ \sigma)(h^\varepsilon \circ \sigma)^{q-1} (\varphi^\varepsilon)^q\, |\nabla w|^q\, dx \leq \frac{c\mu_2}{\theta^{q-1}}\, \varepsilon^{q-1} \int_{\Sigma_2^\varepsilon} |\nabla w|^q\, dx \to 0 ,$$

so that

$$\limsup \int_{\Sigma_2^\varepsilon} H^\varepsilon(x,\, |\nabla v^\varepsilon|)\, dx \leq (1-\theta) \limsup \int_{\Sigma_2^\varepsilon} H^\varepsilon\left(x,\, \frac{|w \nabla \varphi^\varepsilon|}{1-\theta}\right) dx$$

for any $\theta \in (0, 1)$, and letting θ tend to zero

$$\limsup \int_{\Sigma_2^\varepsilon} H^\varepsilon(x, |\nabla v^\varepsilon|)\, dx \leqslant \limsup \int_{\Sigma_2^\varepsilon} H^\varepsilon(x, |w|\, |\nabla \varphi^\varepsilon|)\, dx . \tag{5.3}$$

Now on Σ_2^ε,

$$|\nabla \varphi^\varepsilon| = \left| -\frac{\nabla t}{h^\varepsilon \circ \sigma} + \frac{t \nabla(h^\varepsilon \circ \sigma)}{(h^\varepsilon \circ \sigma)^2} \right| \leqslant \frac{|\nabla t|}{h^\varepsilon \circ \sigma} + \frac{t}{h^\varepsilon \circ \sigma} \frac{|\nabla (h^\varepsilon \circ \sigma)|}{h^\varepsilon \circ \sigma} \leqslant$$

$$\frac{1}{h^\varepsilon \circ \sigma} + \frac{|\nabla(h^\varepsilon \circ \sigma)|}{h^\varepsilon \circ \sigma}$$

which implies, by the same convexity argument, that

$$\int_{\Sigma_2^\varepsilon} H^\varepsilon(x, |w|\, |\nabla \varphi^\varepsilon|)\, dx \leqslant \int_{\Sigma_2^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G(|w| + |w|\, |\nabla (h^\varepsilon \circ \sigma)|)\, dx \leqslant$$

$$\theta \int_{\Sigma_2^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G\left(\frac{|w|}{\theta} \right) dx + (1-\theta) \int_{\Sigma_2^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G\left(\frac{|w|\, |\nabla(h^\varepsilon \circ \sigma)|}{1-\theta} \right) dx .$$

Using (2.3), (2.6), (2.7), the last integral is bounded by

$$\frac{\mu_2}{(1-\theta)^q} \int_{\Sigma_2^\varepsilon} a^\varepsilon \circ \sigma \frac{|\nabla (h^\varepsilon \circ \sigma)|^q}{h^\varepsilon \circ \sigma} |w|^q\, dx \leqslant \frac{c}{(1-\theta)^q} \int_{\Sigma_2^\varepsilon} |w|^q\, dx$$

and therefore it tends to zero with ε: one obtains by letting θ tend to one

$$\limsup \int_{\Sigma_2^\varepsilon} H^\varepsilon(x, |w|\, |\nabla \varphi^\varepsilon|)\, dx \leqslant \limsup \int_{\Sigma_2^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G(|w|)\, dx . \tag{5.4}$$

Finally thanks to the usual diffeomorphism argument

$$\int_{\Sigma_2^\varepsilon} \frac{a^\varepsilon \circ \sigma}{h^\varepsilon \circ \sigma} G(|w|)\, dx =$$

$$\int_{\Gamma_2} \int_0^{h^\varepsilon(\sigma)} \frac{a^\varepsilon(\sigma)}{h^\varepsilon(\sigma)} G(|w(\sigma + t\nu(\sigma))|)(1 + \Phi(\sigma, t)\, t)\, dt\, d\sigma = \int_{\Gamma_2} a^\varepsilon(\sigma)\, b^\varepsilon(\sigma)\, d\sigma$$

where

$$b^\varepsilon(\sigma) = \frac{1}{h^\varepsilon(\sigma)} \int_0^{h^\varepsilon(\sigma)} G(|w(\sigma + t\nu(\sigma))|)(1 + \Phi(\sigma, t)\, t) dt .$$

