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Unirational Quartic Hypersurfaces.

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Sunto. – *Dopo aver ricordato i principali risultati concernenti l'unirazionalità dell'ipersuperficie quartica generale X_4 di $\boldsymbol{P}^n$ (definita su un corpo K qualsiasi) si illustra la costruzione geometrica che permette di provare l'esistenza di una superficie razionale in ogni X_4 di $\boldsymbol{P}^n$, con $n \geqslant 4$, e di trovare altri esempi di ipersuperficie quartiche lisce che sono unirazionali oltre a quello dato da B. Segre nel 1960. Si mostra poi come l'analisi delle superficie quartiche monoidali (cioè contenenti un punto triplo come unica singolarità) ad asintotiche separabili sia utile per la determinazione di famiglie di ipersuperficie quartiche lisce unirazionali in $\boldsymbol{P}^4$ e $\boldsymbol{P}^5$. Vengono infine segnalati alcuni possibili sviluppi e problemi ancora aperti in questo tipo di questioni.*

Introduction.

The study of the rationality of the algebraic varieties is one of the most fascinating and in the same time one of the most difficult problem of the birational geometry. From the end of the last century a lot of algebraic geometers studied this kind of questions starting from the hypersurfaces X_d of degree d in $\boldsymbol{P}^n$ and their intersections which are the simplest varieties that one can consider. First of all we give a quick survey of the main known results on the unirationality of the quartic hypersurfaces X_4 in $\boldsymbol{P}^n$ defined over any feld K. Then we explain the geometric construction which allows us to prove the existence of a rational surface in every X_4 in $\boldsymbol{P}^n$, $n \geqslant 4$, and to find other examples of quartic smooth hypersurfaces which are unirational further the one given by B. Segre in 1960. In the third part we will show how the quartic monoidal surfaces (i.e. containing a triple point as unique singularity) with separable asymptotics can be useful to determine families of quartic unirational smooth hypersurfaces in $\boldsymbol{P}^4$ and $\boldsymbol{P}^5$. Finally we give a list of conjectures and open problems.

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1. – Main known results on the unirationality of the quartic hypersurfaces X_4 in $\boldsymbol{P}^n$.

Let X_4 be a smooth quartic hypersurface defined over a field K.

DEFINITION 1.1. – i) $X_4 \subseteq \boldsymbol{P}^n$ *is called unirational if there exists a rational generically surjective* (*i.e. dominant*) *map*

$$\varphi : \boldsymbol{P}^{n-1} \dashrightarrow X_4,$$

ii) $X_4 \subseteq \boldsymbol{P}^n$ *is called rational if there exists a birational map*

$$\varphi : \boldsymbol{P}^{n-1} \overset{\sim}{\dashrightarrow} X_4.$$

If $n = 2$ X_4 is a smooth plane quartic of genus 3 hence it is not rational. By Lüroth Theorem $X_4 \subseteq \boldsymbol{P}^2$ is not unirational.

If $n = 3$ X_4 is a smooth quartic surface in $\boldsymbol{P}^3$. X_4 is not rational and if K is an algebrically closed field and $K(\boldsymbol{P}^2)/K(X_4)$ is a separable extension then X_4, by the rationality criterion of Castelnuovo, is not unirational. The two hypotesis on K are essential infact there exist several counterexamples for istance the one given by Shioda in [29] of $X_4 \subseteq \boldsymbol{P}^3(K)$, where K is an algebraic closed field of characteristic $p \neq 0$ and $p \equiv 3 \pmod 4$, which is a K3 (hence not rational) unirational surface.

For $n \geqslant 4$ the two notions of unirationality and rationality don't coincide, moreover there are no criterions of unirationality or rationality. For these reasons the answer to the problem «*Is* X_4 *in* $\boldsymbol{P}^n$ *unirational?*» is not so obvious.

If $n \geqslant 7$ U. Morin in the 1936 in [14] proved the following

THEOREM 1.1. – *The generic quartic hypersurface defined over any field K in* $\boldsymbol{P}^n$ *with* $n \geqslant 7$ *is unirational.*

Always Morin in 1940 in [15] proved the following

THEOREM 1.2. – *Given a hypersurface* $X_d \subseteq \boldsymbol{P}^n$, *there exists a constant* $c(d)$ *such that the generic* $X_d \subseteq \boldsymbol{P}^n$ *is unirational if* $n - 1 > c(d)$.

