Andrea Cianchi

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Optimal Integrability of the Jacobian of Orientation Preserving Maps.

ANDREA CIANCHI

Sunto. – *Dato un qualsiasi spazio invariante per riordinamenti* $X(\Omega)$ *su un insieme aperto* $\Omega \subset \mathbb{R}^n$, *si determina il più piccolo spazio invariante per riordinamenti* $Y(\Omega)$ *con la proprietà che se* $u: \Omega \to \mathbb{R}^n$ *è una applicazione che mantiene l'orientamento e* $|Du|^n \in X(\Omega)$, *allora* $\det Du$ *appartiene localmente a* $Y(\Omega)$.

1. – Introduction and results.

Let Ω be an open bounded subset of $\mathbb{R}^n$, $n \geq 2$, and let $u: \Omega \to \mathbb{R}^n$ be a weakly differentiable function whose gradient will be denoted by Du. Consider any Banach function space $X(\Omega)$ of real-valued functions on Ω (see e.g. [BS], Chap. 1) and assume that $|Du|^n \in X(\Omega)$. Then, owing to the monotonicity of the norm in $X(\Omega)$ under pointwise inequality between functions, $\det Du$, the Jacobian of u, is also in $X(\Omega)$. When $X(\Omega) = L^1(\Omega)$, a well-known theorem by S. Müller [Mü] states that if, in addition, u is assumed to be *orientation preserving* (*o.p.*), i.e. $\det Du \geq 0$ a.e. on Ω, then $\det Du$ is not merely in $L^1(\Omega)$, but locally belongs to $L \log L(\Omega)$ as well. After this result, several spaces $X(\Omega)$ have been exhibited enjoing the analogous property that a Banach function space $Y(\Omega)$, strictly contained in $X(\Omega)$, exists such that, for every compact subset E of Ω,

$$\|\chi_E \det Du\|_{Y(\Omega)} \leq C \| |Du|^n \|_{X(\Omega)} \tag{1.1}$$

for all o.p. maps u with $|Du|^n \in X(\Omega)$. Here, $\|\cdot\|_{X(\Omega)}$ and $\|\cdot\|_{Y(\Omega)}$ denote the norms in $X(\Omega)$ and $Y(\Omega)$, respectively, χ_E is the characteristic function of E and C is a constant independent of u. We shall call the spaces $X(\Omega)$ enjoing this property *integrability improving for the Jacobian of o.p. maps* or, briefly, *integrability improving*. For example, the Zygmund space $L \log^\alpha L(\Omega)$ is integrability improving for every $\alpha \geq 0$, since (1.1) holds with $Y(\Omega) = L \log^{\alpha+1} L(\Omega)$ (this is the result of [Mü] when $\alpha = 0$; see [GI] for $\alpha = 1$

and [Mi] for $\alpha \geqslant 0$); more general integrability improving Orlicz spaces are exhibited in [Mos], [GIM] (see also [IS], [BFS] and [G] for related results). On the other hand, most Banach function spaces are not integrability improving. This is not the case, for instance, when $X(\Omega) = L^p(\Omega)$ with $p > 1$. Thus, the following (related) problems arise:

i) *How can integrability improving spaces be characterized?*

ii) *Given $X(\Omega)$, which is the smallest space $Y(\Omega)$ that renders* (1.1) *true?*

In the present note we answer these questions in the framework of *rearrangement invariant Banach function spaces* (*r.i. spaces*), namely Banach function spaces $X(\Omega)$ whose norm satisfies

$$\|f\|_{X(\Omega)} = \|g\|_{X(\Omega)} \qquad \text{whenever } f^* = g^*; \tag{1.2}$$

here f^* stands for the decreasing rearrangement of f, i.e. the non-increasing right-continuous function from $[0, \infty)$ into $[0, \infty)$ which is equidistributed with f (we refer to [BS] for an exhaustive treatment of r.i. spaces). In a sense, our results show that the maximal function approach of [Mü] to higher integrability properties of the Jacobian of o.p maps is sharp. Actually, roughly speaking, it turns out that the r.i. integrability improving spaces are precisely those where the norm of a function f and the norm ot its maximal function Mf are not equivalent, and that the gain in the integrability between $|Du|^n$ and $\det Du$ for o.p. maps is always exactly the same as that between Mf and f. In particular, we recover, as special cases, the above quoted contributions to the subject and prove their optimality in the setting of all r.i. spaces (see section 3).

