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On Almost Normal Subgroups of Supersoluble Groups

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Sunto. – *Un sottogruppo H di un gruppo G si dice «almost normal» se ha soltanto un numero finito di coniugati in G, e ovviamente l'insieme* an (G) *costituito dai sottogruppi almost normal di G è un sottoreticolo del reticolo* $\mathfrak{L}(G)$ *di tutti i sottogruppi di G. In questo articolo vengono studiati gli isomorfismi tra reticoli di sottogruppi almost normal, provando in particolare che se G è un gruppo supersolubile e* $\overline{G}$ *è un gruppo FC-risolubile tale che i reticoli* an (G) *e* an $(\overline{G})$ *sono isomorfi, allora anche* $\overline{G}$ *è supersolubile, e inoltre G e* $\overline{G}$ *hanno la stessa lunghezza di Hirsch.*

1. – Introduction.

A subgroup H of a group G is said to be *almost normal* if it has only finitely many conjugates in G, i.e. if its normalizer $N_G(H)$ has finite index in G. A famous result of B.H. Neumann [4] states that all subgroups of a group G are almost normal if and only if the centre factor group $G/Z(G)$ is finite, while it is clear that *FC-groups* (i.e. groups in which every element has finitely many conjugates) are precisely those groups in which cyclic subgroups are almost normal. These facts suggest to consider groups which are rich of almost normal subgroups in some natural sense. In general, the set an (G) of all almost normal subgroups of a group G is a sublattice of the lattice $\mathfrak{L}(G)$ of all subgroups of G, and some properties of this lattice have been recently studied (see [3],[5],[6]). The aim of this article is to consider lattice isomorphisms $\varphi : \mathrm{an}(G) \to \mathrm{an}(\overline{G})$ between the lattices of almost normal subgroups of two groups G and $\overline{G}$, investigating the consequences for the group $\overline{G}$ of the imposition of group theoretical conditions on the group G. This problem is also suggested by the strong development of the theory of projectivities between groups (i.e. isomorphisms between lattices of all subgroups) over the last fifty years. We will use the recent monograph [9] as a general reference on this subject. Of course, every subgroup of a finite group is almost normal, and hence dealing with the above problem it is natural to restrict the attention to situations that have already been studied in the case of projectivities between finite groups.

It is well-known that there exist (finite) non-nilpotent groups whose subgroups lattices are isomorphic to that of an abelian group, while for a group G the property of being supersoluble can be detected from the behaviour of the lattice $\mathfrak{L}(G)$ (see [9], Theorem 6.4.11). Moreover, every supersoluble group is nilpotent-by-finite, so that it has many almost normal subgroups, and hence satisfactory results can be expected studying lattice isomorphisms $\varphi\colon \mathrm{an}(G) \to \mathrm{an}(\overline{G})$, when G is a supersoluble group. On the other hand, an infinite simple group cannot contain proper non-trivial almost normal subgroups, so that in order to avoid this case, the group $\overline{G}$ must be required to satisfy a (weak) suitable condition. A group is called *FC-soluble* if it has a finite series whose factors are FC-groups. The main result of this article is the following.

THEOREM. – *Let G be a supersoluble group, $\overline{G}$ an FC-soluble group and $\varphi\colon \mathrm{an}(G) \to \mathrm{an}(\overline{G})$ a lattice isomorphism. Then $\overline{G}$ is supersoluble and it has the same Hirsch length as G.*

Recall here that if G is a polycyclic-by-finite group, the number of infinite factors of a finite series of G whose factors either are finite or cyclic is an invariant, called the Hirsch length of G. Most of the notation is standard and can be found in [7] or [9].

I would like to express my gratitude to Maria Cristina Cirino Groccia for many useful discussions on this subject.

2. – Proof of the Theorem.

