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Divided Differences and Symmetric Functions.

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Sunto. – *L'operatore di differenze multivariate è utilizzato per stabilire varie formule di somme riguardanti le funzioni simmetriche, le quali hanno uno stretto legame con le identità del «termine costante».*

1. – Multivariate divided differences.

For a given complex function $f(y)$, its divided difference of the first order at two distinct complex numbers u and v is defined by

$$\Delta[u, v]\, f(y) = \frac{f(u) - f(v)}{u - v}. \tag{1.1a}$$

In general, for n distinct complex numbers $\{x_k\}_{k=1}^n$, we may repeat this process for $f(y)$ and define its $(n-1)$th divided difference by

$$\Delta[x_1, x_2, \ldots, x_n]\, f(y) = \Delta[x_{n-1}, x_n]\, \Delta[x_{n-2}, y] \tag{1.1b}$$

$$\ldots \Delta[x_2, y]\, \Delta[x_1, y]\, f(y). \tag{1.1c}$$

The result is independent of the order of $\{x_k\}_{k=1}^n$ and may be expressed as

$$\Delta[x_1, x_2, \ldots, x_n]\, f(y) = \sum_{i=1}^{n} \frac{f(x_i)}{\prod_{\substack{j=1 \\ j \neq i}}^{n} (x_i - x_j)}. \tag{1.1d}$$

In order to compute the divided difference of monomials $\{y^m\}$, consider ex-

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pansion in partial fractions

$$\prod_{k=1}^{n} \frac{1}{1-x_k y} = \sum_{i=1}^{n} \frac{\lambda_i}{1-x_i y}, \qquad \text{where} \qquad \lambda_i = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x_i}{x_i - x_j}$$

which may be restated as

LEMMA 1. – *Let* $\{x_k\}_{k=1}^{n}$ *be arbitrary distint complex numbers, and* y *a complex variable. We then have*

$$\sum_{i=1}^{n} \frac{1}{1-x_i y} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x_i}{x_i - x_j} = \prod_{k=1}^{n} \frac{1}{1-x_k y}. \tag{1.2}$$

Its power series expansions with respect to y at $y=0$ and $y=\infty$, respectively, read as a summation theorem on symmetric functions.

PROPOSITION 2 (Biedenharn and Louck [1]). – *Denote by* $h_p(x)$ *and* $h_p(1/x)$ *the* p *th complete symmetric functions, respectively, in* $\{x_k\}_{k=1}^{n}$ *and their reciprocals. We then have*

$$\sum_{i=1}^{n} x_i^p \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x_i}{x_i - x_j} = \begin{cases} \dfrac{(-1)^{n-1} h_{-p-n}(1/x)}{x_1 x_2 \dots x_n}, & (-\infty < p \leqslant -n) \\ 0, & (-n < p < 0) \\ h_p(x), & (0 \leqslant p < \infty). \end{cases} \tag{1.3}$$

This approach is much simpler than the original analytic proof due to Biedenharn and Louck [1].

2. – Constant term identities.

Denote by N_0 the set of nonnegative integers. For $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in N_0^n$, and n variables $\{x_k\}_{k=1}^{n}$, let $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}$ be a monomial and $[x^\lambda] f(x)$ the coefficient of x^λ in the Laurent series expansion of $f(x)$, which is a multivariate function of $\{x_k\}_{k=1}^{n}$.

The Dyson [4] conjecture on constant term identities may be stated as

$$[x^0] \prod_{1 \leqslant i \neq j \leqslant n} (1 - x_i/x_j)^{m_i} = \binom{m_1 + m_2 + \dots + m_n}{m_1, m_2, \dots, m_n}. \tag{2.1}$$

It was confirmed independently by Gunson and Wilson in the same volume of Journal of Mathematical Physics where Dyson announced his conjecture.

A shorter proof due to Good [5] is based on a special case of Lemma 1 when $y = 0$.