Precise statements of the above assertions require the introduction of the space $X_0(\Omega)$ associated with $X(\Omega)$ as in the definition below. Such a definition makes use of the notion of the *representation space* $\overline{X}(0, |\Omega|)$ of $X(\Omega)$, namely the unique r.i. space on $(0, |\Omega|)$ such that

$$\|f\|_{X(\Omega)} = \|f^*\|_{\overline{X}(0, |\Omega|)} \tag{1.3}$$

for every $f \in X(\Omega)$, where $|\Omega|$ denotes the Lebesgue measure of Ω. Notice that, for customary r.i. spaces, such as Lebesgue, Lorentz and Orlicz spaces, the norm in the representation space can be immediately computed from the original one; however, a general formula for $\|\cdot\|_{\overline{X}(0, |\Omega|)}$ is available (see e.g. [BS], Chap. 2, proof of Thm. 4.10). Another ingredient in the definition of $X_0(\Omega)$ is the function f^{**} given for $f\colon \Omega \to \mathbb{R}$ by $f^{**}(s) = (1/s)\int_0^s f^*(r)\,dr$. Recall that, by Hertz' theorem, f^{**} is equivalent to $(Mf)^*$, i.e. $c_1 f^{**} \leqslant (Mf)^* \leqslant c_2 f^{**}$, where c_1 and c_2 are positive constants depending only on n.

DEFINITION. – *Let $X(\Omega)$ be a r.i. space on Ω. We call $X_0(\Omega)$ the space of all real-valued functions f on Ω for which the quantity*

$$\|f\|_{X_0(\Omega)} = \|f^{**}\|_{\overline{X}(0,\,|\Omega|)} \tag{1.4}$$

is finite.

It is a routine task to verify that $\|\cdot\|_{X_0(\Omega)}$ is a norm and that $X_0(\Omega)$, equipped with such a norm, is a r.i. space. Observe that, since $f^* \leqslant f^{**}$ for every f, then $\|\cdot\|_{X(\Omega)} \leqslant \|\cdot\|_{X_0(\Omega)}$, so that $X_0(\Omega) \subseteq X(\Omega)$, but the inclusion may be strict.

Problem ii) is solved by the following

THEOREM. – *Let Ω be an open bounded subset of $\mathbb{R}^n$, $n \geqslant 2$. Let $X(\Omega)$ be any r.i. space on Ω and let $X_0(\Omega)$ be the r.i. space defined as above. Then, for every compact subset E of Ω, a constant C exists such that*

$$\|\chi_E \det Du\|_{X_0(\Omega)} \leqslant C\, \|\,|Du|^n\|_{X(\Omega)} \tag{1.5}$$

for all o.p. maps $u\colon \Omega \to \mathbb{R}^n$ for which $|Du|^n \in X(\Omega)$.

