Bollettino
Unione Matematica Italiana

S. Dragotti, G. Magro, L. Parlato

On the groups $\Theta_n^{\mathcal{F}}$ of a sphere

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **3-B** (2000), n.2, p. 337–346.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2000_8_3B_2_337_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2000.

Bollettino U. M. I.
(8) 3-B (2000), 337-346

On the Groups $\Theta_n^{\mathcal{F}}$ of a Sphere.

S. Dragotti - G. Magro - L. Parlato (*)

Sunto. – *In questo articolo studiamo i gruppi $\Theta_h^{\mathcal{F}}$ di una sfera S^n e proviamo che il gruppo $\Theta_n^{\mathcal{F}}(S^n, x_0)$ è isomorfo all'ennesimo gruppo di omotopia di (S^n, x_0), nell'ipotesi che $\mathcal{F}$ sia una classe coconnessa di links ammissibili.*

Introduction.

This paper is concerned with the properties of a functor $\Theta^{\mathcal{F}}$ associated to a manifold class $\mathcal{F}$, and its action on standard spaces.

A manifold class is a graded collection $\mathcal{F} = \{\mathcal{F}_h\}_{h \geq 0}$ of compact polyhedra, defined up to a PL-isomorphism, closed under link and join, and such that $S^0 \in \mathcal{F}_0$ ($S^n =$ standard PL-sphere).

The collection $\mathcal{C}$ of the geometric cycles without boundary is so, and for each $\mathcal{F}$ such that $\mathcal{F}_0 = \{S^0\}$ we have $\mathcal{F} \subseteq \mathcal{C}$.

The collection $\mathcal{PL}$ of the standard PL-spheres is so, and $\mathcal{PL} \subseteq \mathcal{F}$ for each manifold class $\mathcal{F}$.

A polyhedron $\Sigma \in \mathcal{F}_h$ is called $\mathcal{F}_h$-sphere, a polyhedron P of the form $\Sigma - \overset{\circ}{st}(x, \Sigma)$ is called $\mathcal{F}_h$-pseudodisc. $\mathcal{F}_h$-spheres and $\mathcal{F}_h$-pseudodiscs are allowable links for a theory of generalized manifolds: the $\mathcal{F}$-manifolds, and a subsequent cobordism theory: the $\mathcal{F}$-cobordism.

A manifold class $\mathcal{F}$ with some additional property determines geometrically a covariant functor $\Theta^{\mathcal{F}}$ which assigns to every pointed pair of topological spaces (X, A, x_0) a graded group $\Theta^{\mathcal{F}}(X, A, x_0)$, just as $\mathcal{P}L$ determines the classical functor π using $\mathcal{F}$-spheres and $\mathcal{F}$-pseudodiscs instead of PL-spheres and discs, and just as $\mathcal{C}$ determines the classical homology functor H (see [3], [9]).

Every $\Theta^{\mathcal{F}}$ satisfies the first six axioms of E.S. (excision is excluded).

If $\mathcal{F}' \subseteq \mathcal{F}$ there exists a canonical homomorphism (forgetful) of graded groups $\psi_{\mathcal{F}', \mathcal{F}}$: $\Theta^{\mathcal{F}'}(X, A, x_0) \to \Theta^{\mathcal{F}}(X, A, x_0)$ which allows to factorize the

(*) 1991 *Mathematics Subject Classification*: 55 P 65, 57 Q 20, 55 P 40.
Research supported by M.U.R.S.T.(40%).

classical Hurewicz homomorphism

$$\begin{tikzcd}[column sep=small] \pi(X, x_0) \arrow[rr, "\psi_{\mathcal{PL}, \mathcal{C}}"] \arrow[dr, "\psi_{\mathcal{PL}, \mathcal{F}}"'] & & H(X, x_0) \\ & \Theta^{\mathcal{F}}(X, x_0) \arrow[ur, "\psi_{\mathcal{F}, \mathcal{C}}"'] & \end{tikzcd} \tag{1}$$