Let G and $\overline{G}$ be groups, and let $\varphi\colon \mathrm{an}(G) \to \mathrm{an}(\overline{G})$ be a lattice isomorphism. If N is a normal subgroup of G whose image N^{φ} is normal in $\overline{G}$, then φ induces a lattice isomorphism $\mathrm{an}(G/N) \to \mathrm{an}(\overline{G}/N^{\varphi})$. If H is a subgroup of finite index of G, then φ also induces an isomorphism between the lattices $\mathrm{an}(H)$ and $\mathrm{an}(H^{\varphi})$, provided that the subgroup H^{φ} has finite index in $\overline{G}$. We shall prove that this property holds when the group $\overline{G}$ is FC-soluble. This result should be seen in relation to the relevant theorem of Zacher [10] stating that the image of any subgroup of finite index under a projectivity has likewise finite index.

LEMMA 2.1. – *Let G be an FC-soluble group having finitely many almost normal subgroups. Then G is finite.*

PROOF. – Consider a series of minimal length

$$\{1\} = G_0 < G_1 < \ldots < G_t = G$$

of G whose factors are FC-groups. Then G/G_{t-1} is an FC-group with finitely many almost normal subgroups, so that in particular G/G_{t-1} has finitely many cyclic subgroups and hence it is finite. It follows that also G_{t-1} has a finite number of almost normal subgroups, so that G_{t-1} is finite by induction on t, and hence G is a finite group. ■

LEMMA 2.2. – *Let G and $\overline{G}$ be groups and let $\varphi\colon \mathrm{an}(G) \to \mathrm{an}(\overline{G})$ be a lattice isomorphism. If $\overline{G}$ is FC-soluble and H is a subgroup of finite index of G, then the image H^{φ} of H has finite index in $\overline{G}$.*

PROOF. – Replacing H by its core, it can be assumed without loss of generality that H is a normal subgroup of G. Since H^{φ} is almost normal in $\overline{G}$, its normalizer $\overline{K} = N_{\overline{G}}(H^{\varphi})$ has finite index in $\overline{G}$, and hence φ induces a lattice isomorphism $\mathrm{an}(\overline{K}^{\varphi^{-1}}/H) \to \mathrm{an}(\overline{K}/H^{\varphi})$. Thus the FC-soluble group $\overline{K}/H^{\varphi}$ has finitely many almost normal subgroups, and so it is finite by Lemma 2.1. Therefore the subgroup H^{φ} has finite index in $\overline{G}$. ■

COROLLARY 2.3. – *Let G and $\overline{G}$ be groups, and let $\varphi\colon \mathrm{an}(G) \to \mathrm{an}(\overline{G})$ be a lattice isomorphism. If $\overline{G}$ is FC-soluble and J is the finite residual of G, then the subgroup J^{φ} contains the finite residual of $\overline{G}$. In particular, if G is residually finite, also $\overline{G}$ is residually finite.*

PROOF. – Let $\overline{J}$ be the finite residual of $\overline{G}$, and let H be any subgroup of finite index of G. Then by Lemma 2.2 the image H^{φ} of H is a subgroup of finite index of $\overline{G}$, so that $\overline{J}$ is contained in H^{φ} and hence H contains $\overline{J}^{\varphi^{-1}}$. It follows that $\overline{J}^{\varphi^{-1}} \leqslant J$, and so $\overline{J}$ is contained in J^{φ}. ■

LEMMA 2.4. – *Let G and $\overline{G}$ be groups, and let $\varphi\colon \mathrm{an}(G) \to \mathrm{an}(\overline{G})$ be a lattice isomorphism. If G is cyclic and $\overline{G}$ is FC-soluble, then also $\overline{G}$ is cyclic.*

PROOF. – Let $\overline{N}$ be any normal subgroup of finite index of $\overline{G}$ and put $N = \overline{N}^{\varphi^{-1}}$. Then φ induces a projectivity between the finite groups G/N and $\overline{G}/\overline{N}$, and hence also $\overline{G}/\overline{N}$ is cyclic. On the other hand, the group $\overline{G}$ is residually finite by Corollary 2.3, so that $\overline{G}$ is abelian and φ is a projectivity between G and $\overline{G}$. Therefore $\overline{G}$ is a cyclic group. ■

LEMMA 2.5. – *Let G be a free abelian group of finite rank, $\overline{G}$ an FC-soluble group and $\varphi\colon \mathrm{an}(G) \to \mathrm{an}(\overline{G})$ a lattice isomorphism. Then $\overline{G}$ contains a normal subgroup of finite index that is isomorphic to G.*

