Bollettino
Unione Matematica Italiana

L. Beretta, A. Tognoli

Some remarks about proper real algebraic maps

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **3-B** (2000), n.1, p. 117–133.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2000_8_3B_1_117_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2000.

Bollettino U. M. I.
(8) **1**-B (2000), 117-134

Some Remarks about Proper Real Algebraic Maps.

L. BERETTA - A. TOGNOLI

Sunto. – *Nel presente lavoro si studiano le applicazioni polinomiali proprie*

$$\varphi: \mathbb{R}^n \to \mathbb{R}^q.$$

In particolare si prova:

1) *se* $\varphi: \mathbb{R}^n \to \mathbb{R}$ *è un'applicazione polinomiale tale che* $\varphi^{-1}(y)$ *è compatto per ogni* $y \in \mathbb{R}$, *allora* φ *è propria;*

2) *se* $\varphi: \mathbb{R}^n \to \mathbb{R}^q$ *è polinomiale a fibra compatta e* $\varphi(\mathbb{R}^n)$ *è chiuso in* $\mathbb{R}^q$ *allora* φ *è propria;*

3) *l'insieme delle applicazioni polinomiali proprie di* $\mathbb{R}^n$ *in* $\mathbb{R}^q$ *è denso, nella topologia* C^∞, *nello spazio delle applicazioni* C^∞ *di* $\mathbb{R}^n$ *in* $\mathbb{R}^q$.

Introduction.

Let $\varphi: X \to Y$ be a continuous map between topological spaces, φ is called proper if for every compact $H \subset Y$ the set $\varphi^{-1}(H)$ is also compact.

In the first part of this article we study polynomial and analytic proper maps.

In particular we prove:

1) Let $\varphi: \mathbb{R}^n \to \mathbb{R}$ be a polynomial map such that $\varphi^{-1}(x)$ is compact for any $x \in \mathbb{R}$, then φ is proper.

2) Let $\varphi: \mathbb{R}^n \to \mathbb{R}^q$ be an analytic map such that: $\varphi(\mathbb{R}^n)$ is closed and $\varphi^{-1}(x)$ is compact, for any $x \in \mathbb{R}^q$, then φ is proper.

In the last part we study some improvements of the classical Weierstrass approximation theorem that asserts the density of the set $P(n, q)$ of polynomial maps $\varphi: \mathbb{R}^n \to \mathbb{R}^q$ in the space $C^\infty(n, q)$ of C^∞ maps $\psi: \mathbb{R}^n \to \mathbb{R}^q$ (endowed with the usual C^∞ topology). In particular we prove

3) The set $P_R(n, q)$ of proper polynomial maps is dense in $C^\infty(n, q)$.

Moreover if $q \geq n$ the set of polynomial maps $\varphi: \mathbb{R}^n \to \mathbb{R}^q$ that have a proper polynomial extension $\tilde{\varphi}: C^n \to C^q$ is dense in $C^\infty(n, q)$.

Finally we prove a relative version of 3), see Theorem 4.

1. – Proper analytic maps $\varphi\colon \mathbb{R}^n \to \mathbb{R}^q$.

Let X be a topological space and $\varphi\colon X \to \mathbb{R}$ a continuous map, we shall denote:

$$X_\alpha = \varphi^{-1}(\alpha) \qquad \alpha \in \mathbb{R}$$

$$X_\alpha^+ = \varphi^{-1}(]\alpha + \infty[) \qquad X_\alpha^- = \varphi^{-1}(]-\infty\,,\,\alpha,[)\,.$$

If $\varphi\colon X \to \mathbb{R}$ is a continuous map between topological spaces we shall say that φ has *compact fibers* if for any $y \in Y$, $\varphi^{-1}(y)$ is compact; φ shall be called *proper* if $\varphi^{-1}(K)$ is compact for any compact set K of Y.

The map $\varphi\colon X \to Y$ shall be called *weakly proper* if for any $y \in \varphi(X)$ the neighbourhoods of $\varphi^{-1}(y)$ of type $U = \varphi^{-1}(B_y)$, B_y neighbourhood of y, are a foundamental system.

All spaces considered in the following are *metric* and *locally compact.*

LEMMA 1.1. – *Let $\varphi\colon X \to Y$ be a continuous map between metric locally compact spaces, and let us suppose φ has compact fibers, then the following conditions are equivalent*:

(i) φ *is proper*;

(ii) $\varphi(X)$ *is closed and* φ *is weakly proper*;

(iii) $\varphi(X)$ *is closed and for any $y \in Y$, $\varphi^{-1}(y)$ has a compact neighbourhood U in X, of type $U = \varphi^{-1}(B_y)$, B_y neighbourhood of y in Y.*

PROOF. – (i) $\Rightarrow$ (ii). It is known that a proper map is also closed; (X and Y are metric spaces), so $\varphi(X)$ is closed.

If B_y is a compact neighbourhood of y in Y, then $U = \varphi^{-1}(B_y)$ is a compact neighbourhood of $\varphi^{-1}(y)$ in X.

