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Superposition of Imbeddings and Fefferman's Inequality (*).

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Dedicated to memory of Filippo Chiarenza

Sunto. – *In questo lavoro si studiano condizioni sufficienti sulla funzione peso V, espresse in termini di integrabilità, per la validità della disuguaglianza*

$$\left(\int_B u^2(x)\, V(x)\, dx\right)^{1/2} \leqslant c\left(\int_B (\nabla u(x))^2\, dx\right)^{1/2},$$

dove B denota una sfera in $\mathbb{R}^N$. Usando una tecnica di decomposizione di immersioni si dimostrano condizioni sufficienti in termini di appartenenza a spazi di Lebesgue, Lorentz-Orlicz e/o di tipo debole. Come applicazioni vengono fornite condizioni sufficienti per la proprietà forte di prolungamento unico per $|\Delta u| \leqslant V|u|$ nelle dimensioni 2 e 3.

1. – Introduction.

Fefferman's inequality [6]

$$\text{(1.1)}\qquad \left(\int_{\mathbb{R}^N} u^2(x)\, V(x)\, dx\right)^{1/2} \leqslant c\left(\int_{\mathbb{R}^N} (\nabla u(x))^2\, dx\right)^{1/2}, \qquad u \in W^{1,2},$$

has turned out to be a very powerful tool to handle many topical problems in the PDEs including the strong unique continuation property (the SUCP in the sequel), distribution of eigenvalues and so on.

Our goal is to establish efficient and manageable condition for the function V, guaranteeing validity of a local version of (1.1), that is,

$$\text{(1.2)}\qquad \left(\int_B u^2(x)\, V(x)\, dx\right)^{1/2} \leqslant c\left(\int_B (\nabla u(x))^2\, dx\right)^{1/2}, \qquad u \in W_0^{1,2}(B),$$

where B is a bounded domain in $\mathbb{R}^N$, say, a ball, $|B| = 1$. We shall use a natural idea of a decomposition of the imbedding in (1.2) into an imbedding of $W_0^{1,2}$

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into a suitable target space and an imbedding from this target into $L^2(V)$; we invoke imbedding theorems for the Sobolev space $W_0^{1,2}$ — the classical Sobolev theorem and a refinement in terms of Lorentz spaces in the role of target spaces in the dimension $N \geq 3$, and the limiting imbedding theorem due to Brézis-Wainger [2] (see also [17], Lemma 2.10.5) in the dimension $N = 2$, which can be viewed as an analogous refinement of Trudinger's celebrated limiting imbedding [15]. The method suggested for proving (1.2) is a kind of a generator of N-dimensional Hardy inequalities or, alternatively, of weighted imbeddings $W_0^{1,2} \hookrightarrow L^2(V)$: general results of this nature will appear elsewhere. It is rather surprising that working with superpositions of imbeddings we do not lose much and that combining our conditions for validity of (1.2) with [4] we recover or generalize some of known results about the strong unique continuation property for $|\Delta u| \leq V|u|$ in dimensions 2 and 3. In fact all the above imbeddings of the Sobolev spaces are sharp in given scale of spaces and the same is true for the weighted imbeddings. In the latter case we shall use only Hölder's inequality, nevertheless, we actually use conditions which are necessary as well.

For the sake of applications we shall pay a special attention the so called «*smallness condition*» (see (1.3) below), playing a important role in the study of the strong unique continuation property for $|\Delta u| \leq V|u|$: Let $T(V)$ denote the imbedding in (1.1), let $B(x, r)$ stand for a ball centered at x and of radius r and let Ω be a bounded, open and connected subset of $\mathbb{R}^N$, $N=2$ or $N=3$; if

$$\limsup_{r \to 0_+} \|T(V\chi_{B(x,r)})\| \leq \varepsilon \tag{1.3}$$

with a sufficiently small $\varepsilon > 0$ for all $x \in B$, then any solution $u \in W_{\mathrm{loc}}^{2,2}$ of the inequality $|\Delta u| \leq V|u|$ in Ω has the SUCP—see Chanillo and Sawyer [4].