J. P. Murre in 1979 in [17] discussed and wrote in modern language the proof of the previous result. In 1980 in [4] C. Ciliberto, for K algebraic closed field of characteristic zero, gave a new proof of the previous theorem finding as particular case the Theorem 1.1.

Always the Theorem 1.2 was extended to the complete intersecions by A. Predonzan in [20] in 1949 and this extension was discussed and generalized in modern language by L. Ramero in [25] and by K. Paranjape and V. Srinivas in [18].

For $n = 6$ U. Morin in 1952 proved in [16] the following

THEOREM 1.3. – *The generic quartic hypersurface in $\boldsymbol{P}^6$ defined over any field K is unirational.*

In 1998 A. Conte and J. P. Murre in [6] gave a new and much simpler proof of the unirationality of the quartic fivefold using in an essential way a theorem of B. Segre of the 1954 not available to Morin.

For $n = 4$ and $n = 5$ the problem of the unirationality of the generic quartic hypersurface is still open, i.e. it is still unknown if the generic X_4 of dimension 3 and 4 is unirational or not. This is considered one of the most interesting and difficult open problem in this kind of questions.

Nevertheless B. Segre in 1960 in [28] gave an example of a particular smooth quartic hypersurface which is unirational. It has equation

$$x_0^4 + x_0 x_4^3 + x_1^4 - 6x_1^2 x_2^2 + x_2^4 + x_3^4 + x_3^3 x_4 = 0 ,$$

where $(x_0 \colon x_1 \colon x_2 \colon x_3 \colon x_4)$ are the homogeneous coordinates in $\boldsymbol{P}^4$.

In 1971 V. A. Iskovskikh and Yu. I. Manin in [10] proved the following

THEOREM 1.4. – *The quartic hypersurface X_4 in $\boldsymbol{P}^4$ is not rational.*

They showed that the group of the birational automorphisms, which is a birational invariant, of the X_4 is finite. Since Bir$(\boldsymbol{P}^3)$ is the Cremona group and it is infinite it follows the non-rationality of the quartic threefold.

By the previous example given by B. Segre V. A. Iskovskikh and Yu. I. Manin gave a negative answer in dimension three to the Lüroth problem formulated by Lüroth in 1861 and which asks «*Is an unirational variety necessarly rational?*».

In 1972 other counterexamples to the Lüroth problem in dimension three were given by H. Clemens and Ph. A. Griffiths in [5] proving the non-rationality of the cubic hypersurface in $\boldsymbol{P}^4$ and by M. Artin and D. Mumford in [2] building unirational varieties with torsion in $H_3(\boldsymbol{Z})$ different from zero hence not rational.

In 1996 J. Harris, B. Mazur and R. Pandharipande in [8] proved the following

THEOREM 1.5. – *Every hypersurface X_d of degree d in $\boldsymbol{P}^n$ is unirational if the codimension of the singular locus Sing X_d is sufficiently big with respect to d and n.*

Recently some progresses were made. First of all in 1997 A. V. Pukhlikov, in [24], extending the tecniques of proof of Iskovskikh and Manin, proved that the generic Fano hypersurface X_M in $\boldsymbol{P}^M$ with $M \geqslant 4$ is not rational.

In 1998 in [9] J. Harris and Yu. Tschinkel studied the rational points over the quartics and in particular proved the following

THEOREM 1.6. – *Let $X_4 \subseteq \boldsymbol{P}^n$ be a quartic smooth hypersurface defined over K. If $n \geqslant 4$, then for any finite extension K' of K the set $X_4(K')$ of the K'-rational points of X_4 is dense in the Zariski topology.*

2. – Examples of smooth quartic unirational hypersurfaces in $\boldsymbol{P}^4$ and $\boldsymbol{P}^5$.

To find examples of smooth quartic hypersurfaces which are unirational we extend the tecniques of Conte-Murre used in [6] to prove the unirationality of the quartic fivefold.

