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Modular invariant theory and the iterated total power operation.

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Sunto. – *L'operazione coomologica totale iterata in coomologia ordinaria a coefficienti in $\boldsymbol{Z}/p$ ha una sua espressione a seconda della base fissata nell'algebra di Steenrod $\mathcal{A}_p$. Fissato un primo p dispari, vengono qui calcolati i coefficienti dell'operazione totale doppia iterata quando si sceglie in $\mathcal{A}_p$ la base dei monomi ammissibili. Si fornisce inoltre una dimostrazione alternativa di una versione normalizzata di un teorema di Mùi, ottenuta considerando una particolare successione di funzioni, in analogia al caso $p=2$.*

1. – Introduction.

Fix an odd prime p and let H^* be the reduced ordinary cohomology theory over $\boldsymbol{F}_p$ the Galois field of order p. The Steenrod algebra $\mathcal{A}_p$ is the algebra of all stable operations in H^*. Its generators β, P^i, $i \geq 0$, can be defined through the ring homomorphism

$$T: H^*(X) \to H^*(\boldsymbol{Z}/p) \otimes H^*(X),$$

where X is a CW complex. As it is well known, the cohomology ring of an elementary abelian p – group of rank m is

$$H^*((\boldsymbol{Z}/p)^m) = E[x_1, \ldots, x_m] \otimes \boldsymbol{F}_p[y_1, \ldots, y_m],$$

where $E[x_1, \ldots, x_m]$ is the exterior algebra on m generators $x_1, \ldots, x_m$, each having degree 1, and $\boldsymbol{F}_p[y_1, \ldots, y_m]$ is a polynomial ring with generators $y_1, \ldots, y_m$ in grading 2.

T is known as the total power operation and it has been extensively studied by Steenrod in [10]:

$$T(z) = \mu(q) \sum_{\varepsilon = 0, 1} (-1)^{\varepsilon + i} x_1^{\varepsilon} y_1^{(q-2i)h - \varepsilon} \otimes \beta^{\varepsilon} P^i(z),$$

where $z \in H^q(X)$, $h = (p-1)/2$, $\mu(q) = (h!)^q (-1)^{hq(q-1)/2}$. Other operations

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are obtained by iterating T. For each $m \geq 1$, we have

$$T_m : H^*(X) \to H^*((\boldsymbol{Z}/p)^m) \otimes H^*(X),$$

which multiplies the degrees by p^m. There is a natural action of $GL_m = GL(m, \boldsymbol{F}_p)$ upon $H^*((\boldsymbol{Z}/p)^m)$ and the invariant elements rings are closely related to $\mathcal{A}_p$; in fact, from the geometric construction of T_m, it follows that

$$Im(T_m) \subset (H^*((\boldsymbol{Z}/p)^m))^{\widetilde{SL}_m},$$

$\widetilde{SL}_m$ being the subgroup consisting of those matrices $\omega \in GL_m$ such that $(det\ \omega)^h = 1$. Fixed any linear basis $\mathcal{B}$ in $\mathcal{A}_p$, we get an expression of the form

$$T_m(z) = \sum_{b \in \mathcal{B}} f(b) \otimes b(z),$$

with $f(b) \in H^*((\boldsymbol{Z}/p)^m)^{\widetilde{SL}_m}$. The coefficients $f(b)$ have been computed when $\mathcal{B} = \mathcal{B}_{Mil}$, the Milnor basis of $\mathcal{A}_p$ (see [8]). After recalling some basic facts about the geometric setting of $\mathcal{A}_p$ and the modular invariant theory in Section 1, in Section 2 we consider the basis $\mathcal{B}_{Adm}$ of admissible monomials and show how the coefficients $f(b)$ appear when $m = 2$. (The case $p = 2$ has been treated in [4]). The last Section is devoted to providing another proof of the normalized version of Mùi's Theorem [3, Th. 2.9]. We proceed in a way analogous to [6], where the case $p = 2$ has been dealt with. In our case , the corresponding sequence of maps is $\delta_m : \mathcal{A}_p^* \to \Delta_m$, where $\Delta_m = \Phi_m^{B_m}$. Here Φ_m is the localization of $H^*((\boldsymbol{Z}/p)^m)$ out of its Euler class e_m and B_m is the Borel subgroup of GL_m.