Let us recall that a locally integrable function u is said to have a zero of infinite order at x_0 if

$$\lim_{r \to 0_+} r^{-k} \int_{|x - x_0| < r} |u(x)|^2 \, dx = 0$$

for all $k = 1, 2, \ldots$. If every solution of an elliptic equation, with a zero of infinite order, vanishes identically, then the corresponding operator is said to satisfy the strong unique continuation property. As to non-analytic setting of the problem let us recall that in 1939 Carleman [3] proved that the operator $-\Delta + V$ has the strong unique continuation property provided $V \in L_{\mathrm{loc}}^{\infty}$, that is, he showed that under this assumption a solution of the equation $-\Delta u + V(x)u = 0$ with a zero of infinite order vanishes identically. There is a lot of results concerning the SUCP, with various assumptions on the potential V and also on coefficients in the case of a more general elliptic operator in question. Here we shall go along the lines of sufficient conditions in terms of integrability of the potential with no apriori assumptions on its pointwise behaviour. Let us recall Jerison and Kenig [8], Stein [14], where the SUCP is proved for $V \in L_{\mathrm{loc}}^{N/2}$ or for V locally small in the Marcinkiewicz space $L^{N/2,\,\infty}$, $N \geqslant 3$, and Pan [13] with the pointwise growth condition $V(x) \leqslant M/|x|^2$, $N \geqslant 2$, and without the size conditions for V. Wolff [16] has constructed counterexamples for $N = 3$ and $N = 2$, showing that the assumption about the local smallness of the imbedding norm in (1.1) cannot be removed in general. For $N = 2$ there is the result due to Gossez and Loulit [7] with the sufficient condition $V \in L^1 \log L$.

We shall need some basic facts from the Orlicz, Lorentz–Zygmund and Orlicz–Lorentz spaces theory. Let us agree that all the spaces in the sequel will be considered on a ball $B \subset \mathbb{R}^N$ with the unit measure, $N \geqslant 2$, or on the interval $(0,\,1)$; we shall usually omit the appropriate symbol for the domain since it will be clear from the context. Let us observe that then only the asymptotic behaviour of Young functions at infinity is relevant. An even and convex function $\Phi\colon \mathbb{R} \to [0,\,\infty)$ such that $\lim\limits_{t\to 0} \Phi(t) = \lim\limits_{t\to\infty} 1/\Phi(t) = 0$ is called a *Young function*. If Φ is a Young function, then $\Psi(t) = \sup\limits_{s>0}\{s|t| - \Phi(s)\}$ is the Young function *complementary* to Φ. A convex and even function Φ_1 is called the *major part of a Young function* Φ if $\Phi = \Phi_1$ near infinity. For brevity we shall often use only major parts of Young functions in symbols for spaces under consideration.

A Young function Φ_1 is dominated by a Young function Φ_2 if there is a constant $k > 0$ such that $\Phi_1(t) \leqslant \Phi_2(kt)$ near infinity. Two Young functions Φ_1 and Φ_2 are equivalent (we shall write $\Phi_1 \sim \Phi_2$) if each of them is dominated by the other. If $\Phi_1 \sim \Phi_2$, then the same relation holds for the complementary functions.

A general reference for the (non-weighted) theory of Orlicz spaces is [9], more general modular spaces are subject of [12].

We shall also need a finer scale of spaces, which includes Orlicz spaces in a rather same manner as Lorentz spaces include Lebesgue spaces. We refer to Montgomery-Smith [11]: Let Φ and Ψ be Young functions. For a function g even on $\mathbb{R}^1$ and positive on $(0,\,\infty)$ let us put

$$\tilde{g}(t) = \begin{cases} 1/g(1/t)\,, & t > 0\,, \\ \tilde{g}(-t)\,, & t < 0\,, \\ g(0)\,, & t = 0\,. \end{cases}$$

Let V be a weight in B and let f_V^* denote the non-increasing rearrangement of f with respect to the measure $V(x)\,dx$. An *Orlicz-Lorentz space* $L^{\Phi,\,\Psi}(V)$ is the set of all measurable f on B for which the *Orlicz-Lorentz functional*

$$(1.4)\qquad \|f\|_{\Phi,\,\Psi;\,V} = \|f_V^* \circ \widetilde{\Phi} \circ \widetilde{\Psi}^{-1}\|_{\Psi} = \inf\left\{\lambda > 0;\ \int_0^\infty \Psi\left(\frac{f_V^*(\widetilde{\Phi}(\widetilde{\Psi}^{-1}(t)))}{\lambda}\right) dt \leqslant 1\right\}$$

is finite. A measurable function f defined on B belongs to a *weak Orlicz* (or *Orlicz–Marcinkiewicz*) space $L^{\Phi,\,\infty}(V)$ if its *Orlicz–Marcinkiewicz functional*

$$(1.5)\qquad \|f\|_{\Phi,\,\infty;\,V} = \sup_{\xi > 0} \widetilde{\Phi}^{-1}(\xi)\, f_V^*(\xi)$$

is finite. If $V \equiv 1$, we shall simply write $L^{\Phi,\,\Psi}$ and $L^{\Phi,\,\infty}$ instead of $L^{\Phi,\,\Psi}(1)$ and $L^{\Phi,\,\infty}(1)$, resp.