Now we remark that: if $\{U_\lambda\}_{\lambda \in \Delta}$ is a set of compact neighbourhoods of $\varphi^{-1}(y)$ such that $\bigcap_\lambda U_\lambda = \varphi^{-1}(y)$ and the family $\{U_\lambda\}$ is closed under the finite intersection, then $\{U_\lambda\}_{\lambda \in \Delta}$ is a foundamental system of neighbourhoods of $\varphi^{-1}(y)$.

In fact if V is an open neighbourhood of $\varphi^{-1}(y)$, then the sets $U'_\lambda = U_\lambda - V$ are compact and $\bigcap_\lambda U'_\lambda = \emptyset$ hence there exists $\lambda_1 \dots \lambda_q$ such that $\bigcap_{i=1}^{q} U'_{\lambda_i} = \emptyset$, and this proves that $\{U_\lambda\}$ is a foundamental system of neighbourhoods. Clearly the family $\{U_\lambda\}$ of compact neighbourhoods of type $U_\lambda = \varphi^{-1}(B_y^\lambda)$, B_y^λ compact, satisfies the above condition and this proves that φ is weakly proper.

(ii) $\Rightarrow$ (iii). Let $y \notin \varphi(X)$, then there exists a neighbourhood $U_y \ni y$, such that

$$\varphi^{-1}(U_y) = \emptyset .$$

Let now $y \in \varphi(X)$ and U_y a compact neighbourhood of $\varphi^{-1}(y)$, such a neighbourhood exists because $\varphi^{-1}(y)$ is compact and X is locally compact.

φ is weakly proper, hence there exists a compact neighbourhood B_y of y in Y such that $\varphi^{-1}(B_y) \subset U_y$, $\varphi^{-1}(B_y)$ is closed and hence compact.

So the claim is proved.

(iii) $\Rightarrow$ (i). For any $y \in Y$ there exists a compact neighbourhood B_y such that $\varphi^{-1}(B_y)$ is compact.

If $K \subset Y$ is a compact set, then K can be covered with $B_{y_1} \ldots B_{y_q}$ such that $\varphi^{-1}(B_{y_i})$ is compact, clearly this implies that $\varphi^{-1}(K)$ is compact.

The lemma is proved.

LEMMA 1.2. – *Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}^q$ be an analytic map with compact fibers, such that $\varphi(\mathbb{R}^n)$ is closed, then φ is proper.*

PROOF. – If we prove that φ is weakly proper the Lemma 1 proves the claim.

The fact that φ is weakly proper is a consequence of Lojasiewicz inequalities. Let

$$y^0 = (y_1^0 \ldots y_q^0) \in \varphi(\mathbb{R}^n) \qquad \text{and} \qquad d(y_1 \ldots y_n) = \sum_{i=1}^{q} (y_i - y_i^0)^2$$

then

$$\Psi = d \circ \varphi\colon \mathbb{R}^n \to \mathbb{R}$$

is an analytic function such that

$$\{x \in \mathbb{R}^n \mid \Psi(x) = 0\} = \varphi^{-1}(y^0) .$$

Lojasiewicz's inequalities assure that the sets

$$U_\varepsilon = \{x \in \mathbb{R}^n \mid |\Psi(x)| \leq \varepsilon\} = \varphi^{-1}(\{\{y_i\} \in \mathbb{R}^q \mid \sum (y_i - y_i^0)^2 < \varepsilon\}) \qquad \varepsilon > 0$$

are a foundamental system of neighbourhoods of $\varphi^{-1}(y^0)$.

COROLLARY 1. – *Let* $\varphi\colon \mathbb{R}^n \to \mathbb{R}$ *be a polynomial map, if there exists* $\alpha \in \mathbb{R}$ *such that* $\mathbb{R}^n_\alpha = \varphi^{-1}(\alpha)$ *is compact and non-empty, then, if* $n > 1$:

$$\varphi(\mathbb{R}^n) = \begin{cases} [\beta, +\infty[\\ ]-\infty, \beta] \end{cases} \quad \textit{for some } \beta \in \mathbb{R}\,.$$

If φ *has compact fibers, then* φ *is a proper map.*

PROOF. – Let us suppose $\mathbb{R}^n_\alpha$ compact and non-empty and let consider the open sets

$$\mathbb{R}^{n\,+}_\alpha = \varphi^{-1}(]\alpha, +\infty[) \qquad \mathbb{R}^{n\,-}_\alpha = \varphi^{-1}(]-\infty, \alpha[)\,.$$

Clearly we have that $\mathbb{R}^n_\alpha$ contains the boundary of $\mathbb{R}^{n\,+}_\alpha$ and $\mathbb{R}^{n\,-}_\alpha$, a path from $x \in \mathbb{R}^{n\,+}_\alpha$ to $y \in \mathbb{R}^{n\,-}_\alpha$ contains a point of $\mathbb{R}^n_\alpha$.

These facts implies that one and only one of the sets $\overline{\mathbb{R}^{n\,+}_\alpha}$, $\overline{\mathbb{R}^{n\,-}_\alpha}$ is bounded and hence compact.