Moreover, $X_0(\Omega)$ is the smallest r.i. space rendering an inequality of type (1.5) *true, in the sense that if* (1.1) *holds for some r.i. space $Y(\Omega)$, then $\|\cdot\|_{Y(\Omega)} \leqslant \mathrm{Const}\, \|\cdot\|_{X_0(\Omega)}$.*

This result enables us to characterize the r.i. integrability improving spaces and thus to answer problem i). As stated above, such a characterization can be given in terms of (un)boundedness properties of the maximal function operator M; recall that, for a locally integrable function $f\colon \Omega \to \mathbb{R}$, Mf is defined by

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\,dy\,,$$

where the supremum is extended over all cubes having sides parallel to the coordinate axes and f is continued by 0 outside Ω. Alternative characterizations make use of the one-dimensional Hardy operator H, defined as

$$H\phi(s) = \frac{1}{s} \int_0^s \phi(r)\,dr$$

on $\phi\colon [0,\,\infty) \to \mathbb{R}$, or of the (upper) Boyd index $I(X(\Omega)) \in [0,\,1]$ of $X(\Omega)$ given by

$$I(X(\Omega)) = \lim_{t \to \infty} \frac{\log \|D_t\|}{\log t}\,, \tag{1.6}$$

where $\|D_t\|$ is the norm of the dilation operator $D_t\colon \overline{X}(0,\,|\Omega|) \to \overline{X}(0,\,|\Omega|)$ defined, for $t \geqslant 1$, as $D_t \phi(s) = \phi(s/t)$, $s \in (0, |\Omega|)$. Let us mention that the index I plays a role in the theory of interpolation. Formulas for such index for customary spaces are known (see e.g. [BS], [BF], [Mo2]); for instance, $I(L^p(\Omega)) = 1/p$ if $p \in [1, \infty]$.

COROLLARY. – *Let Ω, $X(\Omega)$ and $X_0(\Omega)$ be as in the Theorem. Then the following conditions are equivalent*:

i) *$X(\Omega)$ is integrability improving for the Jacobian of o.p. maps.*

ii) *$X_0(\Omega)$ is strictly contained in $X(\Omega)$.*

iii) *The maximal function operator M is unbounded on $X(\Omega)$.*

iv) *The Hardy operator H is unbounded on $\overline{X}(0, |\Omega|)$.*

v) *The upper Boyd index $I(X(\Omega)) = 1$.*

2. – Proofs.

PROOF OF THE THEOREM. – Let u be as in the statement. Then, in particular, u belongs to the Sobolev space $W^{1,n}(\Omega', \mathbb{R}^n)$ for every smooth open subset Ω' of Ω. Indeed, since Ω has finite measure, then $X(\Omega) \subseteq L^1(\Omega)$ ([BS], Thm. 6.6, Chap. 2); hence $|Du| \in L^n(\Omega)$ and, by the Sobolev embedding theorem, $u \in L^n(\Omega')$. Thus, the following inequality of [Mi], rephrasing the result of [Mü] in terms of rearrangements, holds

$$(2.1) \qquad \frac{1}{s}\int_0^s (\chi_{Q/2} \det Du)^*(r)\, dr \leqslant c \left(\frac{1}{s}\int_0^s [(\chi_Q |Du|^n)^*(r)]^{1/n'}\, dr \right)^{n'}, \qquad \text{for } 0 < s \leqslant |Q|/2 .$$

Here, Q denotes any cube contained in Ω, $Q/2$ stands for the cube concentric with Q whose sides are parallel to those of Q and have half length, c is a constant depending only on n, and $n' = n/(n-1)$.

Consider now the operator T defined by

$$T\phi(s) = \left(\frac{1}{s}\int_0^s |\phi(r)|^{1/n'}\, dr \right)^{n'}, \qquad \text{for } s \in (0, |\Omega|)$$

on functions $\phi : (0, |\Omega|) \to \mathbb{R}$. The operator T is quasilinear, since for all $\lambda \in \mathbb{R}$ and all functions ϕ and ψ, $|T(\lambda\phi)| = |\lambda||T\phi|$ and $|T(\phi + \psi)| \leqslant 2^{n'-1}(|T\phi| + |T\psi|)$. Furthermore, T is bounded on $L^\infty(0, |\Omega|)$ with norm $\leqslant 1$ and, by Hardy's inequality ([BS], Chap. 3, Lemma 3.9) it is also bounded on $L^1(0, |\Omega|)$ with norm $\leqslant n^{n'}$. Thus, since $\overline{X}(0, |\Omega|)$ is a r.i. space, an interpolation theorem by Calderón ([BS], Thm. 2.12, Chap. 3) ensures that T is bounded on $\overline{X}(0, |\Omega|)$ and that