When $m_1 = m_2 = \ldots = m_n = 1$, it may be specified as

$$[(x_1 x_2 \ldots x_n)^{n-1}] \prod_{1 \leqslant i \neq j \leqslant n} (x_i - x_j) = n!. \tag{2.2}$$

Denote by S_n the set of permutations of $\{0, 1, \ldots, n-1\}$. For each $\sigma \in S_n$, define a sign function $\varepsilon(\sigma) = \pm 1$ according to its parity. If $\theta = (n-1, n-2, \ldots, 1, 0)$ and $\vartheta = (0, 1, \ldots, n-2, n-1)$, then the product of the Vandermonde determinants yields

$$[(x_1 x_2 \ldots x_n)^{n-1}] \prod_{1 \leqslant i \neq j \leqslant n} (x_i - x_j), \tag{2.3a}$$

$$= [x^{\theta + \vartheta}] \sum_{\sigma \in S_n} \varepsilon(\sigma)\, x^{\sigma\theta} \sum_{\tau \in S_n} \varepsilon(\tau)\, x^{\tau\vartheta}, \tag{2.3b}$$

$$= [x^{\theta + \vartheta}] \sum_{\sigma,\, \tau \in S_n} \varepsilon(\sigma\tau)\, x^{\sigma\theta + \tau\vartheta}, \tag{2.3c}$$

$$= [x^{\theta + \vartheta}] \sum_{\sigma \in S_n} \varepsilon(\sigma^2)\, x^{\sigma(\theta + \vartheta)}, \tag{2.3d}$$

$$= \sum_{\sigma \in S_n} 1 = n!, \tag{2.3e}$$

which provides an alternate proof for (2.2).

3. – Symmetric functions.

For a fixed natural number n and the corresponding set $[n] = \{1, 2, \ldots, n\}$ of the first n natural numbers, let $\Lambda = \{k_1 < k_2 < \ldots < k_p\} \subset [n]$ be its subset with the cardinality $p = |\Lambda|$ and the complement Λ^c in $[n]$. Then the m-th elementary and complete symmetric functions in $\{x_k \mid k \in \Lambda\}$ and their reciprocals, will be denoted, respectively, by $e_m(x|\Lambda)$, $e_m(1/x|\Lambda)$, $h_m(x|\Lambda)$ and $h_m(1/x|\Lambda)$. When $\Lambda = [n]$, it will be omitted from the symmetric function notation. Similarly, for a multivariate complex function $f(y_1, y_2, \ldots, y_p)$, the replacement $\{y_\iota = x_{k_\iota}\}_{\iota=1}^{p}$ will be denoted by $f(x|\Lambda)$ instead of $f(x_{k_1}, x_{k_2}, \ldots, x_{k_p})$. The tensor product of divided differences of $f(y_1, y_2, \ldots, y_p)$ with respect to the same point-set $\{x_k\}_{k=1}^{n}$ for each variable is given by

$$\Delta^p[x_1, x_2, \ldots, x_n]\, f(y_1, y_2, \ldots, y_p), \tag{3.1a}$$

$$= \sum_{\Lambda \in [n]^p} f(x|\Lambda) \Big/ \prod_{i \in \Lambda} \prod_{j \neq i} (x_i - x_j). \tag{3.1b}$$

Comparing it with the symmetric summation defined by

$$\text{(3.2a)}\qquad \sum_{\substack{A\subset[n]\\|A|=p}}\prod_{i\in A}\frac{x_i^{n-m}}{\prod_{j\notin A}(x_i-x_j)}=\sum_{\substack{A\subset[n]\\|A|=p}}\prod_{i\in A}\frac{x_i^{n-m}}{\prod_{j\neq i}(x_i-x_j)}\prod_{i\neq j\in A}(x_i-x_j)\,,$$

$$\text{(3.2b)}\qquad =\{1/p!\}\sum_{A\in[n]^p}\prod_{i\in A}\frac{x_i^{n-m}}{\prod_{j\neq i}(x_i-x_j)}\prod_{i\neq j\in A}(x_i-x_j)\,,$$

we find that the last sum is equal to the multivariate divided difference

$$\Delta^p[x_1, x_2, \ldots, x_n]\Big\{(y_1y_2\ldots y_p)^{n-m}\prod_{i\neq j\in A}(y_i-y_j)\Big\}\,,$$

where $(y_1y_2\ldots y_p)^{n-m}\prod_{i\neq j\in A}(y_i-y_j)$ is a symmetric polynomial of degree $p(p-1+n-m)$ in $\{y_k\}_{k=1}^p$, whose multivariate divided difference at $\{x_1, x_2, \ldots, x_n\}$ vanishes for $p<m\leqslant n$. When $m=p$, it reduces to the coefficient of the monomial $(y_1y_2\ldots y_p)^{n-1}$ in the Laurent expansion of $(y_1y_2\ldots y_p)^{n-p}\prod_{i\neq j\in A}(y_i-y_j)$, i.e., the constant term of $\prod_{i\neq j\in A}(1-y_i/y_j)$. From (2.2), we recover the following summation formulae