So we have proved that

$$\varphi(\mathbb{R}^n) = \begin{cases} [\beta, +\infty[\\ ]-\infty, \beta] \end{cases}$$

because any non constant polynomial map $\varphi\colon \mathbb{R} \to \mathbb{R}$ is not bounded.

Let now suppose φ has compact fibers; we have proved that $\varphi(\mathbb{R}^n)$ is closed, so, from Lemma 2, we know that φ is weakly proper, hence from Lemma 1 we deduce that φ is proper.

2. – Some remarks about the dimension of the fibers of a polynomial map.

We shall denote by K the real or the complex field.

If n, m, p are positive integers we shall denote by $P_K(n, m, p)$ the vector space of polynomial maps

$$\varphi = (\varphi_1 \dots \varphi_m)\colon K^n \to K^m$$

such that

$$\deg \varphi_i \leq p \qquad i = 1, \dots, m\,.$$

If $V \subset K^s$ is an affine variety we shall denote by $P_K(V, m, p)$ the quotient space of $P_K(s, m, p)$ given by the maps $V \to K^m$ that are restriction of elements of $P_K(s, m, p)$.

We shall also consider

$$P_K(n, m) = \bigcup_p P_K(n, m, p) \qquad P_K(V, m) = \bigcup_p P_K(V, m, p).$$

The sets $P_K(n, m)$, $P_K(V, m)$ are endowed with the usual C^∞ topology of uniform convergence on compact sets.

The sets $P_K(n, m, p)$ are, in a natural way, Euclidean spaces and they have the topology induced by these (i.e. we parametrize the polynomials using the coefficients).

On $P_K(n, m, p)$ we can also consider the induced Zariski topology and on $P_K(V, m, p)$ the quotient Zariski topology.

From the fact that $P_K(n, m, p)$, $P_K(V, m, p)$ are Hausdorff topological vector spaces of finite dimension we have

THEOREM 1. – *The natural injections* $P_K(V, m, p) \to P_K(V, m)$ *are homeomorphisms onto the images for any affine variety* $V \subset K^s$.

If $\varphi: V \to W$ is a map between affine varieties, we shall say that φ is algebraic if it is a rational regular map.

If $\varphi: V \to W$ is an algebraic map between affine varieties, we shall say that φ is dominating if $W =$ Zariski closure of $\varphi(V)$.

Now we recall two well-known results.

THEOREM 2 (Algebraic Sard Theorem). – *Let* $\varphi: V \to W$ *be an algebraic dominating map between complex, irreducible affine varieties.*

Then the set

$$W^r = \{y \in W \mid \varphi^{-1}(y) \cap V_{\text{reg}} \ \textit{is regular}\}$$

contains a Zariski open dense subset of W. Moreover the set

$$W^1 = \{y \in W \mid \dim \varphi^{-1}(y) = \dim V - \dim W\}$$

is a dense open Zariski subset of W.

See [2] page 42-46.

THEOREM 3 (Local Triviality of Semi-Algebraic Maps). – *Let X, Y be semi-algebraic sets and* $\varphi: X \to Y$ *a continuous semi-algebraic map.*

Then there exists:

(i) *a finite semi-algebraic stratification* $\{Y_1 \ldots Y_K\}$ *of* Y;

(ii) *a collection of semi-algebraic sets* $\{F_1 \ldots F_K\}$;

(iii) *a collection of semi-algebraic homeomorphisms*

$$g_i: F_i^I \to Y_i \times F_i \qquad F_i^I = \varphi^{-1}(Y_i) \qquad i = 1 \dots K$$

such that for every $i = 1 \dots K$ *the following diagram is commutative*

$$\begin{array}{ccc} \varphi^{-1}(Y_i) = F_i^I & \xrightarrow{g_i} & Y_i \times F_i \\ & \searrow \varphi \qquad p_i \swarrow & \\ & Y & \end{array}$$

where p_i *is the natural projection.*

Moreover we may suppose that φ *is a trivial (i.e. equivalent to a projection) semi-algebraic map over each connected component of* $Y - Y^1$, *where* Y^1 *is the union of the strata of dimension lower than* $p = \dim Y$.

See [1] page 98.

If X is a semi-algebraic set we shall say that X is pure p-dimensional if it has dimension p in any point. From the above results we deduce:

LEMMA 3. – *Let* $\varphi: V \to W$ *be an algebraic dominating map between real affine irreducible varieties, then there exists an open dense subset, (dense in the usual topology),* W' *of* W_{reg} *such that* $\varphi^{-1}(y) \cap V_{\text{reg}}$ *is regular for any* $y \in W'$ *and dim* $\varphi^{-1}(y) \leq \dim V - \dim W$.

Moreover there exists a nonempty open semi-algebraic subset $W'' \subset W'$ *such that*

$$\dim \varphi^{-1}(y) = \dim V - \dim W \qquad \text{if } y \in W''.$$

PROOF. – Let $\tilde{\varphi}: \tilde{V} \to \tilde{W}$ be a complexification of φ and

$$\widetilde{W'} = \{y \in \tilde{W} \mid \tilde{\varphi}^{-1}(y) \cap \tilde{V}_{\text{reg}} \text{ is regular}\}.$$

By Theorem 2 $\widetilde{W'}$ contains a dense, in the usual topology, Zariski open subset of $\tilde{W}$.