$$\|T\phi\|_{\overline{X}(0, |\Omega|)} \leqslant 2^{n'-1} n^{n'} \|\phi\|_{\overline{X}(0, |\Omega|)} \tag{2.2}$$

for all $\phi \in \overline{X}(0, |\Omega|)$. On setting $\phi = (\chi_Q |Du|^n)^*$ in (2.2), observing that (2.1) holds for all $s \in (0, |\Omega|)$ if c is replaced by $c(2|\Omega|/|Q|)^{n'-1}$ and recalling (1.4) yields

$$\|\chi_{Q/2} \det Du\|_{X_0(\Omega)} \leqslant c 4^{n'-1} n^{n'} (|\Omega|/|Q|)^{n'-1} \| |Du|^n \|_{X(\Omega)}. \tag{2.3}$$

Inequality (1.5) obviously follows from (2.3).

As for the optimality of $X_0(\Omega)$, suppose by contradiction that there exists a r.i. space $Y(\Omega)$ such that (1.1) holds, but

$$\sup_f \frac{\|f\|_{Y(\Omega)}}{\|f\|_{X_0(\Omega)}} = \infty , \tag{2.4}$$

where the supremum extends over all $f \in X_0(\Omega)$ which do not vanish identically. Given any such f, consider the function $v \colon \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$v(x) = \phi(|x|) \frac{x}{|x|}, \qquad \text{where} \qquad \phi(r) = \left(\frac{1}{C_n} \int_0^{C_n r^n} f^*(s)\, ds \right)^{1/n}$$

and $C_n = \pi^{n/2}/\Gamma(1 + n/2)$, the measure of the n-dimensional unit ball. On setting $r = |x|$, we have

$$\det Dv(x) = \frac{1}{n r^{n-1}} \frac{d}{dr} (\phi(r)^n) = f^*(C_n r^n) \geqslant 0 \qquad \text{for} \qquad x \in \mathbb{R}^n; \tag{2.5}$$

moreover,

$$\begin{aligned} |Dv|^n(x) &= (\operatorname{tr} Dv^T Dv)^{n/2} = \left((n-1) \frac{\phi^2(r)}{r^2} + \left(\frac{d\phi}{dr}(r) \right)^2 \right)^{n/2} \leqslant \\ &2^{(n/2)-1} \left(\frac{(n-1)^{n/2} \phi^n(r)}{r^n} + \left(\frac{d\phi}{dr}(r) \right)^n \right) = \\ &2^{(n/2)-1} \left(\frac{(n-1)^{n/2}}{C_n r^n} \int_0^{C_n r^n} f^*(s)\, ds + \left(\frac{1}{C_n r^n} \int_0^{C_n r^n} f^*(s)\, ds \right)^{1-n} (f^*(C_n r^n))^n \right) \leqslant \\ &\frac{K}{C_n r^n} \int_0^{C_n r^n} f^*(s)\, ds , \qquad \text{for } x \in \mathbb{R}^n, \end{aligned} \tag{2.6}$$

where $K = 2^{n/2-1}((n-1)^{n/2} + 1)$. Notice that the last inequality is due to the inequality $f^* \leqslant f^{**}$. Hence,

$$(|Dv|^n)^*(s) \leqslant Kf^{**}(s) \qquad \text{for } s>0\,. \tag{2.7}$$

Now, define $u\colon \Omega \to \mathbb{R}^n$ by $u = v\chi_\Omega$ and choose E in (1.1) to be a ball B (which may be assumed, without loss of generality, to be centered at the origin). Let $N \in \boldsymbol{N}$ be such that $N|B| \geqslant |\Omega|$. It is easily seen that, if we set $g(x) = f^*(C_n|x|^n)$, then

$$(g\chi_B)^{**}(s) \geqslant \frac{1}{N}(g\chi_{\mathbb{R}^n\setminus B})^{**}(s) \qquad \text{for } s>0\,.$$