About the groups $\Theta_h^{\mathcal{F}}$ associated to a standard PL-sphere $S^n (n \geqslant 2)$ we know them of dimension smaller than n: they are the null group. This follows, by some easy expedient, from the $(n-1)$-connectivity of S^n. Almost nothing is known about this groups. It is natural to expect that $\Theta_h^{\mathcal{F}}(S^n, *)$ is isomorphic to $\boldsymbol{Z}$ (as it happens for homotopy and homology). It is rather surprising that this is not true: in [6] is given a manifold class $\overline{\overline{\mathcal{F}}}$ such that the forgetful homomorphism $\psi_{\overline{\overline{\mathcal{F}}}, \mathcal{C}}\colon \Theta_n^{\overline{\overline{\mathcal{F}}}}(S^n, *) \to H_n(S^n, *)$ is an epimorphism not injective, and hence $\Theta_n^{\overline{\overline{\mathcal{F}}}}(S^n, *)$ cannot be isomorphic to $\boldsymbol{Z}$.

In this paper we prove that $\Theta_n^{\mathcal{F}}(S^n, *) \simeq \boldsymbol{Z}$ in the case which $\mathcal{F}$ satisfies the additional property called coconnection (definition in 2). However this condition is sufficient but not necessary because the class $\mathcal{C}$ of the geometric cycles is not coconnected (but $\Theta_n^{\mathcal{C}}(S^n, *) \simeq H_n(S^n, *) \simeq \boldsymbol{Z}$!).

The metod of doing the result is that to prove that for $X = S^n$ and $n \geqslant 1$ the homomorphism $\psi_{\mathcal{PL}, \mathcal{F}}\colon \pi_n(S^n, *) \to \Theta_n^{\mathcal{F}}(S^n, *)$ of the commutative diagram (1) is an isomorphism (theorem 3.2). Being in this case $\psi_{\mathcal{PL}, \mathcal{F}}$ injective, it suffices to prove only that it is onto. This is achieved by using the surjectivity of the map s which in the classical homotopy theory is called suspension homomorphism

$$\Theta_{n-1}^{\mathcal{F}}(S^{n-1}, x_0) \underset{\sim}{\overset{\partial}{\longleftarrow}} \Theta_n^{\mathcal{F}}(D_+^n, S^{n-1}, x_0) \xrightarrow{i} \Theta_n^{\mathcal{F}}(S^n, D_-^n, x_0) \underset{\sim}{\overset{j}{\longleftarrow}} \Theta_n^{\mathcal{F}}(S^n, x_0)$$

$$s = j^{-1} \circ i \circ \partial^{-1}$$

where the boundary homomorphism ∂ and j also are isomorphisms by standard properties of the involved spaces and by the homotopy and exacteness axioms of $\Theta^{\mathcal{F}}$.

Hence, the probleme is again reduced to state that the homomorphism i is onto (theorem 3.1).

If we exclude the above discussion and a few others, the tecniques employed are entirely geometric (lemma 2.1 and theorem 3.1 also).

The original construction of the functor $\Theta^{\mathcal{F}}$ is developed in [3]. The papers [4], [5], [6] contain more closely investigations about their basic properties, and their behaviour in interesting special cases.

In order to make the current paper self-contained enough that the main results can be understood we include a section that provides the definition of the functors $\Theta^{\mathcal{F}}$.

1. – The functor associated to a manifold class.

A manifold class $\mathcal{F}$ is said to be connected if the polyhedron obtained attaching two $\mathcal{F}_h$-pseudodiscs, by a PL-homeomorphism between their boundaries (if there exists), is an $\mathcal{F}_h$-sphere.

Let $\mathcal{F}$ be a connected manifold class such that $\mathcal{F}_0 = \{S^0\}$. The last hypothesis implies that any $\mathcal{F}$-manifold M is a geometric cycle, so it makes sense to define M to be orientable if M is orientable as geometric cycle. If M denotes an oriented $\mathcal{F}$-manifold, then $-$M will denote the same manifold with the opposite orientation.

An $\mathcal{F}$-cobordism between two oriented $\mathcal{F}_h$-spheres Σ_1 and Σ_2 is an oriented $\mathcal{F}$-manifold W such that:

a) ∂W is the disjoint union of Σ_1 and $-\Sigma_2$;

b) $W \cup c_1 * \Sigma_1 \cup c_2 * \Sigma_2$ is an $\mathcal{F}_{h+1}$-sphere.