We recall that in $\tilde{W}$ the usual closure coincides with the Zariski closure, this implies that $W' = \widetilde{W'} \cap W$ contains an open and dense subset of W_{reg}.

It is known that if $\tilde{\varphi}^{-1}(y) \cap \tilde{V}_{\text{reg}}$ is regular, then $\varphi^{-1}(y) \cap V_{\text{reg}}$ is regular and clearly

$$\dim_C \tilde{\varphi}^{-1}(y) \geq \dim_R \varphi^{-1}(y)$$

so we have proved the first part of the claim.

To prove the last part it is enought to remember that φ is dominating, hence $\varphi(V)$ must contain an open set of W.

Now Theorem 3 applied to the map $\varphi^{-1}(W') \to W'$ proves the lemma.

REMARK 1. – Let $V \subset K^n$ be a cone with vertex in the origin and $P_K^h(V, m, p)$ the set of $\varphi = (\varphi_1 \dots \varphi_m) \in P_K(V, m, p)$ such that any φ_i is homogeneous of degree p. It is possible to extend the previous result to the homogeneous case.

For example we have that there exists an open dense subset B of $P_C^h(V, m, p)$, such that for any $\varphi = (\varphi_1 \dots \varphi_m) \in B$ we have

$$\dim V \cap (\cap \{\varphi_i = 0\}) = \dim V - m .$$

Infact we can take φ_1 such that $\{\varphi_1 = 0\} \not\supset V$, φ_2 in such a way that

$$\{\varphi_2 = 0\} \not\supset V \cap \{\varphi_1 = 0\} \text{ (1) } \dots$$

LEMMA 4. – *Let $\varphi\colon X \to Y$ be a semi-algebraic map between semi-algebraic sets of pure dimension p, q and let us suppose φ is dominating, then there exists an open dense subset, $Y' \subset Y$ such that*

$$y \in Y' \;\Rightarrow\; \varphi^{-1}(y) \quad \textit{is pure } p - q\textit{-dimensional}$$

Moreover if Y is pure q-dimensional, φ is an open map and for $y \in Y'$, $\varphi^{-1}(y)$ is pure $p - q$-dimensional, with Y' dense subset, then X is pure p-dimensional.

PROOF. – The claim of the lemma follow easily from Theorem 3, in fact there exists an open dense subset $Y' \subset Y$ such that on any connected component Y_i' of Y' we have

$$\varphi^{-1}(Y_i') \cong Y_i' \times F_i .$$

So if X is pure p-dimensional, F_i must be pure $p - q$-dimensional.

On the converse if φ is open and F_i is pure $p - q$-dimensional then the set $\bigcup_i \varphi^{-1}(Y_i')$ must be dense in X, and this implies that X is pure p-dimensional.

A criterion to recognize when a map is dominating is given in the following.

LEMMA 5. – *Let $\varphi\colon V \to W$ be an algebraic map between real irreducible affine varieties and $\widetilde{\varphi}\colon \widetilde{V} \to \widetilde{W}$ a complexification. If there exists $y_0 \in W$ such that*

$$\dim_C \widetilde{\varphi}^{-1}(y_0) = \dim_R V - \dim_R W$$

then φ is dominating.

(1) Where $\not\supset$ means: doesn't contain any irreducible component.

PROOF. – The map

$$\tilde{\varphi}\colon \tilde{V} \to \widehat{\tilde{\varphi}(\tilde{V})}$$

is dominating, where $\widehat{\ }$ is the closure in the Zariski topology. If we prove that

$$\dim_C \widehat{\tilde{\varphi}(\tilde{V})} = \dim_C \tilde{W} = \dim_R W$$

then we have $\tilde{W} = \widehat{\tilde{\varphi}(\tilde{V})}$ (because W is irreducible) and clearly φ is dominating.

By a classical result (see [2] page 45) we know that

$$\dim_C \tilde{\varphi}^{-1}(y) \geqslant \dim_C \tilde{V} - \dim_C \widehat{\tilde{\varphi}(\tilde{V})}$$

for any $y \in \tilde{\varphi}(\tilde{V})$.

So the hypothesis

$$\dim_C \tilde{\varphi}^{-1}(y_0) = \dim_C \tilde{V} - \dim_C \tilde{W}$$

implies

$$\dim \widehat{\tilde{\varphi}(\tilde{V})} = \dim \tilde{W}$$

and hence the claim.

Let $V \subset K^s$ be an affine variety, we shall denote by $P_K^h(V, n, p)$ the subset of $P_K(V, n, p)$ of the elements that are restriction to V of homogeneous polynomials of degree p.