The quantities in (1.4) and (1.5) are not generally norms. Nevertheless, they are quasinorms in many relevant cases; cf. Montgomery-Smith [11], and Krbec and Lang [10]. Let us observe that $L^{\Phi,\,\Phi} = L^{\Phi}$, the *Orlicz space*. If $\Phi(t) = |t|^p$ and $\Psi(t) = t^q$, then $L^{\Phi,\,\Psi} = L^{p,\,q}$, the *Lorentz space*, $L^{\Phi,\,\infty} = L^{p,\,\infty}$, the Marcinkiewicz space; analogously for the weighted variants.

Special cases of the Orlicz-Lorentz spaces are also the Lorentz-Zygmund spaces, that is, logarithmic Lorentz spaces, investigated by Bennett and Rudnick [1]. For $0 < p, q \leqslant \infty$ and $\alpha \in \mathbb{R}^1$, the Lorentz-Zygmund space $L^{p,\,q}(\log L)^{\alpha}$ consists of functions f with the finite functional

$$\|f\|_{L^{p,\,q}(\log L)^{\alpha}} = \left(\int_0^1 [t^{1/p}(\log(e/t))^{\alpha} f^*]^q \,\frac{dt}{t}\right)^{1/q}, \qquad \text{for } q < \infty\,,$$

$$\|f\|_{L^{p,\,\infty}(\log L)^{\alpha}} = \sup_{0<t<1} t^{1/p}(\log(e/t))^{\alpha} f^*(t)\,, \qquad \text{for } q = \infty$$

(we put $t^{1/\infty} = 1$). It is easy to check that these spaces increase with decreasing p, increasing q and decreasing α.

We shall not pursue the relationship between $L^{p,\,q}(\log L)^{\alpha}$ and $L^{\Phi,\,\Psi}$ in details here. Note only that later we shall also need the spaces of the form $L^{\exp t^{r'},\, t^r}$, where $1/r + 1/r' = 1$. It turns out that they coincide (see [5]) with spaces characterized by the condition (see [2], [17], Lemma 2.10.5)

$$\int_0^1 \left(\frac{f^*(t)}{\log(e/t)}\right)^r dt < \infty\,,$$

which equal to $L^{\infty,\,r}(\log L)^{-1}$ in the [1] notation. Also, the Zygmund space $L\log L$ equals to $L^{1,\,1}\log L$ and it is nothing but $L^{t\log t,\, t\log t}$.

REMARK 1.1. – We recall that $L^{p_1, q_1}(\log L)^{\alpha_1} \subset L^{p_2, q_2}(\log L)^{\alpha_2}$ if any of the following conditions holds:

(i) $p_1 > p_2$;

(ii) $p_1 = p_2$, $q_1 > q_2$, and $\alpha_1 + 1/q_1 > \alpha_2 + 1/q_2$;

(iii) $p_1 = p_2 < \infty$, $q_1 \leqslant q_2$, and $\alpha_1 \geqslant \alpha_2$;

(iv) $p_1 = p_2 < \infty$, $q_1 \leqslant q_2$, and $\alpha_1 + 1/q_1 > \alpha_2 + 1/q_2$

(see [1], Theorems 9.1 and 9.3 and 9.5).

REMARK 1.2. – According to the limiting imbedding theorem due to Brézis and Wainger [2] we have, for $N = 2$,

$$W_0^{1,2} \hookrightarrow L^{\infty, 2}(\log L)^{-1}. \tag{1.7}$$

The latter space, as was observed above, is the Orlicz-Zygmund space $L^{\exp t^2, t^2}$, a space smaller than $L^{\exp t^2} = L^{\exp t^2, \exp t^2}$, and this interpretation of the target space in (1.7) gives a natural analogue to the (sublimiting) imbeddings of Sobolev spaces into Lebesgue spaces and their Lorentz refinements.

2. – Decomposition of imbeddings.

Let B be a bounded domain in $\mathbb{R}^N$ with a sufficiently smooth boundary, say, with an extension property with respect to Sobolev spaces. Let us recall our agreement that for the sake of simplicity we shall suppose that B is a ball, $|B| = 1$. We shall usually omit the symbol of the domain. We are going to establish sufficient conditions for (1.2) and (1.3); we shall even prove a condition stronger than (1.3), namely,

$$\lim_{\delta \to 0} \sup_{\substack{A \subset B \\ |A| < \delta}} \|T(V\chi_A)\| = 0\,. \tag{2.1}$$

First we shall separately consider the scale of Lorentz spaces.