Owing to the subadditivity of the operator «**» and to the last inequality,

$$(g^*)^{**}(s) = g^{**}(s) = (g\chi_B + g\chi_{\mathbb{R}^n\setminus B})^{**}(s) \leqslant (g\chi_B)^{**}(s) + (g\chi_{\mathbb{R}^n\setminus B})^{**}(s) \leqslant (N+1)(g\chi_B)^{**}(s) = (N+1)((g\chi_B)^*)^{**}(s) \qquad \text{for } s>0\,. \tag{2.8}$$

On making use of (2.5), (2.8) and of Corollary 4.7, Chap. 2 of [BS] we get

$$\|\chi_B \det Du\|_{Y(\Omega)} = \|\chi_B g\|_{Y(\Omega)} = \|(\chi_B g)^*\|_{\overline{Y}(0,\,|\Omega|)} \geqslant \frac{1}{N+1}\|g^*\|_{\overline{Y}(0,\,|\Omega|)} = \frac{1}{N+1}\|f^*\|_{\overline{Y}(0,\,|\Omega|)} = \frac{1}{N+1}\|f\|_{Y(\Omega)}. \tag{2.9}$$

On the other hand, since $(|Du|^n)^* \leqslant (|Dv|^n)^*$, inequality (2.7) and the monotonicity of $\|\cdot\|_{\overline{X}(0,\,|\Omega|)}$ imply that

$$\||Du|^n\|_{X(\Omega)} = \|(|Du|^n)^*\|_{\overline{X}(0,\,|\Omega|)} \leqslant K\|f^{**}\|_{\overline{X}(0,\,|\Omega|)} = K\|f\|_{X_0(\Omega)}. \tag{2.10}$$

From (2.9), (2.10) and (2.4) we deduce that

$$\sup_{u\text{ o.p.}} \frac{\|\chi_B \det Du\|_{Y(\Omega)}}{\||Du|^n\|_{X(\Omega)}} \geqslant \frac{1}{K(N+1)} \sup_{f\in X_0(\Omega)} \frac{\|f\|_{Y(\Omega)}}{\|f\|_{X_0(\Omega)}} = \infty\,, \tag{2.11}$$

a contradiction. ∎

Proof of the Corollary. – The equivalence of i) and ii) is a straightforward consequence of the Theorem. The equivalence of iii), iv) and v) is well-known (see e.g. [BS], Chap. 3). Thus, we only need to prove that ii) and iv) are equivalent. Since both $X(\Omega)$ and $X_0(\Omega)$ are Banach function spaces (and $X_0(\Omega) \subseteq X(\Omega)$), then, by Thm. 1.8, Chap. 1 of [BS], $X(\Omega) = X_0(\Omega)$ if and only if there exists a constant $k>0$ such that $\|\cdot\|_{X_0(\Omega)} \leqslant k\|\cdot\|_{X(\Omega)}$, i.e. if and only if

$$\|f^{**}\|_{\overline{X}(0,\,|\Omega|)} \leqslant k\|f^*\|_{\overline{X}(0,\,|\Omega|)}, \qquad \text{for all } f \text{ in } X(\Omega)\,. \tag{2.12}$$