Now we consider the affine varieties

$$\Gamma_K(V, n, p) = \{(\alpha, x) \in P_K(V, n, p) \times V \,|\, \alpha(x) = 0\}$$

$$\Gamma_K^h(V, n, p) = \{(\alpha, x) \in P_K^h(V, n, p) \times V \,|\, \alpha(x) = 0\}$$

and the natural projections

$$p\colon \Gamma_K(V, n, p) \to P_K(V, n, p).$$

PROPOSITION 1. – *Let $V \subset C^s$ be an affine irreducible variety with $\dim V \geqslant n$. Then there exists a dense open Zariski subset U of $P_C(V, n, p)$ such that for any $\alpha \in U$ we have*

$$\dim p^{-1}(\alpha) = \dim V - n$$

and

$$\dim \Gamma_C(V, n, p) = \dim P_C(V, n, p) + \dim V - n .$$

A similar result holds for the projection

$$p_h\colon \Gamma^h_C(V, n, p) \to P^h_C(V, n, p).$$

Proof. – The hypothesis $\dim V \geq n$, $K = C$ and the classical elimination theory assure that the maps p and p_h are dominating. It is known that

1) $q = \dim \Gamma_C(V, n, p) \geq \dim P_C(V, n, p) + \dim V - n$.

2) There exists an open dense Zariski subset U' of $P_C(V, n, p)$ such that if $\alpha \in U'$ then

$$\dim p^{-1}(\alpha) = q - \dim P_C(V, n, p)$$

(see [M] page 46).

Now we shall show that for a dense subset $U'' \subset P_C(V, n, p)$ we have

3) $\alpha \in U'' \Rightarrow \dim p^{-1}(\alpha) = \dim V - n$

If we prove 3) then from 1) and 2) it follows that

$$q = \dim P_C(V, n, p) + \dim V - n$$

and hence from 2) and Theorem 2, that there exists a Zariski open dense subset U of $P_C(V, n, p)$ such that

$$\alpha \in U \Rightarrow \dim p^{-1}(\alpha) = \dim V - n\,.$$

To prove 3) let us consider $\alpha = (\alpha_1 \dots \alpha_n) \in P_C(V, n, p)$.

If α_1 is constant we can approximate α_1 with a non-constant element $\beta_1 \in P_C(V, 1, p)$.

Let now $x'_1 \dots x'_{n_1}$ be a finite set of points such that any irreducible component of $V \cap \{\beta_1 = 0\}$ contains one x'_j. If α_2 is identically zero on some irreducible component of $V \cap \{\beta_1 = 0\}$ we approximate α_2 with β_2 such that

$$\beta_2(x'_j) \neq 0 \qquad \forall j\,.$$

So finally we approximate $(\alpha_1 \dots \alpha_n)$ with $(\beta_1 \dots \beta_n)$ and by construction we have

$$\dim V \cap \left(\bigcap_j \{\beta_j = 0\}\right) = \dim V - n\,.$$

The proposition is proved for the map p, the demonstration runs in a similar way for p_h.

Corollary 2. – *Under the hypothesis of the proposition, there exists a Zariski open dense subset U of $P_C(V, n, p)$ such that*

(I) *If* $P = (P_1 \dots P_n) \in U$ *then* P_j *is not constant on*

$$\bigcap_{K \neq j} \{P_K = 0\} \qquad j = 1 \dots n$$

PROOF. – If condition (I) is not satisfied then

$$\bigcap_{K=1}^{n} \{P_K = 0\} = \emptyset$$

or

$$\dim \bigcap_{K=1}^{n} \{P_K = 0\} > \dim V - n \, .$$

So the proposition proves the corollary.

If $P_1, \dots, P_q$ are elements of $\mathbb{R}[x_1 \dots x_n]$, then the set $V = \bigcap_{j=1}^{q} \{P_j = 0\}$ may have codimension greater than q and in general V is not purely dimensional.

Let now consider $C^n = \{z_1 = x_1 + iy_1 \dots z_n = x_n + iy_n\}$ and for any $P \in C[z_1 \dots z_n]$ the decomposition $P = P' + iP''$ into the real and the imaginary part.

DEFINITION 1. – *An element*

$$P \in \mathbb{R}[x_1 \dots x_n, y_1 \dots y_n]$$

shall be called real (imaginary)part *if there exist* $Q \in \mathbb{R}[x_1 \dots x_n, y_1 \dots y_n]$ *such that* $P + iQ(Q + iP)$ *is element of* $C[z_1 \dots z_n]$.

We shall denote $(\mathbb{R}_{\mathbb{R}}[x, y])_n^q$ the vector space of q-uple of elements of

$$\mathbb{R}[x_1 \dots x_n, y_1 \dots y_n]$$

that are real parts. We have

PROPOSITION 2. – *There exists an open dense subset* U *of* $(\mathbb{R}_{\mathbb{R}}[x, y])_n^q$, $q \leqslant n$ *such that*

$$(P'_1 \dots P'_q) \in U \;\Rightarrow\; V = \bigcap_{j=1}^{q} \{P'_j = 0\}$$

is pure $2n - q$*-dimensional.*

PROOF. – We recall some well-known facts:

1) if $P' \in \mathbb{R}_{\mathbb{R}}[x, y]$ the element P'' such that $P' + iP'' \in C[z]$ is determined, up to an additive constant, (one can reduce the proof to the case $n = 1$ and use the Cauchy-Riemann conditions).