An $\mathcal{F}$-cobordism between two oriented $\mathcal{F}_h$-pseudodiscs P_1 and P_2 is an oriented $\mathcal{F}$-manifold W such that:

a') $\partial W = P_1 \cup P_2 \cup W_0$, where W_0 is a cobordism between ∂P_1 and ∂P_2;

b') $W \cup c_1 * P_1 \cup c_2 * P_2$ is an $\mathcal{F}_{h+1}$-pseudodisc.

Let (X, x_0) be a pointed topological space. A singular $\mathcal{F}$-sphere of (X, x_0) is a triple (Σ, Δ, f), where Σ is an oriented $\mathcal{F}$-sphere, $\Delta \subseteq \Sigma$ is a top dimensional simplex and $f:(\Sigma, \Delta) \to (X, x_0)$ a continuous map.

Two singular $\mathcal{F}_h$-spheres $(\Sigma_1, \Delta_1, f_1)$ and $(\Sigma_2, \Delta_2, f_2)$ are said $\mathcal{F}$-cobordant if there exists a triple (W, W', g), called $\mathcal{F}$-cobordism, where W is an $\mathcal{F}$-cobordism between Σ_1 and Σ_2, $W' \subseteq W$ is a PL-disc such that $W' \cap \Sigma_i = \Delta_i$, i=1,2 , and $g:(W, W') \to (X, x_0)$ is a continuous map, such that: $g/\Sigma_i = f_i$, i=1,2.

The $\mathcal{F}$-cobordism between $\mathcal{F}$-spheres in an equivalence relation.

Let $\Theta_h^{\mathcal{F}}(X, x_0)$ denote the set of the $\mathcal{F}$-cobordism classes of singular $\mathcal{F}_h$-spheres of (X, x_0).

As in the case of the homotopy theory we can geometrically define an addition in $\Theta_n^{\mathcal{F}}(X, x_0)$ $(h \geqslant 1)$ which give a group structure.

Let (X, A) be a pair of topological spaces and let x_0 be a point of A. By relative $\mathcal{F}_h$-sphere of (X, A, x_0) we mean a triple (P, Δ, f), where P is an oriented $\mathcal{F}_h$-pseudodisc, $\Delta \subseteq P$ is a top-dimensional simplex meeting ∂P in a top-dimensional simplex, and $f:(P, \Delta) \to (X, x_0)$ is a map which carries ∂P to A.

Given a relative $\mathcal{F}_h$-sphere (P, Δ, f) of (X, A, x_0), $(\partial P, \Delta \cap \partial P, f/)$ is a singular $\mathcal{F}_{h-1}$-sphere of (A, x_0) which will be denoted by $\partial(P, \Delta, f)$.

Two relative $\mathcal{F}$-spheres (P_i, Δ_i, g_i), i=1,2, of (X, A, x_0) are called $\mathcal{F}$-cobordant if there exists a triple (V, V', G) where V is an $\mathcal{F}$-cobordism between P_1 and P_2, $V' \subseteq V$ a $\mathcal{PL}$-cobordism between Δ_1 and Δ_2, and

$G:(V, V') \to (X, x_0)$ is a continuous map such that the following conditions hold:

(1) $V' \cap P_i = \Delta_i$, i=1,2;

(2) Let $W = \partial V - (\mathring{P}_1 \cup \mathring{P}_2)$ (1) and $W' = W \cap V'$. Then $(W, W', G/)$ is an $\mathcal{F}$-cobordism between $\partial(P_1, \Delta_1, g_1)$ and $\partial(P_2, \Delta_2, g_2)$ with $G(W) \subseteq A$.

The $\mathcal{F}$-cobordism between relative spheres is an equivalence relation.

Let $\Theta_h^{\mathcal{F}}(X, A, x_0)$ denote the set of the $\mathcal{F}$-cobordism classes of relative $\mathcal{F}_h$-spheres of (X, A, x_0). As before, we can introduce in $\Theta_h^{\mathcal{F}}(X, A, x_0)$ $(n \geq 2)$ a group structure.