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2. – Preliminaries.

Let A_{p^m} be the alternating group on $\boldsymbol{F}_p^m$, G an even permutation group containing an elementary abelian p – group of rank m, and X a based CW complex. So we have the Steenrod power map

$$P_G : H^q(X) \to H^{p^m q}(EG^+ \wedge_G X^{(p^m)}),$$

which sends z to $1\otimes z^{p^m}$ at the cochain level, the diagonal homomorphism:

$$d_G^*: H^*(EG^+\wedge_G X^{(p^m)})\to H^*(BG)\otimes H^*(X),$$

induced by the G − homomorphism

$$EG^+\wedge_G X\to EG^+\wedge_G X^{(p^m)}$$

via the diagonal $X\to X^{(p^m)}$ ($H^*(EG^+\wedge_G X)=H^*(BG)\otimes H^*(X)$ by the Künneth formula,) and the restriction homomorphism

$$Res((\boldsymbol{Z}/p)^m, G): H^*(G)\to H^*((\boldsymbol{Z}/p)^m)$$

induced by the inclusion $(\boldsymbol{Z}/p)^m\subset G$. The resulting composition of these three homomorphisms does not depend on the group G containing $(\boldsymbol{Z}/p)^m$ and contained in A_{p^m}; it gives rise to the iterated total power operation T_m. The fact that $Im(T_m)\subset H^*((\boldsymbol{Z}/p)^m)^{\widetilde{SL}_m}\otimes H^*(X)$ comes from the construction above. We need to recall some facts about modular invariant theory. Let

$$V_k=\prod_{\lambda_i\in\boldsymbol{F}_p}(\lambda_1 y_1+\ldots+\lambda_{k-1}y_{k-1}+y_k),$$

$$L_m=V_1\ldots V_m,\qquad \widetilde{L}_m=L_m^h,\qquad Q_{m,s}=Q_{m-1,s}V_m^{p-1}+Q_{m-1,s-1}^p;$$

conventionally, $Q_{s,s}=1$ for each $s\geqslant 0$ and $Q_{m,s}=0$ if either $s<0$ or $s>m$. The $Q_{m,s}$, called Dickson's invariants, arise when we consider the polynomial part of $H^*((\boldsymbol{Z}/p)^m)$. Concerning with the exterior part, we set

$$[k;e_{k+1},\ldots,e_m]=\frac{1}{k!}det\begin{pmatrix} x_1 & \cdots & x_m\\ \cdots & \cdots & \cdots\\ x_1 & \cdots & x_m\\ y_1^{p^{e_{k+1}}} & \cdots & y_m^{p^{e_{k+1}}}\\ \cdots & \cdots & \cdots\\ y_1^{p^{e_m}} & \cdots & y_m^{p^{e_m}}\end{pmatrix},$$

where $e_{k+1},\ldots,e_m$ are non negative integers, $0\leqslant k\leqslant m$, and $M_{m;s_1,\ldots,s_k}=[k;0,1,\ldots,\hat{s}_1,\ldots,\hat{s}_k,\ldots,m-1]$. As usual, $\hat{s}_j$ means that s_j is omitted. We have

$$M_{m;s_1}^2=0;\qquad M_{m;s_1}\ldots M_{m;s_k}=(-1)^{k(k-1)/2}M_{m;s_1,\ldots,s_k}L_m^{k-1},$$