2) Let $P_j = P_j' + iP_j'' \in C[z]$, $j = 1 \ldots q$; if $\widetilde{V} = \bigcap_{j=1}^{q} \{P_j = 0\}$ is regular of complex dimension $n - q$, then $V = \bigcap_{j=1}^{q} \{P_j' = 0\}$ is regular of dimension $2n - q$ (the problem is local and we can use the implicit function theorem).

3) If $\widetilde{W}$ is a complex affine variety, then the set of regular points is open and dense, in the usual topology, in $\widetilde{W}$ (see [M]).

4) By Proposition 1 there exists an open dense subset $\widetilde{U}$ of $P_C(n, q, p)$ such that

$$(P_1 \ldots P_q) \in \widetilde{U} \Rightarrow \dim_C \bigcap_{j=1}^{q} \{P_j = 0\} = n - q .$$

Moreover we may suppose, (see Corollary 2), $(P_1 \ldots P_q) \in \widetilde{U} \Rightarrow$ for any $j = 1 \ldots q$, P_j is not constant on $\bigcap_{K \neq j} \{P_K = 0\}$.

Let us consider the natural map

$$p \colon P_C(n, q) \to (\mathbb{R}_{\mathbb{R}}[x, y])_n^q$$

that associate to an element $(P_1 \ldots P_q) \in P_C(n, q)$ the real parts

$$(P_1' \ldots P_q') \in (\mathbb{R}_{\mathbb{R}}[x, y])_n^q .$$

Clearly p is surjective and open, hence from Corollary 2, we deduce that there exists an open dense subset U of $(\mathbb{R}_{\mathbb{R}}[x, y])_n^q$ such that if $(P_1' \ldots P_q') \in U$ then for any choice of $(P_1'' \ldots P_q'')$ such that $(P_1 + iP_1'' \ldots P_q + iP_q'') \in P_C(n, q)$ we have

$$\dim \widetilde{V} = \dim_C \bigcap_{j=1}^{q} \{P_j = 0\} = n - q \qquad P_j = P_j' + P_j'' .$$

Let now $x^0 \in V = \bigcap_{j=1}^{q} \{P_j' = 0\}$, $(P_1' \ldots P_q') \in U$, then we can choose the additive constants in such a way that

$$x^0 \in \bigcap_{j=1}^{q} \{P_j'' = 0\} \qquad \text{with} \qquad P_j' + iP_j'' \in C[z]$$

Now the point x^0 is limit of regular points of $\widetilde{V}$ and hence from the above points 2) and 3) we know that V is of dimension $2n - q$ in the point x^0. The proposition is proved.

REMARK 2. – A similar result, with the same proof, holds for the imaginary part of elements of $C[z]$.

3. – Some improvements to the Weierstrass approximation theorem.

The classical Weierstrass approximation theorem states that any C^∞ map

$$\varphi\colon \mathbb{R}^n \to \mathbb{R}^q$$

can be approximated, in the C^∞ topology, by polynomial maps.

The following are natural questions:

(I) any C^∞ map $\varphi\colon \mathbb{R}^n \to \mathbb{R}^q$ can be approximated by proper polynomial maps $\psi_\lambda\colon \mathbb{R}^n \to \mathbb{R}^q$?

(II) when a C^∞ map $\varphi\colon \mathbb{R}^n \to \mathbb{R}^q$ can be approximated by polynomial maps ψ_λ such that $\psi_\lambda\colon C^n \to C^q$ is proper?

(III) Let $V \subset \mathbb{R}^n$ be a compact affine variety and $\varphi\colon \mathbb{R}^n \to \mathbb{R}^q$ a C^∞ map such that φ/V is polynomial, when can we approximate φ with polynomial proper maps $\psi_\lambda\colon \mathbb{R}^n \to \mathbb{R}^q$ such that

$$\psi_\lambda/V = \varphi/V?$$

The purpose of this paragraph is to give some positive answer to the above questions.

In the following we shall consider K^n canonically embedded in $P_n(K)$

$$i\colon K^n \hookrightarrow P_n(K)$$

and we shall denote

$$P_n^\infty(K) = P_n(K) - i(K^n).$$

Let $\varphi\colon V_1 \to V_2$ be an algebraic map defined between two affine K-varieties, we shall call $\widehat{\varphi}\colon \widehat{V}_1 \to \widehat{V}_2$ a projective compactification of φ if there exist:

1) two algebraic embedding

$$i_j\colon V_j \to K^{n_j} \qquad j = 1, 2$$

such that $\widehat{V}_j =$ Zariski closure of $i_j(V_j)$ in $P_{n_j}(K)$

2) an algebraic map

$$\widehat{\varphi}\colon \widehat{V}_1 \to \widehat{V}_2$$

such that the following diagram is commutative

$$\begin{array}{ccc} V_1 & \xrightarrow{\varphi} & V_2 \\ \downarrow & & \downarrow \\ \widehat{V}_1 & \xrightarrow{\widehat{\varphi}} & \widehat{V}_2 . \end{array}$$

As before we shall denote $\widehat{V}_j^{\infty} = \widehat{V}_j - i_j(V_j)$.