THEOREM 2.1. – *Let* $N \geqslant 3$.

(1) *Let* $V \in L^{N/2, r}$, $N/2 \leqslant r < \infty$. *Then* (1.2) *and* (2.1) *hold.*

(2) *Let* $V \in L^{N/2, \infty}$. *Then* (1.2) *holds.*

PROOF. – In both cases the inequality (1.2) follows by combining the refined Sobolev imbedding $W^{1,2} \hookrightarrow L^{2N/(N-2), s}$ where $s \geqslant 2$ (cf. e.g. [17], Theorem 2.10.2). To prove (2.1) fix $r \in [N/2, \infty)$ and put $p = 2r/(r-1)$. Then

$p \leqslant 2N/(N-2)$ and by Hölder's inequality,

$$\left(\int_B u^2(x)\, V(x)\, \chi_A(x)\, dx \right)^{1/2} \leqslant \|u\|_{L^{2N/(N-2),\, p}} \|V\chi_A\|_{L^{N/2,\, p/(p-2)}}.$$

Repeating this for $p = 2$ and $A = B$ yields also the imbedding in (2) itself when $V \in L^{N/2,\, \infty}$. ■

We shall pass to Lorentz-Zygmund spaces now and prove a theorem, establishing a general sufficient condition for (1.2) and various sufficient conditions for (2.1); let us observe that the situation is not straightforward since three parameters can change. The first parameter will be kept fixed, equal to 1; its changes lead to changes too big for the fine tuning we need.

THEOREM 2.2. – *Let* $N = 2$.

(1) *The inequality* (1.2) *holds provided* $V \in L^{1,\, \infty}(\log L)^2$.

(2) *Let* $V \in L^{1,\, s}(\log L)^{\beta}$, *where either*

$$0 < s \leqslant 1\,, \qquad \beta \geqslant 1\,, \tag{2.2}$$

or

$$1 < s < \infty\,, \qquad \beta \geqslant 2 - 1/s\,, \tag{2.3}$$

or

$$s = \infty\,, \qquad \beta > 2\,. \tag{2.4}$$

Then (1.2) *and* (2.1) *hold.*

PROOF. – *Step 1.* We shall prove (1). By Hölder's inequality we have

$$\int_B u^2(x)\, V(x)\, dx \leqslant \int_0^1 [(\log(e/t))^{-1} u^*(t)]^2\, t (\log(e/t))^2\, V^*(t)\, \frac{dt}{t} \leqslant$$
$$\|u\|^2_{L^{\infty,\, 2}(\log L)^{-1}} \|V\|_{L^{1,\, \infty}(\log L)^2}.$$

This together with Remark 1.2 yields (1.2).

Step 2. Let us assume that (2.2) holds. Then by Remark 1.1 (iii) we have

$$L^{1,\, s}(\log L)^{\beta} \hookrightarrow L^{1,\, 1}(\log L)^1. \tag{2.5}$$

It follows from (2.1) and Remark 1.1 (iv) that

$$W_0^{1,\, 2} \hookrightarrow L^{\infty,\, 2}(\log L)^{-1} \hookrightarrow L^{\infty,\, \infty}(\log L)^{-1/2}. \tag{2.6}$$

Using Hölder's inequality we get

$$\int_B u^2(x)(V\chi_A)(x)\,dx \leqslant \int_0^1 [(\log(e/t))^{-1/2} u^*(t)]^2 t \log(e/t)(V\chi_A)^*(t)\,\frac{dt}{t} \leqslant$$

$$\|u\|^2_{L^{\infty,\infty}(\log L)^{-1/2}} \|V\chi_A\|_{L^{1,1}(\log L)^1}.$$

This combined with (2.5) and (2.6) implies the inequality (1.2) and the size condition (2.1).

Step 3. Let us now suppose that (2.4) is true. Then by Remark 1.1 (iii) we have

$$V \in L^{1,s}(\log L)^{2-1/s}. \tag{2.7}$$

Furthermore, from (1.7) and Remark 1.1 (iv),

$$W_0^{1,2} \hookrightarrow L^{\infty,2s/(s-1)}(\log L)^{(1/2s)-1}.$$

By Hölder's inequality we have

$$\int_B u^2(x)(V\chi_A)(x)\,dx \leqslant \|u\|^2_{L^{\infty,2s/(s-1)}(\log L)^{(1/2s)-1}} \|V\chi_A\|_{L^{1,s}(\log L)^{2-(1/s)}}.$$

This together with (2.6) and (2.7) gives (1.2) and (2.1).