where $0 \leqslant s_1 < \ldots < s_k \leqslant m-1$. We set

$$\widetilde{M}_{m;\, s_1,\, \ldots,\, s_k} = M_{m;\, s_1,\, \ldots,\, s_k} L_m^{h-1}, \qquad R_{m;\, s_1,\, \ldots,\, s_k} = M_{m;\, s_1,\, \ldots,\, s_k} L_m^{p-2} \tag{1}$$

and

$$e_m = \prod(\lambda_1 y_1 + \ldots + \lambda_m y_m) \qquad \text{(the Euler class)},$$

where the product runs over all nontrivial m-tuples of elements of $\boldsymbol{F}_p$. We observe that

$$Q_{m,\,0} = L_m^{p-1} = \widetilde{L}_m^2 = (-1)^m e_m .$$

We invert the Euler class in $H^*((\boldsymbol{Z}/p)^m)$ and get the ring

$$\Phi_m = H^*((\boldsymbol{Z}/p)^m)[e_m^{-1}]$$

upon which the action of GL_m on $H^*((\boldsymbol{Z}/p)^m)$ extends. As it is well known:

$$\Gamma_m = \Phi_m^{GL_m} = E[R_{m;\,0}, \ldots, R_{m;\,m-1}] \otimes \boldsymbol{F}_p[Q_{m,\,0}^{\pm 1}, Q_{m,\,1}, \ldots, Q_{m,\,m-1}],$$

$$\widetilde{\Gamma}_m = \Phi_m^{\widetilde{SL}_m} = E[\widetilde{M}_{m;\,0}, \ldots, \widetilde{M}_{m;\,m-1}] \otimes \boldsymbol{F}_p[\widetilde{L}_m^{\pm 1}, Q_{m,\,1}, \ldots, Q_{m,\,m-1}].$$

In Φ_m, we have defined particular elements which can be assumed as generators of $\Phi_m^{B_m}$. We set:

$$\begin{cases} v_1 = V_1, \quad v_{k+1} = V_{k+1}/Q_{k,\,0}, \quad k \geqslant 0 \\ u_k = M_{k,\,k-1}/(v_1^{p^{k-2}} v_2^{p^{k-3}} \ldots v_{k-1} v_k), \quad k \geqslant 1\,; \end{cases} \tag{2}$$

the gradings of v_k and u_k are 2 and -1 respectively.

The following relations hold:

$$\begin{cases} V_k = v_1^{(p-1)\,p^{k-2}} v_2^{(p-1)p^{k-3}} \ldots v_{k-1}^{(p-1)} v_k \\ L_k = v_1^{p^{k-1}} v_2^{p^{k-2}} \ldots v_{k-1}^{p-1} v_k . \end{cases}$$

Further, let w_k be v_k^{p-1}.

PROPOSITION 1. – $\Phi_m^{B_m} \cong E[u_1, \ldots, u_m] \otimes \boldsymbol{F}_p[w_1^{\pm 1}, \ldots, w_m^{\pm 1}]$.

PROOF. – From [5, Prop. 7.5], we know that

$$\Phi_m^{B_m} \cong E[N_1, \ldots, N_m] \otimes \boldsymbol{F}_p[W_1^{\pm 1}, \ldots, W_m^{\pm 1}],$$

where $N_k = L_k^{p-1} M_{k;\,k-1}$ and $W_k = V_k^{p-1}$. Easy calculations lead to

$$W_1 = w_1$$
$$W_k = (W_1 \dots W_{k-1})^{p-1} w_k$$
$$N_k = u_k W_k .$$

From [9, Lemma 5.4], we know that

$$M_{m;\,s} = \sum_{r=s+1}^{m} M_{r;\,r-1} V_{r+1} \dots V_m Q_{r-1,\,s} .$$

Combining this relation with the second of (1) and the (2), we get:

$$(3) \qquad R_{m;\,s} = M_{m;\,s} L_m^{p-1} = Q_{m,\,0} \sum_{r=s+1}^{m} \frac{M_{r;\,r-1}}{v_1^{p^{r-1}} v_2^{p^{r-2}} \dots v_{r-1}^{p} v_r} Q_{r-1,\,s}$$

$$= Q_{m,\,0} \sum_{r=s+1}^{m} u_r \frac{V_r}{v_r} Q_{r-1,\,s} = Q_{m,\,0} \sum_{r=s+1}^{m} u_r Q_{r-1,\,0}^{-1} Q_{r-1,\,s} .$$

3. – On the double power operation.