2.1. – *Existence of a rational surface.*

In [6] Conte and Murre proved that the generic $X_4 \subseteq \boldsymbol{P}^n$ with $n \geqslant 5$, contains a rational surface. It is possible to prove that also the generic quartic of dimension 3 in $\boldsymbol{P}^4$ contains a rational surface, more precisely we prove the following

PROPOSITION 2.1. – *Every $X_4 \subseteq \boldsymbol{P}^4(K)$ contains a rational surafce S^0 and moreover if $P^* \in X_4$ is a fixed point of X_4 we can take S^0 going through it.*

See [12] for the proof.

2.2. – *Construction of the quadric bundle and unirationality of the X_4 in $\boldsymbol{P}^4$ and $\boldsymbol{P}^5$.*

Consider $X_4 \subseteq \boldsymbol{P}^{m+1}$, with $m \geqslant 3$ and S^0 the rational surface contained in X_4. Fix $R \in S^0$ and H^0 a hyperplane in $\boldsymbol{P}^{m+1}$ and take the tangent cone $C_R(X_4)$ to X_4 in R. Let $Q_R = C_R(X_4) \cap H^0$ be the quadric hypersurface of dimension $m-2$ obtained intersecting $C_R(X_4)$ with H^0. Consider the quadric bundle

$$\pi : X^+ \to S^0$$

with $X^+ = \{(R, P')/R \in S^0,\ P' \in Q_R\}$ and $\pi((R, P')) = R$. X^+ is an irreducible variety defined over K_0 of dimension m. If $m = 4$ Q_R is a quadric in $\boldsymbol{P}^3$ while if $m = 3$ Q_R is a conic in $\boldsymbol{P}^2$ and hence $X^+ \to S^0$ is a conic bundle.

If $m \geqslant 5$ the existence of the rational surface S^0 in X_4 is sufficient to prove, applying in an essential way the Segre's theorem, that the previous quadric bundle has a rational section and hence to prove the unirationality of the $X_4 \subseteq \boldsymbol{P}^{m+1}$.

If $m = 3, 4$ the existence of the rational surface S^0 is not, alone, sufficient to conclude that the quadric bundle admits a rational section because it is not possible to apply the Segre's theorem. We note that if all conic bundles over a rational surface were unirational then the previous construction would imply automatically the unirationality of the quartic $X_4 \subseteq \boldsymbol{P}^4$. Unfortunately this problem, also if it is not known if there exist examples of conic bundles which are not unirational, is very far to be solved and it is, togheter the one of the unirationality or not of the quartic hypersurface in $\boldsymbol{P}^4$ and $\boldsymbol{P}^5$, the most important open problem in these rationality questions. It seems that the answer to this problem is negative and a possible counterexample should be given by the hypersurface of degree n, X_n, in $\boldsymbol{P}^4$ containing a line of molteplicity $n-2$ for $n \geq 5$.

Nevertheless there are special rational surfaces S^0 such that the previous quadric bundle admits a rational section. We will study some of these particular cases.

i) Surfaces with separable asymptotics.

Let F_n be a surface in $\boldsymbol{P}^3(K)$, irreducible, of order $n \geq 3$, which is not a developable ruled surface. Let x be a generic point of F_n and Π_x the tangent plane to F_n in x. The intersection $F_n \cap \Pi_x$ is a curve with a double point in x and the lines $l \subseteq \Pi_x$ such that $x \in l$ are the tangent lines to F_n in x with

$$\mathrm{molt}_x(l \cap F_n) \geq 2 .$$

If Q_x is the polar quadric to F_n in x, the intersection $Q_x \cap \Pi_x$ is a conic C_x with a double point in x. Then in $K(x)$ or in its quadratic extension

$$C_x = l'_x + l''_x$$

where l'_x and l''_x are two tangent lines to F_n in x, more precisely the two aymptotics lines. Let S be the congruence, defined over K, generated by the asymptotic lines to F_n moving x, contained in the Grassmannian of the lines in $\boldsymbol{P}^3 \boldsymbol{G}(1,3)$.

DEFINITION 2.1. – *F_n has separable asymptotics if $S = S' + S''$, that is if S consists of two irreducible components over an algebraic extension of K.*

We have that

$$\mathrm{molt}_x(l'''_x \cap F_n) \geq 3$$

that is $l'''_x \cap F_n = 3x + p_1 + \ldots + p_\alpha$.