PROOF. – Put $G = G_1 \times \ldots \times G_t$, where every G_i is an infinite cyclic group, and for each $i \leqslant t$ consider the subgroup $H_i = Dr_{j \neq i}\, G_j$. Since H_i^{φ} is an almost normal subgroup of $\overline{G}$, its normalizer $\overline{N}_i = N_{\overline{G}}(H_i^{\varphi})$ has finite index in $\overline{G}$, and so also

$$\overline{N} = \bigcap_{i=1}^{t} \overline{N}_i$$

is a subgroup of finite index of $\overline{G}$. Then also the core $\overline{K}$ of $\overline{N}$ in $\overline{G}$ has finite index in $\overline{G}$. Let $K = \overline{K}^{\varphi^{-1}}$ and $L_i = K \cap H_i$ for all $i \leqslant t$. Then G/K is finite and K/L_i is an infinite cyclic group. Moreover, the subgroup $L_i^{\varphi} = \overline{K} \cap H_i^{\varphi}$ is normal in $\overline{K}$, and the map φ induces a lattice isomorphism

$$\mathfrak{L}(K/L_i) \to \mathrm{an}(\overline{K}/L_i^{\varphi}),$$

so that $\overline{K}/L_i^{\varphi}$ is cyclic by Lemma 2.4. Obviously

$$\bigcap_{i=1}^{t} L_i = \{1\}$$

and hence also

$$\bigcap_{i=1}^{t} L_i^{\varphi} = \{1\},$$

so that the group $\overline{K}$ is abelian. Therefore the lattices $\mathfrak{L}(K)$ and $\mathfrak{L}(\overline{K})$ are isomorphic, and also $\overline{K}$ is a free abelian group of rank t. ■

COROLLARY 2.6. – *Let G be a finitely generated abelian-by-finite group, $\overline{G}$ an FC-soluble group and φ*: $\mathrm{an}(G) \to \mathrm{an}(\overline{G})$ *a lattice isomorphism. Then $\overline{G}$ is a finitely generated abelian-by-finite group and it has the same Hirsch length as G.*

PROOF. – Since G is a finitely generated abelian-by-finite group, it contains a free abelian normal subgroup A of finite rank such that G/A is finite. Then A^{φ} has finite index in $\overline{G}$ by Lemma 2.2, and φ induces a lattice isomorphism between $\mathfrak{L}(A)$ and $\mathrm{an}(A^{\varphi})$, so that it follows from Lemma 2.5 that A^{φ} contains a normal subgroup $\overline{B}$ isomorphic to A such that $A^{\varphi}/\overline{B}$ is finite. Therefore $\overline{G}$ is a finitely generated abelian-by-finite group having the same Hirsch length as G. ■

Let G and $\overline{G}$ be groups, φ: $\mathfrak{L}(G) \to \mathfrak{L}(\overline{G})$ a lattice isomorphism, and let N be a normal subgroup of G. It has been proved by G. Busetto that, if $\overline{H}$ and $\overline{K}$ are the normal closure and the core of N^{φ} in $\overline{G}$, respectively, then the subgroups $\overline{H}^{\varphi^{-1}}$ and $\overline{K}^{\varphi^{-1}}$ are normal in G (see [9], Theorem 6.5.6). It follows that φ induces a projectivity from $G/\overline{K}^{\varphi^{-1}}$ onto $\overline{G}/\overline{K}$ and in particular it is possible to reduce many problems concerning projective images of subgroups of finite index to the case of finite groups. Using the same ideas developed in the proof of Busetto's theorem, we will show that a similar situation occurs in the case of isomorphisms between lattices of almost normal subgroups. Recall here that a subgroup H of a group G is said to be *quasinormal* if $HK = KH$ for every subgroup K of G.