From [8], we know the coefficients $f(b)$ when $\mathcal{B} = \mathcal{B}_{Mil}$. Mùi's Theorem reads as follows:

THEOREM 2. – ([8, 1.3]) *Let* $z \in H^q(X)$, $s = (s_1, \dots, s_k)$, $1 \leq s_1 < \dots < s_k \leq m$, $R = (r_1, \dots, r_m)$. *Then*

$$T_m(z) = \mu(q)^m \tilde{L}_m^q \sum_{S,\,R} (-1)^{r(S,\,R)} R_{m,\,s_1} \dots R_{m,\,s_k} Q_{m,\,0}^{r_0} \dots Q_{m,\,m-1}^{r_{m-1}} \otimes St^{S,\,R}(z) ,$$

where $r_0 = -k - (r_1 + \dots + r_m)$, $r(S, R) = k + s_1 + \dots + s_k + r_1 + 2r_2 + \dots + mr_m$ *and* $St^{S,\,R} \in \mathcal{B}_{Mil}$ *(see below).*

In [2] the coefficients $f(b)$ in the double iterated total power operation are computed when we choose in $\mathcal{A}_p$ the classical basis $\mathcal{B}_{Adm}$. We adopt the abbreviated notation $P^I = \beta^{\varepsilon_1} P^{t_1} \dots \beta^{\varepsilon_k} P^{t_k}$ for a typical monomial in $\mathcal{A}_p$, where $I = (\varepsilon_1, t_1, \dots, \varepsilon_k, t_k)$ is a multi-index whose entries ε_i are 0 or 1 and t_i are positive integers (possibly $t_k = 0$ if $\varepsilon_k = 1$). The length of P^I is k if $t_k \neq 0$; it is $k-1$ if $t_k = 0$ and $\varepsilon_k = 1$. A monomial P^I belongs to $\mathcal{B}_{Adm}$ if $t_j \geq pt_{j+1} + \varepsilon_{j+1}$ for each $1 \leq j \leq k-1$. Then an admissible monomial of lenght 2 is of the form $\beta^{\varepsilon_1} P^{pt + \varepsilon_2 + \alpha} \beta^{\varepsilon_2} P^t$, where $\alpha, t \geq 0$ and $\varepsilon_1, \varepsilon_2 = 0, 1$. Leading to the admissible basis, the Adem relations play an important role in determining the $f(b)$, together with comparisons of coefficients in suitable power series.

THEOREM 3. – ([2]) *For each* $z \in H^q(X)$, *X a CW complex*, $q \geq 0$, *we have:*

$$T_2(z) = \mu(q)^2 \tilde{L}_2^q \sum_{t,\,\alpha,\,i} (-1)^{\alpha+i} Q_{2,0}^{pi-t-\alpha-1} Q_{2,1}^{\alpha-pi-i-1} \cdot$$

$$\left\{ \binom{\alpha-pi}{i} [Q_{2,0}Q_{2,1} \otimes P^{pt+\alpha} - R_{2;0,1}Q_{2,1} \otimes \beta P^{pt+\alpha+1}\beta + R_{2;1}Q_{2,1} \otimes P^{pt+\alpha+1}\beta - R_{2;0}Q_{2,1} \otimes \beta P^{pt+\alpha}] + \binom{\alpha-pi-1}{i} R_{2;1}Q_{2,0} \otimes \beta P^{pt+\alpha} \right\} P^t(z).$$

As we can see, the combinatorics involved is complicated since the double iteration. Consider $\mathcal{A}_p$ as graded by the length of monomials. In grading 2, it sufficies to apply once the Adem relations in order to get the admissible expression of any monomial. A similar procedure does not apply to upper length monomials, since there are not explicit non- recursive formulas, neither to obtain an admissible expression of any monomial of length $k > 2$, nor to convert a Milnor basis element to the basis $\mathcal{B}_{Adm}$ (see [7]).