REMARK 2.1. – a) Every (non developable) ruled F_n has separable asymptotics. Moreover $\dim S' = 1$, $\dim S' = 2$ and $S' = \{$generatrices of the ruled surface$\}$. If F_n is a cone or a developable surface $S' = S''$.

b) If $n = 2$, F_2 is a quadric and hence it has always separable asymptotics and $\dim S'' = \dim S'' = 1$.

c) If F_n has separable asymptotics then every surface in $\boldsymbol{P}^3$, obtained starting from F_n by a nondegenerate projectivity of $\boldsymbol{P}^3$, has separable asymptotics.

It is possible to prove the following

THEOREM 2.1. – *Let $S^0 \subseteq X_4 \subseteq \boldsymbol{P}^4$, $\boldsymbol{P}^5$ be a surface with separable asymptotics and let*

$$\pi : X^+ \to S^0$$

be the quadric bundle constructed above. Then the quadric bundle admits a rational section.

PROOF. – Let $R \in S^0$ be a non singular point of S^0, then the tangent plane to S^0 in R intersects its quadric tangent cone $C_R(X_4)$ (i.e. the tangent cone in R to the quartic hypersurafce $X_4 \cap T_R(X_4)$ where $T_R(X_4)$ is the tangent space to X_4 in R) in two generatrices which correspond to the two points in X^+ $((R, P')$ and $(R, P''))$. Let S be the closure of the geometric locus of these couples of points moving R on S^0. S is the congruence generated by the asymptotic lines moving R on S^0 and hence, by hypotesis, consists of two components S' and S'', over an algebraic extension of K, such that each determines a rational section of π. ■

In the next section we will study the monoidal quartic surfaces with separable asymptotics.

ii) If S^0 is a ruled non developable surface of order 4, it contains a triple line. In this case we can find unirational quartic hypersurfaces X_4 cointaing S^0 which are smooth only in $\boldsymbol{P}^5$ and not in $\boldsymbol{P}^4$.

iii) The Fano variety $F_1(X_4)$ of the lines contained in $X_4 \subseteq \boldsymbol{P}^n$ has dimension $2n - 7$ (see [1]) and hence it is a curve (of genus 801, see [30]) for $n = 4$ and a variety of dimension 3 for $n = 5$. In this case to every rational curve contained in $F_1(X_4)$ corresponds a rational normal scroll (called Hirzebruch's surface F_n) contained in X_4 the generatrices of which determine a rational section of the quadric bundle $\pi : X^+ \to S^0$. Hence we have the following

THEOREM 2.2. – *Every $X_4 \subseteq \boldsymbol{P}^5$ which contains a Hirzebruch surface* (*in particular, a plane*) *is unirational.*

It would be interesting to study if there exist always rational curves in $F_1(X_4)$ $(n = 5)$.

iv) More generically it is possible to take as S^0 a generic rational surface containing a pencil of algebraic asymptotics. Examples of such surfaces have been given by B. Segre in [26]. To determine all surfaces of this type could be a problem worthy of attention.

3. – Monoidal quartic surfaces with separable asymptotics and families of smooth quartic unirational hypersurfaces in $\boldsymbol{P}^4$ and $\boldsymbol{P}^5$.

3.1. – *Monoidal quartic surfaces with separable asymptotics.*

We consider the monoidal quartic surfaces F_4 that is the algebraic surfaces which have a triple point as unique singularity. If we suppose that the triple point T has homogeneous coordinates $T = (0:0:0:1)$, the equation of F_4 is

$$x_3 \cdot \beta(x_0, x_1, x_2) - \alpha(x_0, x_1, x_2) = 0$$

where $\alpha(x_0, x_1, x_2)$ and $\beta(x_0, x_1, x_2)$ are homogeneous polynomials, without common factors, of degree four and three respectively.

Among all monoidal quartic surfaces $F_4 \subseteq \boldsymbol{P}^3$ it is possible to characterize the ones with separable asymptotics, to which we are interested for the reasons explained before, in several ways for istance by the following

PROPOSITION 3.1. – *Let F_4 be as above then F_4 has separable asymptotics if and only if for the generic point x of F_4 on K*

$$h(x) = l(x)^2$$

where $h(x)$ is the equation of the Hessian surface of F_4 and $l(x)$ is an element of $K_1(x)$ with K_1 quadratic extension of K (eventually equal to K).

See [21] or [28] for the proof.