Inequality (2.12) is equivalent to the inequality

$$\|H\phi\|_{\overline{X}(0,\,|\Omega|)} \leqslant k\|\phi\|_{\overline{X}(0,\,|\Omega|)} \qquad \text{for all } \phi \text{ in } \overline{X}(0,\,|\Omega|), \tag{2.13}$$

i.e. to the boundedness of H on $\overline{X}(0,\,|\Omega|)$. Indeed, (2.12) follows from (2.13) on choosing $\phi = f^*$. Conversely, assume that (2.12) holds. By Cor. 7.8, Chap. 2 of [BS], for every $\phi \in \overline{X}(0,\,|\Omega|)$ there exists a real-valued function f on Ω such that $\phi^* = f^*$. By Hardy-Littlewood's inequality ([BS], Thm. 2.2, Chap. 2), $\int_0^s \phi(r)\,dr \leqslant \int_0^s \phi^*(r)\,dr$ for $s \geqslant 0$. Therefore,

$$\|H\phi\|_{\overline{X}(0,\,|\Omega|)} \leqslant \|\phi^{**}\|_{\overline{X}(0,\,|\Omega|)} = \|f^{**}\|_{\overline{X}(0,\,|\Omega|)} \leqslant k\|f^*\|_{\overline{X}(0,\,|\Omega|)} = k\|\phi^*\|_{\overline{X}(0,\,|\Omega|)} = k\|\phi\|_{\overline{X}(0,\,|\Omega|)} \tag{2.14}$$

and (2.13) follows. ■

3. – Examples.

Lebesgue spaces. Any of conditions iii), iv), v) of the Corollary tells us that $L^1(\Omega)$ is the only integrability improving Lebesgue space. Moreover, since $\overline{L^1}(\Omega) = L^1(0,\,|\Omega|)$, we obtain from (1.4), via Fubini's theorem, that

$$\|f\|_{(L^1(\Omega))_0} = \int_0^{|\Omega|} f^*(s)\log(|\Omega|/s)\,ds\,,$$

a Lorentz norm which is equivalent to the norm in the Orlicz space $L\log L(\Omega)$ (see Remark 1 below). Thus, the Theorem reproduces the result of [Mü] and also tells us that it is optimal in the framework of all r.i. spaces.

Orlicz spaces. Let A be a Young function, i.e. a non-decreasing convex function from $[0,\,\infty)$ into $[0,\,\infty]$ vanishing at 0. We denote by $L^A(\Omega)$ the Orlicz space of those functions f whose Luxemburg norm, defined as $\|f\|_{L^A(\Omega)} = \inf\{\lambda > 0 : \int_\Omega A(|f(x)|/\lambda)\,dx \leqslant 1\}$, is finite. Condition iii) of the Corollary, combined with a generalization of the Hardy-Littlewood-Wiener maximal theorem, tells us that $L^A(\Omega)$ is integrability improving if and only if the conjugate Young function $\widetilde{A}$ of A, defined by $\widetilde{A}(s) = \sup\{rs - A(r) : r \geqslant 0\}$, does not satisfy the Δ_2-condition near infinity. Recall that this amounts to requiring that $\limsup_{r\to+\infty} \widetilde{A}^{-1}(r)/\widetilde{A}^{-1}(\mu r) \geqslant c$ for every $\mu > 0$, where c is a positive constant independent of μ. Furthermore, the optimal space

$$(L^A(\Omega))_0 = L^{A_0}(\Omega)\,,$$

with equivalent norms, where A_0 is the Young function defined by $A_0(s) =$

$s\int_0^s A(r)\, r^{-2}\, dr$ (and A is modified, if necessary, near 0, in such a way that the integral converges). This is a consequence of results from [BP], [C], [GIM], [K1], [K2]. Thus, the Theorem overlaps with Theorem 2 of [GIM].