Assume for a moment (this will be shortly proved) that

$$b^{\varepsilon} \to G(|w|) \text{ in } L^1(\Gamma_2). \tag{5.5}$$

Then, as $a^{\varepsilon} \rightharpoonup a$ in weak*-$L^{\infty}(\Gamma_2)$ (see (2.6)), one obtains

$$\int_{\Sigma_2^{\varepsilon}} \frac{a^{\varepsilon} \circ \sigma}{h^{\varepsilon} \circ \sigma} G(|w|)\, dx \to \int_{\Gamma_2} a\, G(|w|)\, d\sigma = \int_{\Gamma_2} a\, G(|v|)\, d\sigma$$

and by means of (5.2) to (5.4) this implies

$$\limsup \int_{\Sigma_2^{\varepsilon}} \frac{a^{\varepsilon} \circ \sigma}{h^{\varepsilon} \circ \sigma} G(h^{\varepsilon} \circ \sigma |\nabla v^{\varepsilon}|)\, dx \leqslant \int_{\Gamma_2} a\, G(|v|)\, d\sigma\,, \tag{5.6}$$

so that using (5.1)

$$\limsup \left\{ F(v^{\varepsilon}_{|\Omega}) + \int_{\Sigma^{\varepsilon}} \frac{a^{\varepsilon} \circ \sigma}{h^{\varepsilon} \circ \sigma} G(h^{\varepsilon} \circ \sigma |\nabla v^{\varepsilon}|)\, dx \right\} \leqslant F(v) + \int_{\Gamma_2} a\, G(|v|)\, d\sigma$$

and the proof is complete, except for (5.5). But (5.5) follows from Lebesgue theorem. Actually $b^{\varepsilon}(\sigma) \to G(|w(\sigma)|)$ for a.e. σ of Γ_2 and we are going to bound above b^{ε} by an $L^1(\Gamma_2)$ function independent of ε. This goes as follows. Since w vanishes on $\partial\Sigma_2' \cap \partial\Omega'$

$$|w(\sigma + t\nu(\sigma))|^q = \left| -\int_t^{t'} \nabla w(\sigma + \theta\nu(\sigma)) \cdot \nu(\sigma)\, d\theta \right|^q \leqslant$$

$$(t' - t)^{q-1} \int_t^{t'} |\nabla w(\sigma + \theta\nu(\sigma))|^q\, d\theta \leqslant (t')^{q-1} \int_0^{t'} W\, d\theta$$

where we denote for simplicity

$$W = W(\theta, \sigma) = |\nabla w(\sigma + \theta\nu(\sigma))|^q\,.$$

Hence

$$\frac{1}{h^{\varepsilon}(\sigma)} |w(\sigma + t\nu(\sigma))|^q \leqslant \frac{(t')^{q-1}}{h^{\varepsilon}(\sigma)} \int_0^{t'} W\, d\theta\,,$$

$$\frac{1}{h^{\varepsilon}(\sigma)} \int_0^{h^{\varepsilon}(\sigma)} |w(\sigma + t\nu(\sigma))|^q\, dt \leqslant (t')^{q-1} \int_0^{t'} W\, d\theta$$

and then, by definition of b^ε and by (2.1), (2.3)

$$|b^\varepsilon(\sigma)| \leqslant \frac{c}{h^\varepsilon(\sigma)} \int_0^{h^\varepsilon(\sigma)} G(|w(\sigma + t\nu(\sigma))|)\, dt \leqslant$$

$$\frac{c\mu_2}{h^\varepsilon(\sigma)} \int_0^{h^\varepsilon(\sigma)} |w(\sigma + t\nu(\sigma))|^q\, dt \leqslant c(t')^{q-1} \int_0^{t'} W\, d\theta$$

and $\int_0^{t'} W\, d\theta$ is independent of ε and belongs to $L^1(\Gamma_2)$ since

$$\int_{\Gamma_2} d\sigma \int_0^{t'} W\, d\theta = \int_{\Gamma_2} d\sigma \int_0^{t'} |\nabla w(\sigma + \theta\nu(\sigma))|^q\, d\theta \leqslant c \int_{\Sigma_2'} |\nabla w|^q\, dx < +\infty\, .$$

This completes the proof of (5.5) and the proof of Lemma 3.

6. – Example.

We apply the general result stated above to

$$F(v) = \frac{1}{p} \int_\Omega |\nabla v|^p\, dx + \frac{1}{p} \int_\Omega |v|^p\, dx\,, \qquad G(\xi) = \frac{1}{q}\, \xi^q\, .$$

Using more current notations, we set $a^\varepsilon (h^\varepsilon)^{q-1} = \mu^\varepsilon$ and we assume that h^ε and μ^ε are given functions defined on Γ such that

- $\forall \varepsilon$, $h^\varepsilon \in \mathcal{C}^1(\Gamma)$ and $\forall \sigma \in \Gamma$, $0 < h^\varepsilon(\sigma) \leqslant \varepsilon$;
- $\forall \varepsilon$, $\mu^\varepsilon \in L^\infty(\Gamma)$;
- $\exists \alpha > 0$, a.e. $\sigma \in \Gamma$, $\forall \varepsilon$, $\mu^\varepsilon(\sigma) \geqslant \alpha (h^\varepsilon(\sigma))^{q-1}$;
- there exists a partition Γ_1, Γ_2 of Γ such that

$$\int_{\Gamma_1} h^\varepsilon (\mu^\varepsilon)^{1-q'}\, d\sigma \to 0\,,$$

$$\exists a \in L^\infty(\Gamma_2)\,, \qquad \mu^\varepsilon (h^\varepsilon)^{1-q} \rightharpoonup a \text{ in weak*-} L^\infty(\Gamma_2)\,,$$

$$\exists H > 0\,, \qquad \forall \sigma \in \Gamma_2\,, \qquad \forall \varepsilon\,, \qquad |\nabla h^\varepsilon(\sigma)|^q \leqslant H h^\varepsilon(\sigma)\, .$$