4. – An alternative proof of the normalized total power operation.

We start from the ring homomorphism

$$S_m \colon H^*(X) \to \Phi_m^{B_m} \otimes H^*(X).$$

For each $z \in H^*(X)$, $S_m(z)$ is:

$$S_m(z) = \sum_{\mathcal{E},\,J} u^{\mathcal{E}} w^{-J} \otimes \Theta^{(\mathcal{E},\,J)}(z),$$

where $\mathcal{E} = (\varepsilon_1, \ldots, \varepsilon_m)$, $\varepsilon_i = 0, 1$, $u^{\mathcal{E}} = u_1^{\varepsilon_1} \ldots u_m^{\varepsilon_m}$, $J = (j_1, \ldots, j_m)$, $i = 1, \ldots, m$, $w^{-J} = w_1^{-j_1} \ldots w_m^{-j_m}$, $\Theta^{(\mathcal{E},\,J)} = \beta^{\varepsilon_1} P^{j_1} \ldots \beta^{\varepsilon_m} P^{j_m}$. Up to a sign, S_m is the homomorphism defined in [1]. $S_m(z)$ has the same dimension as z. Following the idea in [6] for $p = 2$, we construct a sequence of maps:

$$\delta_m \colon \mathcal{A}_p^* \to \Delta_m = \Phi_m^{B_m},$$

where $\mathcal{A}_p^*$ denotes the $\boldsymbol{F}_p$ – dual of $\mathcal{A}_p$, and we will use them to give an alternative proof of a normalized version of a result of Mùi (it is quoted here for the $\mathcal{A}_p$-module $H^*(X)$).

THEOREM 4. – ([3, *Th.* 2.9])

$$S_m(z) = \sum_{S,\,R} (-1)^{r(S,\,R)} R_{m;\,s_1} \dots R_{m;\,s_k} Q_{m,\,0}^{r_0} \dots Q_{m,\,m-1}^{r_{m-1}} \otimes St^{S,\,R}(z),$$

where $r_0 = -k - r_1 - \dots - r_m$, $r(S, R) = k + s_1 + \dots + s_k + r_1 + 2r_2 + \dots + mr_m$. We recall that $\mathcal{A}_p^*$ is isomorphic to:

$$E[\tau_0, \tau_1, \dots, \tau_k, \dots] \otimes \boldsymbol{F}_p[\xi_1, \dots, \xi_k, \dots].$$

Here ξ_k and τ_k are dual to $P^{p^{k-1}} P^{p^{k-2}} \dots P^1$ and $P^{p^{k-1}} P^{p^{k-2}} \dots P^1 \beta$ respectively, with respect to the basis of admissible monomials. For sequences $S = (s_1, \dots, s_k)$, $0 \leqslant s_1 < s_2 < \dots < s_k$, $k \geqslant 0$ and $R = (r_1, \dots, r_l)$, $r_i \geqslant 0$, $l \geqslant 0$, let

$$St^{S,\,R} = (\tau_S \xi^R)^* = (\tau_{s_1} \dots \tau_{s_k} \xi_1^{r_1} \dots \xi_l^{r_l})^*$$

with respect to the basis $\{\tau_S \xi^R\}_{S,\,R}$ of $\mathcal{A}_p^*$. These elements form the so called Milnor basis of $\mathcal{A}_p$. We are going to show that