PROPOSITION 3 (Gross and Richards [6, eqs. (3.3-6)]).

$$\text{(3.3)}\qquad \sum_{\substack{A\subset[n]\\|A|=p}}\prod_{i\in A}\frac{x_i^{n-m}}{\prod_{j\notin A}(x_i-x_j)}=\begin{cases}0\,, & p<m\leqslant n\,,\\ 1\,, & m=p\,.\end{cases}$$

THEOREM 4. – *Let* $\{m_1, m_2, \ldots, m_p\}$ *be nonnegative integers. There hold*

$$\text{(3.4a)}\qquad \prod_{\substack{A\subset[n]\\|A|=p}}\ \prod_{i\in A,\ j\notin A}\frac{x_i}{x_i-x_j}\prod_{k=1}^{p}h_{m_k}(x|A)=\prod_{k=1}^{p}h_{m_k}(x)\,,$$

$$\text{(3.4b)}\qquad \sum_{\substack{A\subset[n]\\|A|=p}}\ \prod_{i\in A,\ j\notin A}\frac{x_i}{x_i-x_j}\prod_{k=1}^{p}e_{m_k}(x|A^c)=\delta\Big(0, \sum_{k=1}^{p}m_k\Big)\,,$$

where $\delta(\cdot,\cdot)$ *is the Kronecker delta.*

In particular, when $m_1=m$ and $m_2=m_3=\ldots=m_p=0$, this theorem reduces to

COROLLARY 5. – *Let m be a nonnegative integer. There hold*

$$\sum_{\substack{A\subset[n]\\|A|=p}}\prod_{i\in A,\,j\notin A}\frac{x_i}{x_i-x_j}h_m(x|A)=h_m(x),\tag{3.5a}$$

$$\sum_{\substack{A\subset[n]\\|A|=p}}\prod_{i\in A,\,j\notin A}\frac{x_i}{x_i-x_j}e_m(x|A^c)=\delta(0,m).\tag{3.5b}$$

The first formula in this corollary is due to Gross and Richards [6, eq. (3.13)]. When $m=0$, both identities reduce to the case $m=p$ of (3.3).

Recalling the generating function for complete symmetric functions

$$\prod_{k\in A}\frac{1}{1-x_k y}=\sum_{m=0}^{\infty}h_m(x|A)\,y^m,$$

$$\prod_{k=1}^{n}\frac{1}{1-x_k y}=\sum_{m=0}^{\infty}h_m(x)\,y^m,$$

we may restate (3.4*a*) and (3.4*b*) as

THEOREM 6.

$$\sum_{\substack{A\subset[n]\\|A|=p}}\prod_{i\in A,\,j\notin A}\frac{x_i}{x_i-x_j}\prod_{k\in A}\prod_{l=1}^{p}\frac{1}{1-x_k y_l}=\prod_{k=1}^{n}\prod_{l=1}^{p}\frac{1}{1-x_k y_l},\tag{3.6a}$$

$$\sum_{\substack{A\subset[n]\\|A|=p}}\prod_{i\in A,\,j\notin A}\frac{x_i}{x_i-x_j}\prod_{k\notin A}\prod_{l=1}^{p}(1+x_k y_l)=1.\tag{3.6b}$$

In fact, if we multiply both sides of (3.6*a*) by $\prod_{k=1}^{n}\prod_{l=1}^{p}(1-x_k y_l)$ and replace $\{y_l\}$ by their opposites, we get (3.6*b*). Then (3.4*b*) follows simply from (3.6*b*) on account of the generating functions for elementary symmetric functions

$$\sum_{m\geqslant 0}e_m(x|A)\,y^m=\prod_{k\in A}(1+x_k y),$$

$$\sum_{m\geqslant 0}e_m(x)\,y^m=\prod_{k=1}^{n}(1+x_k y).$$

Therefore it suffices to show (3.4*a*) in order to prove the theorems.