We note that the example of the particular quartic smooth hypersurface which is unirational given by B. Segre in [28] in 1960 contains the monoidal quartic surface with separable asymptotics of equation

$$x_1^4 - 6x_1^2 x_2^2 + x_2^4 + x_3^4 + x_3^3 x_4 = 0 .$$

A. Predonzan in [22] gave a complete projective classification of all monoidal quartic surfaces with separable asymptotics by the following

THEOREM 3.1. – *A monoidal surface F_4 of order* 4, *not ruled, in a projective space $\boldsymbol{P}^3(K)$, with K algebraic closed field of characteristic zero, has separable asymptotics if and only if is one of the following different projective types of canonical equation*

$$\begin{aligned}
&\text{I)} \quad X_0X_1X_2X_3 + X_0^4 + X_1^4 + X_2^4 - 2(X_0^2X_1^2 + X_1^2X_2^2 + X_2^2X_0^2) = 0\\
&\text{II)} \quad X_0X_1X_2X_3 + (X_1^2 - X_2^2)^2 = 0\\
&\text{III)} \quad X_1^2X_2X_3 + X_0^2X_1^2 + X_2^4 = 0\\
&\text{IV)} \quad X_1^2X_2X_3 + X_0^4 = 0\\
&\text{V)} \quad X_0^3X_3 + X_1X_2(X_1^2 - X_2^2) = 0\\
&\text{VI)} \quad X_0^3X_3 + X_1^2X_2^2 = 0\ .
\end{aligned}$$

In the proof of the theorem, A. Predonzan, didn't make explicitly most of the very complicated and painful computations which led him to the result stated. We checked them by using the modern symbolic computation systems Maple [11] and CoCoA [3].

3.2. – *Tethraedral surfaces and families of smooth quartic unirational hypersurfaces in $\boldsymbol{P}^4$ and $\boldsymbol{P}^5$.*

The most general monoidal quartic surface S in $\boldsymbol{P}^3$ with separable asymptotics is the one of equation

$$X_0X_1X_2X_3 + X_0^4 + X_1^4 + X_2^4 - 2(X_0^2X_1^2 + X_1^2X_2^2 + X_2^2X_0^2) = 0$$

and it is called tethraedral surface.

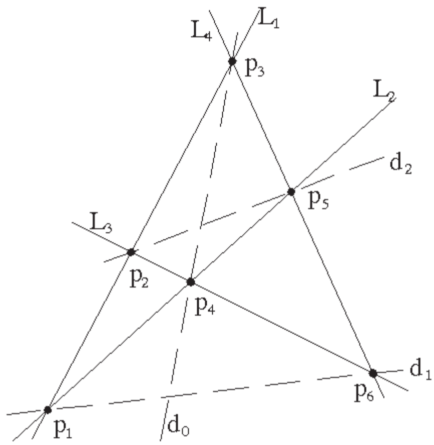
We determined in a rigorous way the equation of this surface by the modern tools of symbolic computation Maple and CoCoA in the following way. The equation of the generic monoidal quartic surface, as we saw before, is

$$x_3 \cdot \beta(x_0, x_1, x_2) - \alpha(x_0, x_1, x_2) = 0\ .$$

$\beta(x_0, x_1, x_2) = 0$ is the equation of the cubic tangent cone to S in the triple point. It is possible to prove that if this surface has separable asymptotics it consists of irreducible linear components i.e. it breaks into planes. The most general case is the one in which the cubic tangent cone consists of three planes not going through a same line. Hence we can suppose that the equation of the tangent cone to the tethraedral surface S in T is

$$x_0x_1x_2 = 0\ .$$

Moreover, by other characterizations of the surfaces with separable asymptotics, it is possible to prove, see [22], that S has 6 double points p_i which are the vertices of a complete plane quadrilateral and the diagonal trilateral of which is the intersection of the plane containing the quadrilateral with the cubic tangent cone. If we take $x_3 = 0$ as the equation of the plane containing the quadrilateral we can consider the following picture



d_0, d_1, d_2 are the lines of equations $x_0 = 0$, $x_1 = 0$, $x_2 = 0$ and p_i are the 6 double points of coordinates, supposed $a_0, a_1, a_2 \neq 0$,

$$p_1\left(\frac{\sqrt{a_2}}{\sqrt{a_0}} : 0 : 1\right), \qquad p_2\left(-\frac{\sqrt{a_1}}{\sqrt{a_0}} : 1 : 0\right), \quad p_3\left(0 \colon \frac{\sqrt{a_2}}{\sqrt{a_1}} : 1\right),$$

$$p_4\left(0 : -\frac{\sqrt{a_2}}{\sqrt{a_1}} : 1\right), \quad p_5\left(\frac{\sqrt{a_1}}{\sqrt{a_0}} : 1 : 0\right), \qquad p_6\left(-\frac{\sqrt{a_2}}{\sqrt{a_0}} : 0 : 1\right).$$