$$S_m(z) = \sum_{R,\,S} \delta_m(\tau_S \xi^R) \otimes St^{S,\,R}(z). \tag{4}$$

Then we prove that $\delta_m(\tau_S \xi^R)$ is just equal to

$$(-1)^{r(S,\,R)} R_{m;\,s_1} \dots R_{m;\,s_k} Q_{m,\,0}^{-k-(r_1+\dots+r_m)} Q_{m,\,1}^{r_1} \dots Q_{m,\,m-1}^{r_{m-1}};$$

hence S_m is the normalized iterated total power operation. We first introduce a map which is formally identical to S_m:

$$S_m : \mathcal{A}_p \to \Delta_m \otimes \mathcal{A}_p$$

$$\Theta \mapsto \sum_{\mathcal{E},\,J} u^{\mathcal{E}} w^{-J} \otimes \Theta^{(\mathcal{E},\,J)} \circ \Theta.$$

DEFINITION 5. – $\delta_m : \mathcal{A}_p^* \to \Delta_m$ *has the following definition: for* $\tau_S \xi^R \in \mathcal{A}_p^*$, *we set*

$$\delta_m(\tau_S \xi^R) := (-1)^{r(S,\,R)} ((id \otimes \tau_S \xi^R) \circ S_m)(1),$$

that is $\delta_m(\tau_S \xi^R)$ *is the image of* $1 \in \mathcal{A}_p$ *under the following composition:*

$$\mathcal{A}_p \xrightarrow{S_m} \Delta_m \otimes \mathcal{A}_p \xrightarrow{id \otimes \tau_S \xi^R} \Delta_m \otimes \boldsymbol{F}_p \cong \Delta_m.$$

As $S_m(1) = \sum_{\mathcal{E}, J} u^{\mathcal{E}} w^{-J} \otimes \Theta^{(\mathcal{E}, J)}$ (an infinite sum!), we have that:

$$\delta_m(\tau_S \xi^R) = (-1)^{r(S, R)}(id \otimes \tau_S \xi^R)\left(\sum_{\mathcal{E}, J} u^{\mathcal{E}} w^{-J} \otimes \Theta^{(\mathcal{E}, J)}\right)$$

$$= \sum_{\mathcal{E}, J}(-1)^{r(S, R)} u^{\mathcal{E}} w^{-J} \langle \tau_S \xi^R, \Theta^{(\mathcal{E}, J)}\rangle,$$

where $\langle \tau_S \xi^R, \Theta^{(\mathcal{E}, J)}\rangle$ is the value of $\tau_S \xi^R$ on $\Theta^{(\mathcal{E}, J)}$. It is easy to check that δ_m is a ring homomorphism.

LEMMA 6. – *Let* $a < pb$ *and* $a + b = p^n + p^{n-1}$. *Then*

(i) *the coefficient of* $P^{p^n} P^{p^{n-1}}$ *in*

$$P^a P^b = \sum_{t=0}^{[a/p]} (-1)^{a+t} \binom{(p-1)(b-t)-1}{a-pt} P^{a+b-t} P^t$$

is zero;

(ii) *the coefficient of* $P^{p^n} = P^{p^n} P^0$ *in* (5) *is zero.*

COROLLARY 7. – *Let* $a_1 + \ldots + a_m = p^{n-1} + p^{n-2} + \ldots 1 = p^n - 1$ $(m \geq n)$. *Then the coefficient of* $P^{p^{n-1}} P^{p^{n-2}} \ldots P^1$ *in the admissible expression of* $P^{a_1} P^{a_2} \ldots P^{a_m}$ *is zero.*

The same argument works to show that $P^{p^{k-1}} P^{p^{k-2}} \ldots P^1 \beta$ does not appear in the admissible expression of any nonadmissible monomial $P^{a_1} P^{a_2} \ldots P^{a_m} \beta$.

COROLLARY 8.