Given a continuous map $f:(X, A, x_0) \to (Y, B, y_0)$, we can define, for each $h \geq 2$, a homomorphism $\Theta^{\mathcal{F}}(f)$: $\Theta_h^{\mathcal{F}}(X, A, x_0) \to \Theta_h^{\mathcal{F}}(Y, B, y_0)$ by setting

$$\Theta^{\mathcal{F}}(f)([(P, \Delta, g)]) = [(P, \Delta, f \circ g)].$$

As proved in [3] the definitions above recalled allow us to build a covariant functor, from the cathegory of pointed pairs of topological spaces to the cathegory of graded groups, satisfying the first six axioms of Eilenberg and Steenrod (excision is excluded).

2. – A geometric lemma.

Our theorem 3.1 depends on a geometric result which allows to engulf (include into a PL-disc) a finite subset of a compact, connected $\mathcal{F}$-manifold in the case when $\mathcal{F}$ satisfies an additional axiom.

In order to state and prove the lemma involved (and theorem 3.1 again) a few preliminary results are necessary. We reporte these briefly. For details and proofs, see [2].

DEFINITION. – *A manifold class $\mathcal{F}$ is said to be coconnected if for every $\mathcal{F}_h$-sphere Σ and $\mathcal{F}_h$-pseudodisc $P \subset \Sigma$, the polyhedron $\Sigma - \mathring{P}$ is an $\mathcal{F}_h$-pseudodisc.*

PROPOSITION a. – *Let $\mathcal{F}$ be a connected, coconnected manifold class. All the $\mathcal{F}$-spheres and $\mathcal{F}$-pseudodiscs of positive dimension are connected. The only $\mathcal{F}_0$-sphere is S^0.*

PROPOSITION b. – *Let $\mathcal{F}$ be a connected, coconnected manifold class, and let $P' \subseteq P$ be $\mathcal{F}_h$-pseudodiscs such that $\partial P \cap \partial P' = P''$ is an $\mathcal{F}_{h-1}$-pseudodisc, then the polyhedron $P - (\mathring{P}' \cup \mathring{P}'')$ is an $\mathcal{F}_h$-pseudodisc.*

(1) $\mathring{P}$ stands for $P - \partial P$.

PROPOSITION c. – *Let $\mathcal{F}$ be a connected, coconnected manifold class. The cylinder $P \times I$ on an $\mathcal{F}_{h-1}$-pseudodisc P is an $\mathcal{F}_h$-pseudodisc.*

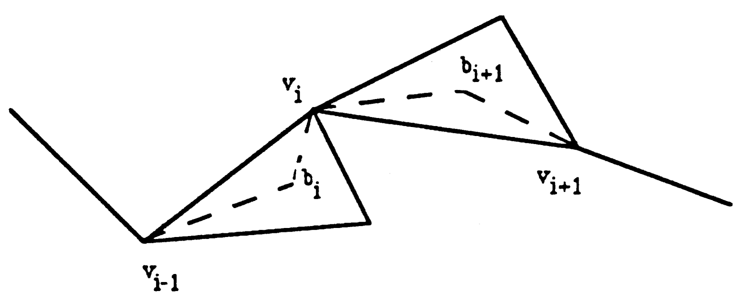
PROPOSITION d. – *Let $\mathcal{F}$ be a connected, coconnected manifold class and let $M' \subseteq M$ be two $\mathcal{F}_h$-manifolds such that $M' \cap \partial M$ is the empty set or an $\mathcal{F}_{h-1}$-manifold of ∂M, then the polyhedron $M - intM'$ is an $\mathcal{F}$-manifold.*

Now we prove the announced result

LEMMA 2.1. – *Let M be a compact, connected $\mathcal{F}$-manifold, where $\mathcal{F}$ is a connected, coconnected manifold class. If x, y are regular*[2] *points of M, there exists a regular polygonal path X in M from x to y which meets the boundary ∂M of M at most in x, y.*

PROOF. – We use induction on the dimension m of M. By the Proposition a, it follows that $\mathcal{F}^0 = \{S^0\}$, $\mathcal{F}^1 = \{S^1\}$ and hence an $\mathcal{F}$-manifold of dimension 1 or 2 is a PL-manifold, then the assertion is true for m=1,2.

Assume the result for $m \leqslant h-1$ and let M be a compact, connected $\mathcal{F}$-manifold of dimension h and x, y regular points of M. Being M a connected polyhedron, there exists a triangulation L of M and a simplicial path of L, Y, joining x and y. Let $x = v_0, \ldots, v_r = y$ be the vertices of Y. If Y satisfies the required properties, we take this. If not, we can modify Y as follows. The 1-simplex $v_{i-1}v_i$ certainly lies in the boundary of some top-dimensional simplex σ_i of L, then we can replace $v_{i-1}v_i$ by the polygonal path $v_{i-1}b_iv_i$ where b_i is the barycentre of σ_i.