PROOF OF (3.4*a*). – We will do it by the induction principle.

When $p=1$, it reduces to (1.3) due to Biedenharn and Louck [1]. Suppose the statement is true for p. Then for $p+1$, let

$$H(x|\Lambda) = h_{m_1}(x|\Lambda)\, h_{m_2}(x|\Lambda) \ldots h_{m_p}(x|\Lambda).$$

From (1.3) we can deduce

$$\Omega := \sum_{|\Lambda| = p+1} \prod_{i \in \Lambda,\, j \notin \Lambda} \frac{x_i}{x_i - x_j} h_{m_0}(x|\Lambda)\, H(x|\Lambda), \tag{3.7a}$$

$$= \sum_{|\Lambda| = p+1} \prod_{i \in \Lambda,\, j \notin \Lambda} \frac{x_i}{x_i - x_j} H(x|\Lambda) \sum_{k \in \Lambda} \frac{x_k^{p+m_0}}{\prod_{l \neq k \in \Lambda} (x_k - x_l)}. \tag{3.7b}$$

Interchanging the summation order and noticing that for a fixed k

$$\frac{x_k^{p+m_0}}{\prod_{l \neq k \in \Lambda} (x_k - x_l)} \prod_{j \notin \Lambda} \frac{x_k}{x_k - x_j} = \frac{x_k^{n-1+m_0}}{\prod_{l \neq k} (x_k - x_l)},$$

we have

$$\Omega = \sum_{k=1}^{n} \frac{x_k^{n-1+m_0}}{\prod_{l \neq k} (x_k - x_l)} \sum_{\substack{|\Lambda| = p \\ \Lambda \subset [n] \setminus \{k\}}} \prod_{i \in \Lambda} \frac{x_i^{n-1-p}}{\prod_{j \notin \Lambda \cup \{k\}} (x_i - x_j)} H(x|\Lambda \cup \{k\}). \tag{3.7c}$$

Noting that

$$h_m(x|\Lambda \cup \{k\}) = \sum_{\nu=0}^{m} x_k^{m-\nu} h_\nu(x|\Lambda),$$

we may combine its tensor product

$$H(x|\Lambda \cup \{k\}) = \prod_{\iota=1}^{p} \sum_{\nu_\iota = 0}^{m_\iota} x_k^{m_\iota - \nu_\iota} h_{\nu_\iota}(x|\Lambda)$$

with the induction hypothesis which asserts that for every p-ple nonnegative integers $\{\nu_1, \nu_2, \ldots, \nu_p\}$, there holds

$$\sum_{\substack{|\Lambda| = p \\ \Lambda \subset [n] \setminus \{k\}}} \prod_{i \in \Lambda} \frac{x_i^{n-1-p}}{\prod_{j \notin \Lambda \cup \{k\}} (x_i - x_j)} \prod_{\iota=1}^{p} h_{\nu_\iota}(x|\Lambda) = \prod_{\iota=1}^{p} h_{\nu_\iota}(x|[n] \setminus \{k\})$$

and conclude that

$$\sum_{\substack{|\Lambda| = p \\ \Lambda \subset [n] \setminus \{k\}}} \prod_{i \in \Lambda} \frac{x_i^{n-1-p}}{\prod_{j \notin \Lambda \cup \{k\}} (x_i - x_j)} H(x|\Lambda \cup \{k\}) = H(x). \tag{3.7d}$$

In view of (1.3), we arrive at

$$\Omega = \sum_{k=1}^{n} \frac{x_k^{n-1+m_0}}{\prod_{j \neq k}(x_k - x_j)} H(x) = h_{m_0}(x)\, H(x)\,. \tag{3.7e}$$

This complete the proof of the Theorems.