(i) $\langle \xi_k, P^{i_1} \ldots P^{i_n}\rangle = 1$ *if and only if* $n = k$ *and* $(i_1, \ldots, i_n) = (p^{k-1}, p^{k-2}, \ldots, 1)$;

(ii) $\langle \tau_k, P^{i_1} \ldots P^{i_n} \beta\rangle = 1$ *if and only if* $n = k$ *and* $(i_1, \ldots, i_n) = (p^{k-1}, p^{k-2}, \ldots, 1)$.

PROPOSITION 9. – $\delta_n(\xi_k) = (-1)^k \sum_J w^{-J}$, *where* J *is a multi-index of the form* $(0, \ldots, 0, p^{k-1}, \ldots, 0, \ldots, p, 0, \ldots 1, 0 \ldots)$ *with* $n - k$ *zeros inserted.*

PROOF. – $\xi_k = (P^{p^{k-1}} \ldots P^p P^1)^*$. From Cor. 3(i), $\langle \xi_k, \Theta^{(\mathcal{E}, J)}\rangle = 1$ if and only if $\Theta^{(\mathcal{E}, J)} = P^{p^{k-1}} \ldots P^p P^1$. The corresponding coefficient is $\sum_J w^{-J}$, where J is as above. ■

PROPOSITION 10. – $\delta_n(\tau_k) = (-1)^{k+1} \sum_{t=k+1}^{n} u_t w^{-J_t}$ *for all* $k = 0, \ldots, n-1$, *where the sequence* J_t *is of the following type*:

$$J_t = (j_1, j_2, \ldots, j_{t-1}) = (0, \ldots, P^{p_{k-1}}, 0, \ldots, P^p, 0, \ldots P^1, 0 \ldots, 0),$$

with $t-1-k$ *zeros inserted.*

PROOF. – $\tau_k = (P^{p^{k-1}} P^{p^{k-2}} \ldots P^p P^1 \beta)^*$. Applying Cor. 3 (ii), we have $\langle \tau_k, \Theta^{(\mathcal{E}, J)} \rangle = 1$ if and only if $\Theta^{(\mathcal{E}, J)} = P^{p^{k-1}} P^{p^{k-2}} \ldots P^p P^1$ and the corresponding summands are those indicated in the statement. ■

PROPOSITION 11. – $\delta_n(\xi_k) = (-1)^k Q_{n,0}^{-1} Q_{n,k} \in \Gamma_n \subset \Delta_n$ *for each* $k \geq 1$.

PROOF. – The relation above holds for $n = k$ since $Q_{n,n} = 1$ and $Q_{n,0}^{-1} Q_{n,n} = Q_{n,0}^{-1} Q_{n,k} = w_1^{-p^{n-1}} w_2^{-p^{n-2}} \ldots w_n^{-1}$. If $k > n$, then $\delta_n(\xi_k) = 0 = Q_{n,0}^{-1} Q_{n,k}$ as, by convention, $Q_{n,k} = 0$ in this case. So let $n > k$ and suppose that $\delta_{n-1}(\xi_k) = Q_{n-1,0}^{-1} Q_{n-1,k}$. The following relations hold:

$$Q_{n,s} = Q_{n-1,0}^{p-1} Q_{n-1,s} w_n + Q_{n-1,s-1}^{p} w_n^0$$

$$Q_{n,0} = Q_{n-1,0}^{p} w_n = w_1^{p^{n-1}} w_2^{p^{n-2}} \ldots w_n$$

$$V_n^{p-1} = Q_{n-1,0}^{p-1} w_n.$$

Hence,

$$Q_{n-1,0} Q_{n,k} = (Q_{n-1,0}^{-p} w_n^{-1})(Q_{n-1,0}^{p-1} Q_{n-1,k} w_n + Q_{n-1,k-1}^{p} w_n^0)$$