Hence we obtain for L_1, L_2, L_3, L_4 respectively the equations:

$$\begin{aligned}
\sqrt{a_0}x_0 + \sqrt{a_1}x_1 - \sqrt{a_2}x_2 = 0\\
\sqrt{a_0}x_0 - \sqrt{a_1}x_1 - \sqrt{a_2}x_2 = 0\\
\sqrt{a_0}x_0 + \sqrt{a_1}x_1 + \sqrt{a_2}x_2 = 0\\
\sqrt{a_0}x_0 - \sqrt{a_1}x_1 + \sqrt{a_2}x_2 = 0\ ,
\end{aligned}$$

and we can put

$$\alpha(x_0, x_1, x_2) = L_1 \cdot L_2 \cdot L_3 \cdot L_4$$

i.e.

$$\alpha(x_0, x_1, x_2) = a_0^2 x_0^4 + a_1^2 x_1^4 + a_2^2 x_2^4 - 2(a_0 a_1 x_0^2 x_1^2 + a_1 a_2 x_1^2 x_2^2 + a_0 a_2 x_0^2 x_2^2).$$

The equation of S becomes

$$x_0 x_1 x_2 x_3 + a_0^2 x_0^4 + a_1^2 x_1^4 + a_2^2 x_2^4 - 2(a_0 a_1 x_0^2 x_1^2 + a_1 a_2 x_1^2 x_2^2 + a_0 a_2 x_0^2 x_2^2) = 0$$

and by the coordinate change

$$\begin{cases} X_0 = \sqrt{a_0} x_0 \\ X_1 = \sqrt{a_1} x_1 \\ X_2 = \sqrt{a_2} x_2 \\ X_3 = \dfrac{x_3}{\sqrt{a_0 a_1 a_2}} \end{cases}$$

it assumes the canonical form I).

In order to find families of quartic smooth hypersurfaces in $\boldsymbol{P}^4$ and $\boldsymbol{P}^5$ which are unirational we want to determine the dimension of the algebraic systems of all quartic hypersurfaces $X_4 \subseteq \boldsymbol{P}^r(K)$ which contain a tethraedral surface (and hence they are unirational over an extension K_1 of K as we saw before).

The dimension of the family of all tethraedral surfaces S in $\boldsymbol{P}^3$ is 15. Denoting $\boldsymbol{P}^r = \boldsymbol{P}^r(K)$, put

$$\Phi = \{S \text{ tethraedral } / \exists\, \boldsymbol{P}^3 \subset \boldsymbol{P}^r \text{ with } \boldsymbol{P}^3 \supseteq S\},$$

$$\Sigma = \{X = X_4 \subseteq \boldsymbol{P}^r / \exists\, S \in \Phi, \quad \text{with } S \subseteq X\},$$

and consider the incidence corrispondence:

$$I = \{(S, X) \in \Phi \times \Sigma / S \subseteq X\}$$

with the relative projections:

$$\begin{array}{ccc} I & \xrightarrow{\sigma} & \Sigma \\ \varphi \downarrow & & \\ \Phi & & \end{array}$$

If the generic $S \in \Phi$, contained in the linear space of dimension three of equation $\{x_4 = \ldots = x_r = 0\}$, has equation

$$\varphi_4(x_0, \ldots, x_3) = 0,$$

where $(x_0 : \ldots : x_r)$ are the homogeneous coordinates of $\boldsymbol{P}^r$, then the generic $X \in \Sigma$ which contains S has equation

$$\sum_{i=4}^{r} x_i f_i(x_0, \ldots, x_r) + \lambda \varphi_4(x_0, \ldots, x_3) = 0 ,$$

where f_i are general homogeneous polynomials of degree 3 and $\lambda \in K^*$.