Step 4. Finally, we assume that (2.4) holds. Then $V \in L^{1,1}(\log L)^1$ by Remark 1.1 (ii) and this case has been considered in Step 2 above. ■

REMARK 2.3. – The space $L^{1,\infty}(\log L)^2$ can be identified with the Orlicz-Marcinkiewicz space $L^{t\log^2 t,\,\infty}$ and $L^{1,s}(\log L)^\beta$, $0 < s < \infty$, with $L^{t\log^\beta t,\,t^s}$. This can be checked easily. Indeed, considering for instance $V \in L^{1,\infty}(\log L)^2$, that is, $\sup_{0<t<1} t(\log(e/t))^2 V^*(t) < \infty$, we have $\widetilde{F}^{-1}(t) = t(\log(e/t))^2$ near the origin, hence $F(\xi) \sim \xi(\log(e/\xi))^2$ for large values of ξ.

By way of applications we give a sufficient condition for the SUCP, relying on a result due to Chanillo and Sawyer [4] recalled in Section 1.

COROLLARY 2.4.

(1) *Let $N = 3$. Let $V \in L^{3/2,r}$, $3/2 \leqslant r < \infty$. Then the inequality $|\Delta u| \leqslant V|u|$ has the* SUCP *in $W^{2,2}_{\mathrm{loc}} \cap W_0^{1,2}$.*

(2) *Let $N = 2$. Let $u \in W^{2,2}_{\mathrm{loc}} \cap W_0^{1,2}$, let $V \in L^{1,s}(\log L)^\beta$, where s and β satisfy any of the conditions* (2.2)-(2.4). *Then the inequality $|\Delta u| \leqslant V|u|$ has the* SUCP *in $W^{2,2}_{\mathrm{loc}} \cap W_0^{1,2}$.*

REMARK 2.5. – The statement in (1) actually says that the size condition from Stein [14] is fulfilled under the given conditions.

If $V \in L^{1,s}(\log L)^{\beta}$, where s and β satisfy either (2.2) or (2.4), then $V \in L^{1,1}(\log L)^1$ (see Remark 1.1) and we recover the SUCP theorem due to Gossez and Loulit [7]. Concerning (2.3) we show that $L^{1,1}(\log L)^1$ and $L^{1,s}(\log L)^{2-(1/s)}$ are incomparable for $1 < s < \infty$. Indeed, let V_{α}, $0 < \alpha \leq 1$, be such that

$$V_{\alpha}^*(t) = \frac{1}{t}(\log(e/t))^{-2}(\log(\log(e/t)))^{-\alpha}, \qquad \text{for } t \text{ small}.$$

Then $V_{\alpha} \notin L^{1,1}(\log L)^1$. On the other hand, if $s > 1/\alpha$, then $V_{\alpha} \in L^{1,s}(\log L)^{2-(1/s)}$. Hence we have

$$L^{1,s}(\log L)^{2-(1/s)} \not\subset L^{1,1}(\log L)^1, \qquad 1 < s < \infty.$$

It remains to prove that

$$L^{1,1}(\log L)^1 \not\subset L^{1,s}(\log L)^{2-(1/s)}. \tag{2.8}$$

Let V_{τ}, $0 < \tau < 1$, be such that $V_{\tau}^*(t) = \chi_{(0,\tau)}(t)$. Then

$$\|V_{\tau}\|_{L^{1,1}(\log L)^1} = \tau(2 - \log \tau), \qquad 0 < \tau < 1.$$

We have

$$\lim_{\tau \to 0} \frac{\|V_{\tau}\|^s_{L^{1,s}(\log L)^{2-(1/s)}}}{\|V_{\tau}\|^s_{L^{1,1}(\log L)^1}} = \lim_{\tau \to 0} \frac{\int_0^{\tau} t^{s-1}(\log(e/t))^{2s-1}\,dt}{\tau^s(2-\log\tau)^s} =$$

$$\lim_{\tau \to 0} \frac{\tau^{s-1}(\log e/\tau)^{2s-1}}{s\tau^{s-1}(2-\log\tau)^{s-1}\log e/\tau} = \infty.$$

Hence $L^{1,1}(\log L)^1$ is not continuously imbedded into $L^{1,s}(\log L)^{2-(1/s)}$ and by the closed graph theorem we get (2.8).

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