Observe that [7, chap. I] every elementary symmetric function $e_m(x|\Lambda)$ can be expressed as a multivariate polynomial of degree m in terms of complete symmetric functions $\{h_k(x|\Lambda)\}_k$, and viceversa. We can deduce from (3.4a) and (3.4b) the following

PROPOSITION 7. – *For p nonnegative integers $\{m_k\}_{k=1}^{p}$ with $0 \leqslant \sum_{k=1}^{p} m_k \leqslant p$, there hold*

$$\sum_{\substack{\Lambda \subset [n] \\ |\Lambda| = p}} \prod_{i \in \Lambda,\, j \notin \Lambda} \frac{x_i}{x_i - x_j} \prod_{k=1}^{p} e_{m_k}(x|\Lambda) = \prod_{k=1}^{p} e_{m_k}(x)\,, \tag{3.8a}$$

$$\sum_{\substack{\Lambda \subset [n] \\ |\Lambda| = p}} \prod_{i \in \Lambda,\, j \notin \Lambda} \frac{x_i}{x_i - x_j} \prod_{k=1}^{p} h_{m_k}(x|\Lambda^c) = \delta\left(0, \sum_{k=1}^{p} m_k\right). \tag{3.8b}$$

In particular, we have the dual formulas of (3.5a) and (3.5b).

COROLLARY 8. – *Let m be a nonnegative integer with $0 \leqslant m \leqslant p$, we have*

$$\sum_{\substack{\Lambda \subset [n] \\ |\Lambda| = p}} \prod_{i \in \Lambda,\, j \notin \Lambda} \frac{x_i}{x_i - x_j} e_m(x|\Lambda) = e_m(x)\,, \tag{3.9a}$$

$$\sum_{\substack{\Lambda \subset [n] \\ |\Lambda| = p}} \prod_{i \in \Lambda,\, j \notin \Lambda} \frac{x_i}{x_i - x_j} h_m(x|\Lambda^c) = \delta(0, m)\,. \tag{3.9b}$$

The last formula is also due to Gross and Richards [6, eq. (3.12)] and may be extended further when m is not limited to $0 \leqslant m \leqslant p$.

PROPOSITION 9. – *For each nonnegative integer m, we have*

$$\sum_{\substack{\Lambda \subset [n] \\ |\Lambda| = p}} \prod_{i \in \Lambda,\, j \notin \Lambda} \frac{x_i}{x_i - x_j} h_m(x|\Lambda^c) = \sum_{k=0}^{p} (-1)^k e_k(x)\, h_{m-k}(x) \tag{3.10a}$$

which specifies, for $m = p+1$, to

$$\sum_{\substack{\Lambda \subset [n] \\ |\Lambda| = p}} \prod_{i \in \Lambda,\, j \notin \Lambda} \frac{x_i}{x_i - x_j} h_{p+1}(x \mid \Lambda^c) = (-1)^p e_{p+1}(x). \tag{3.10b}$$

The corresponding generating functions read as

$$\sum_{\substack{\Lambda \subset [n] \\ |\Lambda| = p}} \prod_{i \in \Lambda,\, j \notin \Lambda} \frac{x_i}{x_i - x_j} \prod_{k \in \Lambda^c} \frac{1}{(1 - x_k y)} = \frac{\sum_{k=0}^{p} (-1)^k e_k(x) y^k}{\prod_{l=1}^{n} (1 - x_l y)}, \tag{3.11a}$$

$$\sum_{\substack{\Lambda \subset [n] \\ |\Lambda| = p}} \prod_{i \in \Lambda,\, j \notin \Lambda} \frac{x_i}{x_i - x_j} \prod_{k \in \Lambda} (1 - x_k y) = \sum_{k=0}^{p} (-1)^k e_k(x) y^k. \tag{3.11b}$$

In fact, similar to the proof of (3.4a), we have

$$\Xi = \sum_{\substack{\Lambda \subset [n] \\ |\Lambda| = p}} \prod_{i \in \Lambda,\, j \notin \Lambda} \frac{x_i}{x_i - x_j} h_m(x \mid \Lambda^c), \tag{3.12a}$$

$$= \sum_{\substack{\Lambda \subset [n] \\ |\Lambda| = p}} \prod_{i \in \Lambda,\, j \notin \Lambda} \frac{x_i}{x_i - x_j} \sum_{k \in \Lambda^c} \frac{x_k^{m+n-p-1}}{\prod_{l \neq k \in \Lambda} (x_k - x_l)}, \tag{3.12b}$$

$$= \sum_{k=1}^{n} (-1)^p \frac{x_k^{m+n-p-1}}{\prod_{l \neq k} (x_k - x_l)} \sum_{\substack{|\Lambda| = p \\ \Lambda \subset [n] \setminus \{k\}}} \prod_{i \in \Lambda} \frac{x_i^{n-p}}{\prod_{j \notin \Lambda \cup \{k\}} (x_i - x_j)}. \tag{3.12c}$$