We shall say that $\widehat{\varphi}$ is a *good (projective) compactification* if

$$\widehat{\varphi}(\widehat{V}_1^{\infty}) \subset \widehat{V}_2^{\infty} .$$

We have the

LEMMA 6. – *Let $\varphi : V \to W$ be an algebraic map between affine K-varieties, then φ has a projective compactification, $\widehat{\varphi}: \widehat{V} \to \widehat{W}$.*
If $\widehat{\varphi}$ is a good compactification, then φ is a proper map.
If $K = C$ and φ is proper, then $\widehat{\varphi}$ is a good compactification.

PROOF. – Let

$$V \subset K^n \qquad W \subset K^m$$

and

$$\Gamma_{\varphi} \subset K^n \times K^m \subset P_n(K) \times P_m(K)$$

the graph of φ.

Let us define $i_V: V \to \Gamma_{\varphi}$, $i_V(x) = (x, \varphi(x))$, $\widehat{V}$ = Zariski closure of Γ_{φ} in $P_N(K) \supset P_n(K) \times P_m(K)$ and $\widehat{\varphi}$ the natural projection

$$P_n(K) \times P_m(K) \to P_m(K)$$

restricted to $\widehat{\Gamma}_{\varphi}$.

$P_n(K)$ is compact, $K = \mathbb{R}, C$, hence if $\widehat{\varphi}$ is a good compactification then φ is a proper map.

On the converse if $K = C$, then the Zariski closure coincides with the usual one, and hence, if φ is proper, then $\widehat{\varphi}$ is a good compactification.

If $V \subset K^n$ and $\varphi: V \to W$ is an algebraic map, then, in general, the projective compactification $\widehat{\varphi}: \widehat{V} \to \widehat{W}$ cannot be found in such a way that $\widehat{V} \subset P_n(K)$. We shall try to find a sufficient condition to realize $\widehat{V}$ in $P_n(K)$.

Let us consider a polynomial map

$$\varphi = (\varphi_1, \ldots, \varphi_m): K^n \to K^m$$

given by

(1) $$y_j = \varphi_j(x_1 \ldots x_n) \qquad j = 1 \ldots m \quad \deg \varphi_j \leq p .$$

If we consider the homogeneous coordinates

$$u_0 \dots u_m \qquad y_j = \frac{u_j}{u_0} \qquad v_0 \dots v_n \qquad x_j = \frac{v_j}{v_0}$$

then the relations (1) give:

$$\frac{u_j}{u_0} = \varphi_j\left(\frac{v_1}{v_0} \dots \frac{v_n}{v_0}\right) \qquad j = 1 \dots m \tag{2}$$

and hence

$$\frac{u_j}{u_0} = \frac{v_0^{d_j} \check{\varphi}_j(v_0 \dots v_n)}{v_0^p} \qquad j = 1 \dots m \tag{3}$$

where

$$p = \sup \deg \varphi_j \qquad d_j = p - \deg \varphi_j$$

$$\check{\varphi}_j(v_0 \dots v_n) = v_0^{d_j'} \varphi_j\left(\frac{v_1}{v_0} \dots \frac{v_n}{v_0}\right) \qquad d_j' = \deg \varphi_j .$$

If we now suppose $d_j' = p$, for all j, then relation (3) are satisfied if

$$\begin{cases} u_j = \check{\varphi}_j(v_0 \dots v_n) & j = 1 \dots m \\ u_0 = v_0^p \end{cases} \tag{4}$$

If $v_0 \neq 0$ relations (4) give the algebraic map $\check{\varphi}$, if they define a regular map

$$\widehat{\varphi}\colon P_n(K) \to P_m(K)$$

then $\widehat{\varphi}$ is the unique algebraic extension of φ, and $\widehat{\varphi}$ is a good projective compactification of φ.

Clearly relations (4) define a regular map in a neighbourhood of $(\overline{v}_0, \dots, \overline{v}_n)$ if one, at least, of the numbers $\check{\varphi}_j(\overline{v}_0 \dots \overline{v}_n)$, $\overline{v}_0$ is different from zero.

Let now $V \subset K^n$ be an affine variety and $\varphi\colon V \to K^m$ a polynomial map.

DEFINITION 2. – *If relations* (4) *define a projective compactification*

$$\widehat{\varphi}\colon \widehat{V} \to P_m(K)$$

then $\widehat{\varphi}$ *shall be called a* very good (*projective*) compactification.

PROPOSITION 3. – *Let $V \subset C^m$ be a complex affine variety of dimension n, if $q \geq n$, then the set of elements of $P_C(V, q)$ that have a very good projective compactification is dense in $P_C(V, q)$.*

In the real case the same result holds for any $q \geq 1$.

PROOF. – Let $\varphi = (\varphi_1 \dots \varphi_q) \in P_C(V, q)$ and

$$\varphi_\varepsilon = (\varphi_1 + \varepsilon\psi_1, \dots, \varphi_q + \varepsilon\psi_q): V \to C^q \qquad \varepsilon \in \mathbb{R}.$$

By Proposition 1 we may choose $(\psi_1 \dots \psi_q)$ in a open dense subset U of some $P_C^h(V, q, p)$ in such a way that φ_ε has a very good projective compactification for any $\varepsilon \in \mathbb{R}'$, $\mathbb{R}'$ open dense subset of $\mathbb{R}$.