LEMMA 2.7. – *Let G and $\overline{G}$ be FC-soluble groups, $\varphi\colon \mathrm{an}(G) \to \mathrm{an}(\overline{G})$ a lattice isomorphism and N a normal subgroup of finite index of G. If $\overline{H}$ and $\overline{K}$ are the normal closure and the core of N^{φ} in $\overline{G}$, respectively, then the subgroups $\overline{H}^{\varphi^{-1}}$ and $\overline{K}^{\varphi^{-1}}$ are normal in G.*

PROOF. – Put $H = \overline{H}^{\varphi^{-1}}$ and $K = \overline{K}^{\varphi^{-1}}$. Assume by contradiction that the subgroup H is not normal in G, and choose a counterexample such that the index $|H : N|$ is minimal. If L is any normal subgroup of G such that $N \leqslant L \leqslant H$, we have that $\overline{H}$ is the normal closure of L^{φ} in $\overline{G}$, and so it follows from the minimal choice of the counterexample that $L = N$. Therefore N is the core of H in G. The subgroup $\overline{N} = N^{\varphi}$ has finite index in $\overline{G}$ by Lemma 2.2, so that also $\overline{K}$ has finite index in $\overline{G}$, and the lattices $\mathfrak{L}(G)$ and $\mathfrak{L}(\overline{G})$ contain the isomorphic intervals $[G/K]$ and $[\overline{G}/\overline{K}]$. Since N is a normal subgroup of G, $\overline{N}$ is a modular element of the lattice $[\overline{G}/\overline{K}]$, and so $\overline{N}$ is a modular subgroup of $\overline{G}$. As in the proof of Lemma 6.5.5 of [9], it can be shown that $\overline{N}$ is actually quasinormal in $\overline{G}$. Since the subgroup $\overline{N}$ is not normal in $\overline{G}$, there exists an element y of $\overline{G}$ such that $\overline{N}^{y} \neq \overline{N}$ and $\langle y\rangle/\langle y\rangle \cap \overline{N}$ has prime-power order. Put $\overline{T} = \langle \overline{N}, y\rangle$ and $T = \overline{T}^{\varphi^{-1}}$. Then the lattice $\mathfrak{L}(T/N)$ is isomorphic to the interval $[\overline{T}/\overline{N}]$ of $\mathfrak{L}(\overline{G})$, and so also to $\mathfrak{L}(\langle y\rangle/\langle y\rangle \cap \overline{N})$. It follows that also T/N is cyclic, and its order is a power of a prime number p. Let x be an element of G such that $T = \langle N, x\rangle$. As in the proof of Lemma 6.5.5 of [9], it can now be proved that $R/N = \Omega(T/N)$ is a quasinormal subgroup of order p of G/N, and it is contained in H/N. Moreover, if A/N is a cyclic q-subgroup of G/N for some prime $q \neq p$ such that $\overline{N}$ is not normal in A^{φ}, a similar argument shows that $\Omega(A/N)$ is a quasinormal subgroup of order q of G/N which is contained in H/N. Now a contradiction can be obtained following the same lines of the proof of Lemma 6.5.5 of [9]. Therefore the subgroup H is normal in G. To prove that also K is a normal subgroup of G, the argument of Theorem 6.5.6 of [9] can be used, observing that the first part of the statement also holds for the isomorphism φ^{-1}, since both groups G and $\overline{G}$ are assumed to be FC-soluble. ■

LEMMA 2.8. – *Let G be a soluble residually finite group with derived length n, $\overline{G}$ an FC-soluble group and $\varphi\colon \mathrm{an}(G) \to \mathrm{an}(\overline{G})$ a lattice isomorphism. Then $\overline{G}$ is soluble with derived length at most $3n-1$.*

PROOF. – Let $\overline{N}$ be any normal subgroup of finite index of $\overline{G}$. If K is the core of $N = \overline{N}^{\varphi^{-1}}$, it follows from Lemma 2.7 that the image $\overline{K} = K^{\varphi}$ of K is a normal subgroup of $\overline{G}$. Moreover, the finite groups G/K and $\overline{G}/\overline{K}$ have isomorphic subgroup lattices, and hence $\overline{G}/\overline{K}$ is soluble with derived length at most $3n-1$

(see [9], Theorem 6.6.3). Therefore $\overline{G}^{(3n-1)}$ is contained in $\overline{N}$ and $\overline{G}$ is soluble with derived length at most $3n-1$, since $\overline{G}$ is residually finite by Corollary 2.3. ∎

If $\mathfrak{X}$ is a class of groups, a group G is called *just-non-*$\mathfrak{X}$ if it is not an $\mathfrak{X}$-group but all its proper homomorphic images belong to $\mathfrak{X}$. The structure of soluble just-non-supersoluble groups has been described by Robinson and Wilson [8], and we will use their results in the proof of our theorem.