According to (3.9a), the inner sum of the last equation is equal to

$$e_p(x \mid [n] \setminus \{k\}) = \sum_{\imath=0}^{p} (-1)^{p-\imath} x_k^{p-\imath} e_\imath(x)$$

which results from the generating functions

$$e_m(x \mid [n] \setminus \{k\}) = [y^m] \left\{ \frac{\prod_{\imath=1}^{n} (1 + x_\imath y)}{1 + x_k y} \right\}$$

$$= [y^m] \sum_{\nu=0}^{\infty} (-1)^\nu x_k^\nu y^\nu \sum_{\imath=0}^{n} e_\imath(x)\, y^\imath$$

$$= \sum_{\nu=0}^{m} (-1)^\nu x_k^\nu e_{m-\nu}(x).$$

Substituting this into (3.12*c*) for Ξ and using (3.9*b*), we have

$$\Xi = \sum_{k=1}^{n}(-1)^{p}\frac{x_k^{m+n-p-1}}{\prod_{l\neq k}(x_k-x_l)}\sum_{\imath=0}^{p}(-1)^{p-\imath}x_k^{p-\imath}e_\imath(x), \tag{3.12d}$$

$$= \sum_{\imath=0}^{p}(-1)^{\imath}e_\imath(x)\sum_{k=1}^{n}\frac{x_k^{m+n-\imath-1}}{\prod_{l\neq k}(x_k-x_l)}, \tag{3.12e}$$

$$= \sum_{\imath=0}^{p}(-1)^{\imath}e_\imath(x)\,h_{m-\imath}(x) \tag{3.12f}$$

which completes the proof of (3.10*a*).

Recall that [7, chap. I] every power-sum symmetric function $\varrho_m(x|\Lambda)$ in $\{x_k\}_{k\in\Lambda}$ can be expressed as a multivariate polynomial of degree m in terms of complete symmetric functions $\{h_k(x|\Lambda)\}_k$. For each term of the expression, apply (3.4*a*) and (3.8*b*). Then their respective sums give rise to the following formulas consequently.

PROPOSITION 10. – *For a nonnegative integer m with $0 \leqslant m \leqslant p$, there hold*

$$\sum_{\substack{\Lambda\subset[n]\\|\Lambda|=p}}\prod_{i\in\Lambda,\,j\notin\Lambda}\frac{x_i}{x_i-x_j}\varrho_m(x|\Lambda) = \varrho_m(x), \tag{3.13a}$$

$$\sum_{\substack{\Lambda\subset[n]\\|\Lambda|=p}}\prod_{i\in\Lambda,\,j\notin\Lambda}\frac{x_i}{x_i-x_j}\varrho_m(x|\Lambda^c) = \delta(0,m). \tag{3.13b}$$

For a partition defined by $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_p)$, denote by λ' its conjugate partition. Then the Schur functions [7, chap. I] read as

$$S_\lambda(x|\Lambda) = \det[h_{j-i+\lambda_i}(x|\Lambda)]_{1\leqslant i,j\leqslant p}, \tag{3.14a}$$

$$S_{\lambda'}(x|\Lambda^c) = \det[e_{j-i+\lambda_i}(x|\Lambda^c)]_{1\leqslant i,j\leqslant p} \tag{3.14b}$$

from which (3.4*a*) and (3.4*b*) may be reformulated as the following identities on the Schur functions.

PROPOSITION 11. – *Let $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_p)$ be a partition. There hold*

$$\sum_{\substack{\Lambda \subset [n] \\ |\Lambda| = p}} \prod_{i \in \Lambda,\, j \notin \Lambda} \frac{x_i}{x_i - x_j} S_\lambda(x|\Lambda) = S_\lambda(x), \tag{3.15a}$$

$$\sum_{\substack{\Lambda \subset [n] \\ |\Lambda| = p}} \prod_{i \in \Lambda,\, j \notin \Lambda} \frac{x_i}{x_i - x_j} S_{\lambda'}(x|\Lambda^c) = \delta\left(0, \sum_{k=1}^{p} \lambda_k\right). \tag{3.15b}$$

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