In fact if $(\psi_1 \dots \psi_q)$ have the property

$$\dim \bigcap_{j=0}^{q} \{\psi_j = 0\} = 0 \qquad \text{and} \qquad \deg \varphi_i < p$$

relation (4) define the desidered compactification.

It is now clear that $\varphi = \lim_{\varepsilon \to 0} \varphi_\varepsilon$, hence the proposition is proved if $K = C$.

In the real case let us define $\psi_1 = \dots = \psi_q = \sum_{j=0}^{m} x_j^{2d}$ with d big enough.

REMARK 3. – In the complex case the hypothesis $q \geq \dim V$ is necessary because if φ has a good projective compactification then it is a proper map, and hence with finite fibers.

In the following we shall state a relative approximation theorem.

Let $V \subset \mathbb{R}^n$ be an affine variety, $W \subset V$ a closed affine subvariety and $\varphi: W \to \mathbb{R}^q$ an algebraic map.

We shall denote $C^\infty(V, \varphi, q)$, $P(V, \varphi, q)$ the spaces of C^∞, and algebraic maps $V \to \mathbb{R}^q$ extending φ.

By $P_C(V, \varphi, q)$ we shall denote the subspace of the algebraic maps that have a good projective compactification and extend φ.

We recall that an affine variety W is called *quasi regular* if in any point the germ of the algebraic complexification coincides with the germ of the analytic complexification.

We have

THEOREM 4. – *Let V be a real affine variety, W a closed affine quasi-regular compact subvariety and $\varphi: W \to \mathbb{R}^q$ an algebraic map.*

Then $P_C(V, \varphi, q)$ is dense in $C^\infty(V, \varphi, q)$.

PROOF. – We wish to prove the following claim:

under the hypothesis of the theorem, for any compact set $H \subset V$ there exists an algebric map $g\colon V \to \mathbb{R}^N$ such that:

1) there exists an open set $U \supset W \cup H$ in V, such that $g\colon U \to g(U)$ is an algebraic isomorphism;

2) there exists a sphere $S^{N-1} \subset \mathbb{R}^N$ such that $g(W) = g(V) \cap S^{N-1}$.

PROOF (of the claim). – a) if $V \subset \mathbb{R}^n$ there exists a polinomial $P \in \mathbb{R}[x_1 \ldots x_n]$ such that

$$W = \{x \in \mathbb{R}^n \mid P(x) = 0\}.$$

Using Segre's map we can find an embedding $i\colon V \to \mathbb{R}^N$ such that there exists an hyperplane $I \subset \mathbb{R}^N$ with the property $i(W) = i(V) \cap I$ (see [2]).

b) If we consider the inverse of the stereographic map

$$\sigma\colon S^N - \{0\} \to \mathbb{R}^N$$

we verify that $\sigma^{-1}\colon \mathbb{R}^N \to S^N$ can be extended to an algebraic map

$$\widehat{\sigma}^{-1}\colon P_N(R) \to S^N$$

(see [T]).

The map $\widehat{\sigma}^{-1}$ is an isomorphism on $\mathbb{R}^N$ and the constant map on $P_N(\mathbb{R})^\infty$.

c) If we consider the composition

$$g = \widehat{\sigma}^{-1} \circ i\colon V \to S^N$$

it easy to verify that $g\colon V \to g(V)$ is an algebraic isomorphism and

$$g(W) = g(V) \cap \widehat{\sigma}^{-1}(I).$$

It is well-known that the inverse of stereographic projection sends linear subspaces into spheres, hence $\widehat{\sigma}^{-1}(I)$ is an $N-1$ sphere and the claim is proved.

PROOF (of the theorem). – We can suppose $V \subset \mathbb{R}^N$ in such a way that

$$W = \left\{x \in V \mid \sum_{j=1}^{N} x_j^2 - 1 = 0\right\}.$$

It is known (see [T]) that $P(V, \varphi, q)$ is dense in $C^\infty(V, \varphi, q)$ because W is quasi regular.

Let now $\psi \in P(V, \varphi, q)$, then

$$\psi_{\varepsilon} = \psi + \varepsilon \left(\sum_{j=1}^{N} x_j^2 - 1 \right)^{2d}$$

is, for small ε, an approximation of ψ and, if d is big enough,

$$\psi_{\varepsilon} \in P_C(V, \varphi, q).$$

The theorem is proved.

REFERENCES

[B.R.] R. Benedetti - J. J. Risler, *Real Algebraic and Semi-Algebraic Sets*, Herman, Paris, 1990.

[M] D. Mumford, *Algebraic Geometry I, Complex Projective Varieties*, Springer Verlag, Berlin, 1976.

[T] A. Tognoli, *Algebraic Approximation of Manifolds and Spaces*, Sem. Bourbaki N. 548, 1979-80.

L. Beretta and A. Tognoli: University of Trento

Pervenuta in Redazione
il 2 luglio 1999