(The data f^ε, g^ε satisfy as in general $f^\varepsilon \rightharpoonup f$ in weak-$L^p(\Omega)$ and $\{\|g^\varepsilon\|_{L^{q'}(\Sigma^\varepsilon)}\}_\varepsilon$ bounded.) Then with F, G chosen as above, the functionals J^ε reduce to

$$J^\varepsilon(v) = \frac{1}{p} \int_\Omega |\nabla v|^p\, dx + \frac{1}{p} \int_\Omega |v|^p dx + \frac{1}{q} \int_{\Sigma^\varepsilon} \mu^\varepsilon \circ \sigma |\nabla v|^q dx - \int_\Omega f^\varepsilon v\, dx - \int_{\Sigma^\varepsilon} g^\varepsilon v\, dx$$

and the solution u^ε of $(\mathscr{P}^\varepsilon)$ is characterized by the variational formulation:

$$
(6.1)\quad \begin{cases} u^\varepsilon \in V^\varepsilon \text{ and for every } v^\varepsilon \in V^\varepsilon , \\ \displaystyle\int_\Omega |\nabla u^\varepsilon|^{p-2} \nabla u^\varepsilon \cdot \nabla v \, dx + \int_\Omega |u^\varepsilon|^{p-2} u^\varepsilon v \, dx + \\ \qquad\qquad \displaystyle\int_{\Sigma^\varepsilon} \mu^\varepsilon \circ \sigma |\nabla u^\varepsilon|^{q-2} \nabla u^\varepsilon \cdot \nabla v \, dx - \int_\Omega f^\varepsilon v \, dx - \int_{\Sigma^\varepsilon} g^\varepsilon v \, dx = 0 \end{cases}
$$

that is u^ε is a weak solution of

$$
(6.2)\quad \begin{cases} -\operatorname{div}(|\nabla u^\varepsilon|^{p-2} \nabla u^\varepsilon) + |u^\varepsilon|^{p-2} u^\varepsilon = f^\varepsilon \text{ in } \Omega , \\ -\operatorname{div}(\mu^\varepsilon \circ \sigma |\nabla u^\varepsilon|^{q-2} \nabla u^\varepsilon) = g^\varepsilon \text{ in } \Sigma^\varepsilon , \\ u^\varepsilon = 0 \text{ on } \partial\Omega^\varepsilon , \\ |\nabla u^\varepsilon|^{p-2} \dfrac{\partial}{\partial \nu} (u^\varepsilon_{|\Omega}) = \mu^\varepsilon \circ \sigma |\nabla u^\varepsilon|^{q-2} \dfrac{\partial}{\partial \nu} (u^\varepsilon_{|\Sigma^\varepsilon}) \text{ on } \Gamma \end{cases}
$$

(having no «discontinuity» on Γ).

The application of Theorem 1 gives as limit problem

$$
(\mathscr{P}) \qquad \operatorname{Inf}\{J(v),\ v \in V\} ,
$$

where as before

$$
V = \{v \in W^{1,p}(\Omega),\ v_{|\partial\Omega \setminus \Gamma_2} = 0,\ v_{|\Gamma_2} \in L^q(\Gamma_2)\} ,
$$

and where

$$
J(v) = \frac{1}{p} \int_\Omega |\nabla v|^p \, dx + \frac{1}{p} \int_\Omega |v|^p \, dx + \frac{1}{q} \int_{\Gamma_2} a |v|^q \, d\sigma - \int_\Omega f v \, dx .
$$

Its solution u is characterized by the variationnal formulation:

$$
(6.3)\quad \begin{cases} u \in V \text{ and for every } v \in V , \\ \displaystyle\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx + \int_\Omega |u|^{p-2} u v \, dx + \int_{\Gamma_2} a |u|^{q-2} u v \, d\sigma - \int_\Omega f v \, dx = 0 \end{cases}
$$

that is u is a weak solution of

$$
(6.4)\quad \begin{cases} -\operatorname{div}(|\nabla u|^{p-2} \nabla u) + |u|^{p-2} u = f \text{ in } \Omega , \\ u = 0 \text{ on } \partial\Omega \setminus \Gamma_2 , \\ |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} + a |u|^{q-2} u = 0 \text{ on } \Gamma_2 . \end{cases}
$$

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