We proved

THEOREM 3.2. – *Let Σ be defined as above. If $r \geqslant 8$, $\Sigma = |\mathcal{O}_{\boldsymbol{P}^r}(4)|$. If $r \leqslant 7$, Σ is an algebraic irreducible system over K and*

$$\dim \Sigma = \binom{r+4}{4} + 4r - 32 .$$

See [13] for the proof.

If $r \geqslant 6$ the theorem doesn't add anything because the generic X_4 is, in this case, unirational. But if $r = 4, 5$ the previous theorem says that $\dim \Sigma = 54, 114$ (while $\dim |\mathcal{O}_{\boldsymbol{P}^r}(4)| = 69, 125$). In this way we constructed tw o families of quartic hypersurfaces in $\boldsymbol{P}^4$ and $\boldsymbol{P}^5$ which are unirational of dimension respectively 54 and 114.

4. – Further possible developments: conjectures and open problems.

As we told in section 2 the most important and significant open problem consits of deciding if there exist or not conic bundles which are not unirational and a negative answer to this problem implies immediately the unirationality of the generic quartic hypersurface in $\boldsymbol{P}^4$. Nevertheless in the 1928 G. Fano in [7] gave the following

CONJECTURE 4.1. – *The hypersurface of degree n X_n in $\boldsymbol{P}^4$ containing a line l of molteplicity $n-2$ is not unirational if $n \geqslant 5$;*

reinforcing furthermore, in handwrite papers kept in the library «G. Peano» of the Department of Mathematics of the University of Turin, saying that the previous hypersurface doesn't contain any surface. It is obvious that the solution to the Fano conjecture would represent an important step in the understanding of these rationality problems.

Concerning the unirationality of the quartic hypersurface in $\boldsymbol{P}^4$ and $\boldsymbol{P}^5$ it seems that in this moment there are no precise opinions neither about their probable unirationality or not-unirationality (also if the majority of the experts seems to prefer no). Moreover there are no ideas which can bring us to define a possible line to attack this problem. It is worth to study in depth the nice results of B. Segre, that we mentioned in section 2 (but also see [27]), on the rational surfaces which contain such a surface, and hence which are unirational as we did in section 3 for the quartic monoidal surfaces. In the same way it appears useful the study of the existence or not of rational curves on the Fano variety (of dimension 3) of the lines of the X_4 in $\boldsymbol{P}^5$ for the reasons that we explained in section 2.

Concerning the generic hypersurface of degree d X_d in $\boldsymbol{P}^n$, we think that probably the more significant open problems are of two types:

(*i*) determine which are effectively the hypersurfaces, for $d \gg n$, which are unirational. With regard to this problem the result of J. Harris, B. Mazur and R. Pandharipande, that we mentioned in section 1, and which says that X_n is unirational if the codimension of its singular locus is sufficiently big with respect to d and n, also if it generalized the Morin's result for the generic X_n it appears not fully satisfactory. By the way there is the paper [23] of A. Predonzan where it is conjectured that every X_d is unirational for $d \gg n$ if X_d is not the locus of a family not unirational of linear subspaces and it is proved to be true for $d = 4$ and $n \geqslant 7$ (this proof is very interesting and it would be worth to analize it in a modern language);

(*ii*) improve the limits $n(d)$ actually known, by which if $n \gg n(d)$ then the generic X_d in $\boldsymbol{P}^n$ is unirational (we remember that $n(4) = 6$, $n(5) = 17$, both could be too hight, and surely are too hight the $n(d)$ known for $d \geqslant 6$). There is the following

CONJECTURE 4.2. – (*A. Conte*) *Every X_d in $\boldsymbol{P}^n$ which contains a linear space of dimensione $d-2$ is unirational.*

This conjecture is true for $d = 4, 5$ and if it is proved it would imply the unirationality of the generic X_d in $\boldsymbol{P}^n$ for

$$(n-d+2)(d-1) \geqslant \binom{2d-2}{d}$$

because if the previous disequality is true, the generic X_d in $\boldsymbol{P}^n$ always contains a linear space of dimension $d-2$ (by [19]).

Of course all these open problems would be easily solved if we had an available criterion about unirationality for the algebraic varieties. But this is a problem very complicated and difficult and seems unassalaible also by the depest and sofisticated tecniques of the modern Algebraic Geometry.

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