PROOF OF THE THEOREM. – The group G is soluble by Lemma 2.8. Let $\overline{N}$ be any normal subgroup of finite index of $\overline{G}$. If K is the core of $\overline{N}^{\varphi^{-1}}$ in G, the image $\overline{K} = K^{\varphi}$ of K is a normal subgroup of $\overline{G}$ by Lemma 2.7, and the finite groups G/K and $\overline{G}/\overline{K}$ have isomorphic subgroup lattices, so that the factor group $\overline{G}/\overline{K}$ is supersoluble (see [9], Corollary 5.3.8), and hence also $\overline{G}/\overline{N}$ is supersoluble. Therefore every finite homomorphic image of $\overline{G}$ is supersoluble. Assume by contradiction that $\overline{G}$ is not supersoluble. Then it follows from a result of Baer that $\overline{G}$ is not even polycyclic (see [1]). The Fitting subgroup F of G is nilpotent, and the factor group G/F is finite. Then the subgroup F^{φ} has finite index in $\overline{G}$, and so it is not polycyclic. Moreover, there exists a lattice isomorphism between $\mathrm{an}(F)$ and $\mathrm{an}(F^{\varphi})$, so that replacing G by F and $\overline{G}$ by F^{φ} we may suppose that G is a finitely generated nilpotent group. Since $\overline{G}$ satisfies the maximal condition on almost normal subgroups, there exists an almost normal subgroup $\overline{H}$ of $\overline{G}$ which is maximal with respect to the condition that the factor group $N_{\overline{G}}(\overline{H})/\overline{H}$ is not polycyclic. Put $H = \overline{H}^{\varphi^{-1}}$ and consider the subgroup of finite index $L = N_G(H) \cap (N_{\overline{G}}(\overline{H}))^{\varphi^{-1}}$ of G. Let $\overline{M}$ be any normal subgroup of $\overline{L} = L^{\varphi}$ properly containing $\overline{H}$. Since $\overline{L}$ has finite index in $\overline{G}$, $\overline{M}$ is almost normal in $\overline{G}$, and hence the group $N_{\overline{G}}(\overline{M})/\overline{M}$ is polycyclic. In particular $\overline{L}/\overline{M}$ is polycyclic, so that all proper quotients of $\overline{L}/\overline{H}$ are polycyclic. Moreover, the group $\overline{L}/\overline{H}$ is not polycyclic, and φ induces a lattice isomorphism from $\mathrm{an}(L/H)$ onto $\mathrm{an}(\overline{L}/\overline{H})$. Replacing now G by L/H and $\overline{G}$ by $\overline{L}/\overline{H}$, it can be assumed without loss of generality that the group $\overline{G}$ is just-non-polycyclic, and so even just-non-supersoluble since all its finite homomorphic images are supersoluble. Thus the Fitting subgroup $\overline{A}$ of $\overline{G}$ is isomorphic with Q_{π} for some finite set π of primes (here Q_{π} denotes the additive group of rational numbers whose denominators are π-numbers), and there exists a free abelian subgroup $\overline{X}$ of $\overline{G}$ such that $\overline{U} = \overline{X}\overline{A}$ has finite index in $\overline{G}$ and $\overline{A} \cap \overline{X} = \{1\}$ (see [8]). For each prime number $p \notin \pi$, the subgroup $\overline{V}_p = \overline{X}\overline{A}^p$ has index p in $\overline{U}$, so that $V_p = \overline{V}_p^{\varphi^{-1}}$ is a maximal almost normal subgroup of the nilpotent group $U = \overline{U}^{\varphi^{-1}}$. Thus V_p is normal in U, and U/V_p has prime order. It follows that $V = \bigcap_{p \notin \pi} V_p$ is a normal subgroup of U, and the factor group U/V is abelian. In particular, V is almost normal in G, and we may consider its image $\overline{V} = V^{\varphi}$. Since