$$= Q_{n-1,0}^{-1} Q_{n-1,k} + (Q_{n-1,0}^{-1} Q_{n-1,k-1})^p w_n^{-1}.$$

By the induction hypothesis, we know that

$$Q_{n-1,0}^{-1} Q_{n-1,k} = \sum w_{j_1}^{-p^{k-1}} w_{j_2}^{-p^{k-2}} \ldots w_{j_k}^{-1},$$

where the sum runs over all integers j_i such that $1 \leq j_1 < \ldots < j_k \leq n-1$. Thus:

$$Q_{n-1,0}^{-1} Q_{n-1,k} + (Q_{n-1,0}^{-1} Q_{n-1,k-1})^p w_n^{-1} =$$

$$(\sum w_{j_1}^{-p^{k-1}} w_{j_2}^{-p^{k-2}} \ldots w_{j_k}^{-1}) + (\sum w_{j_1}^{-p^{k-2}} w_{j_2}^{-p^{k-3}} \ldots w_{j_{k-1}}^{-1})\, w_n^{-1} =$$

$$\sum_J w^{-J} + \sum w_{j_1}^{-p^{k-1}} w_{j_2}^{-p^{k-2}} \ldots w_{j_{k-1}}^{-p} = \sum_J w^{-J} + \sum_{J'} w^{-J'},$$

where the symbol J denote sequences of length n with the last element zero and others $n-1-k$ zeros are inserted among places from 1 to $n-1$, and the symbols J' denote sequences of length n with the last element equal to 1 and others $n-k$ zeros are inserted among places from 1 to $n-1$. Then we get

$$(-1)^k Q_{n,0}^{-1} Q_{n,k} = (-1)^k \sum w_{j_1}^{-p^{k-1}} w_{j_2}^{-p^{k-2}} \dots w_{j_k}^{-1} = \delta_n(\xi_k),$$

the sum being over $(j_1, \dots, j_k)$, where $1 \leqslant j_1 < \dots < j_k \leqslant n$. ∎

PROPOSITION 12. – $\delta_n(\tau_k) = (-1)^{k+1} R_{n;k} Q_{n,0}^{-1}$ *for each* $0 \leqslant k \leqslant n-1$.

PROOF. – From (3), $R_{n;k} Q_{n,0}^{-1} = \sum_{r=k+1}^{n} u_r Q_{r-1,0}^{-1} Q_{r-1,k}$. We want to prove that:

$$R_{n;k} Q_{n,0}^{-1} = \sum_{r=k+1}^{n} u_r w_{j_1}^{-p^{k-1}} w_{j_2}^{-p^{k-2}} \dots w_{j_k}^{-1},$$

with $1 \leqslant j_1 < \dots < j_k \leqslant r-1$. But this directly follows from the previous Proposition, since we have shown that

$$Q_{r-1,0}^{-1} Q_{r-1,k} = \sum_{1 \leqslant j_1 < \dots < j_k \leqslant r-1} w_{j_1}^{-p^{k-1}} w_{j_2}^{-p^{k-2}} \dots w_{j_k}^{-1}. \qquad ∎$$

COROLLARY 13. – *For* $S = (s_1, \dots, s_k)$, $1 \leqslant s_1 < \dots < s_k$ *and* $R = (r_1, \dots, r_l)$, $r_i \geqslant 0$, $l \geqslant 1$,

$$\delta_n(\tau_S \xi^R) = (-1)^{r(S,R)} R_{n;s_1} \dots R_{n;s_k} Q_{n,0}^{r_0} Q_{n,1}^{r_1} \dots Q_{n,l}^{r_l},$$

where $r_0 = -k - (r_1 + \dots + r_l)$.

We have proved the following

THEOREM 14. – $S_n(z) = \sum_{S,R} (-1)^{r(S,R)} R_{n;s_1} \dots R_{n;s_k} Q_{n,0}^{r_0} \dots Q_{n,l}^{r_l} \otimes St^{S,R}(z)$.

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