Lorentz spaces. Given an integrable function $w:(0, |\Omega|) \to [0, \infty)$ and a number $q \geq 1$, the classical Lorentz space $\Lambda_{w,q}(\Omega)$ is the space of those functions f for which the quantity $\|f\|_{\Lambda_{w,q}(\Omega)} = \left(\int_0^{|\Omega|} f^*(s)^q\, w(s)\, ds \right)^{1/q}$ is finite. Observe that, when $w(s) = s^{q/p-1}$, $\Lambda_{w,q}(\Omega) = L^{p,q}(\Omega)$, the usual Lorentz space. In spite of the notation, the expression $\|\cdot\|_{\Lambda_{w,q}(\Omega)}$ need not be a norm. A characterization of those w and q for which $\|\cdot\|_{\Lambda_{w,q}(\Omega)}$ is (equivalent to) a norm, and hence $\Lambda_{w,q}(\Omega)$ is a r.i. space, is known. When $q > 1$, Theorem 4 of [S] tells us that this is the case if and only if the Hardy operator H is bounded on $\Lambda_{w,q}(0, |\Omega|)$. Hence, by the Corollary, no integrability improving r.i. space $\Lambda_{w,q}(\Omega)$ exists for $q > 1$. When $q = 1$, $\|\cdot\|_{\Lambda_{w,1}(\Omega)}$ is equivalent to a norm if and only if

$$\frac{1}{s}\int_0^s w(t)\, dt \leq \mathrm{Const}\, \frac{1}{r}\int_0^r w(t)\, dt \qquad \text{for } 0 < r \leq s \tag{3.1}$$

(see [CGS]); notice that, in particular, (3.1) holds with Const = 1 if w is nonincreasing. Under assumption (3.1), combining the Corollary with the results of [AM] tells us that $\Lambda_{w,1}(\Omega)$ is integrability improving if and only if

$$\limsup_{s \to 0} \frac{s \int_s^{|\Omega|} w(t)\, t^{-1}\, dt}{\int_0^s w(t)\, dt} = \infty\,.$$

An alternative necessary and sufficient condition for $\Lambda_{w,1}(\Omega)$ to be integrability improving, which follows from the Corollary via a result of Boyd (see [S]), is that

$$\lim_{t \to +\infty} \frac{\log\left(\sup_{0 < s < |\Omega|/t} \left(\int_0^{st} w(r)\, dr \Big/ \int_0^s w(r)\, dr \right)\right)}{\log t} = 1\,.$$

Moreover, the optimal space

$$(\Lambda_{w,1})_0(\Omega) = \Lambda_{w_0,1}(\Omega)\,,$$

where $w_0(s) = \int_s^{|\Omega|} w(t)\, t^{-1}\, dt$, as an application of Fubini's theorem shows.

Notice that the only integrability improving $L^{p,q}(\Omega)$ space is $L^{1,1}(\Omega) = L^1(\Omega)$.

REMARK 1. – The last two examples overlap, since Orlicz and Lorentz spaces may coincide. Actually, the Orlicz space $L^A(\Omega)$ is known to agree with a Lorentz space $\Lambda_{w,1}(\Omega)$ (with equivalent norms) if and only if there exists $\sigma > 0$ such that $\int^{\infty} \widetilde{A}(\sigma t)\, \widetilde{A}(t)^{-2}\, d\widetilde{A}(s) < \infty$, and, if this is the case, $w(s) = 1/A^{-1}(1/s)$; a condition of the same kind for the converse to hold is also available ([L]). Thus, in particular, the results where $X(\Omega) = L \log^{\alpha} L(\Omega)$, $\alpha \geqslant 0$, can be recovered from either of the last two examples, on choosing $A(s) = s \log^{\alpha}(1+s)$ and $w(s) = \log^{\alpha}(|\Omega|/s)$, respectively. It can be shown, however, that neither all integrability improving Orlicz spaces are Lorentz spaces nor the converse is true.

REMARK 2. – A common generalization of Orlicz and Lorentz spaces is given by the Orlicz-Lorentz spaces (see e.g. [Mo1]). Integrability improving spaces from this class could be exhibited on making use of the Corollary and of the information on their indices contained in [Mo2].

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Andrea Cianchi: Istituto di Matematica, Facoltà di Architettura
Università di Firenze - Via dell'Agnolo 14, 50122 Firenze, Italy
E-mail: cianchi@cesit1.unifi.it

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