$$\bigcap_{p \notin \pi} \overline{A}^p = \{1\}$$

the subgroup $\overline{V}$ is contained in $\overline{X} = \bigcap_{p \notin \pi} \overline{V}_p$. Let $\overline{W}$ be the core in $\overline{G}$ of the normalizer $N_{\overline{G}}(\overline{V})$. Then

$$[\overline{A} \cap \overline{W}, \overline{V}] \leqslant \overline{A} \cap \overline{V} = \{1\},$$

so that the subgroup of finite index $\langle \overline{A} \cap \overline{W}, \overline{X} \rangle$ centralizes $\overline{V}$, and hence $\overline{V}$ lies in the FC-centre of $\overline{G}$. On the other hand, since the soluble group $\overline{G}$ has no polycyclic non-trivial normal subgroups, its FC-centre is trivial, and so $\overline{V} = 1$. Therefore $V = 1$, so that U is abelian, and the group G is abelian-by-finite. Application of Corollary 2.6 yields now that $\overline{G}$ is abelian-by-finite, and hence polycyclic. This contradiction proves that $\overline{G}$ is a supersoluble group. Let

$$\{1\} = G_0 < G_1 < \ldots G_t = G$$

be a normal series with cyclic factors of G. Since every G_i^{φ} is an almost normal subgroup of $\overline{G}$, the intersection

$$\overline{R} = \bigcap_{i=0}^{t} N_{\overline{G}}(G_i^{\varphi})$$

is a subgroup of finite index of $\overline{G}$, and so $R = \overline{R}^{\varphi^{-1}}$ has finite index in G. Suppose that G_{i+1}/G_i is infinite for some i. Then $G_{i+1} \cap R/G_i \cap R$ is infinite cyclic, and hence the set of almost normal subgroups J of G such that $G_i \cap R \leqslant J \leqslant G_{i+1} \cap R$ is also infinite, and this shows that the Hirsch length of G is less or equal to that of $\overline{G}$. Finally, applying the same argument to the isomorphism φ^{-1}, we obtain that G and $\overline{G}$ have the same Hirsch length. ■

The statement obtained replacing in our theorem the maximal condition by the minimal condition on subgroups is in general false. To see this, consider the following example due to Čarin (see [7], Part 1, p. 152). Let p a prime, and let K be the algebraic closure of the field with p elements. If q is a prime other than p, the multiplicative group K^* of K contains a subgroup X of the type q^{∞}. Let F be the subfield of K generated by X, and let A be the additive group of F. Then every element of X induces by multiplication an automorphism of A, and we may consider the semidirect product $\overline{G} = X \ltimes A$. It is easy to show that the infinite group A is the unique minimal normal subgroup of $\overline{G}$, so that $\overline{G}$ has no proper subgroups of finite index. In particular, every almost normal subgroup of $\overline{G}$ is normal, and hence the lattice an$(\overline{G})$ is isomorphic to the lattice of subgroups of any Prüfer group G. On the other hand, the following can be proved.

PROPOSITION 2.9. – *Let G be a Černikov group whose finite residual has no locally cyclic non-trivial primary components. If $\overline{G}$ is an FC-soluble group and $\varphi: \mathrm{an}(G) \to \mathrm{an}(\overline{G})$ is a lattice isomorphism, then also $\overline{G}$ is a Černikov group.*

PROOF. – Let J be the finite residual of G. Application of Lemma 2.2 yields that the subgroup $\overline{J} = J^{\varphi}$ has finite index in $\overline{G}$ and that $\overline{J}$ has no proper subgroups of finite index. In particular, every almost normal subgroup of $\overline{J}$ is normal, and hence φ induces a lattice isomorphism from the subgroup lattice $\mathfrak{L}(J)$ onto the lattice $\mathfrak{N}(\overline{J})$ of all normal subgroups of $\overline{J}$. Therefore $\overline{J}$ is isomorphic to J (see [2]), and $\overline{G}$ is a Černikov group. ■

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