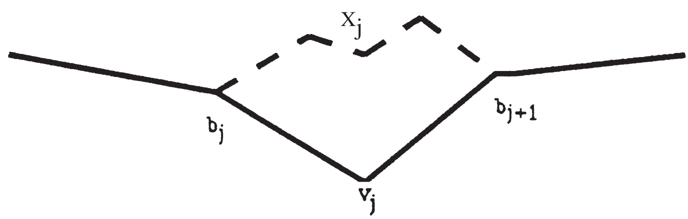
We repeat that for all the line segments of Y. So, we obtain a polygonal path

$$Z = v_0 b_1 v_1 b_2 v_2 \ldots v_r$$

(2) A point $x \in M$ is said to be regular if link(x, M) is PL-homeomorphic to a standard PL-sphere or PL-disc. A subset X is said to be regular if all their points are regular points.

joining again x and y and such that all the points of Z different from the vertices v_i are regular points and lies in interior $\overset{\circ}{M}$ of M.

If necessary, we modifie Z as follows: let v_j be a «bad» vertex of Z and let L_j its link in the first barycentric subdivision L' of L. By Proposition a, L_j is a connected $\mathcal{T}_{h-1}$-manifold (sphere or pseudodisc) for which our induction hypothesis holds. So, there exists a regular polygonal path $X_j \subseteq \overset{\circ}{L}_j$ joining b_j and b_{j+1}.



The points of X_j are regular also in M, because L_j is bicollared in M. Moreover the points of X_j are interior to L_j and hence to M, because $L_j \cap \partial M = \partial L_j$.

Then we replace the polygonal path $b_j v_j b_{j+1}$ by the polygonal path X_j. We repeat that for all vertices of Z for which it is necessary, and finally we obtain a polygonal path X which satisfies our requests. So the inductive step is established. ■

REMARK. – The polygonal path X of the above lemma can be arranged, merely by cutting loops, so that it is collapsible.

On the other hand, being the points of X all regular, a regular neighbourhood N of X in M is a PL-manifold. Hence, if X is collapsible, N is a PL-disc.

3. – The main theorem.

Let D_+^n and D_-^n be the northern and the southern hemispheres of S^n ($n \geq 2$), $p_1 \in \overset{\circ}{D}{}_+^n$ the north pole, $p_2 \in \overset{\circ}{D}{}_-^n$ the south pole, and let x_0 be a fixed point of $S^{n-1} = \dot{D}_+^n = \dot{D}_-^n$.

Consider the pointed pairs (D_+^n, S^{n-1}, x_0), $(S^n - p_2, S^n - (p_1 \cup p_2), x_0)$ and for each $h \geq 2$ the diagram

$$\begin{array}{ccc}
\Theta_h^{\mathcal{T}}(D_+^n, S^{n-1}, x_0) & \xrightarrow{\partial_1} & \Theta_{h-1}^{\mathcal{T}}(S^{n-1}, x_0) \\
\big\downarrow{\scriptstyle i_1} & & \big\downarrow{\scriptstyle j_1} \\
\Theta_h^{\mathcal{T}}(S^n - p_2, S^n - (p_1 \cup p_2), x_0) & \xrightarrow{\partial_2} & \Theta_{h-1}^{\mathcal{T}}(S^n, -(p_1 \cup p_2), x_0))
\end{array}$$

where the vertical maps are induced by inclusion maps. Because D_+^n and $S^n - p_2$ are contractible, the homotopy and the exactness axioms of the functor $\Theta^{\mathcal{T}}$ assure that the connecting homomorphisms ∂_1, ∂_2 are isomorphisms. Moreover S^{n-1} is a strong deformation retract of $S^n - (p_1 \cup p_2)$, and hence j_1 is an isomorphism. From the trivial commutativity of the above diagram it follows that i_1 is an isomorphsm.

A similar argument for the pointed pairs (S^n, D_-^n, x_0), $(S^n, S^n - p_1, x_0)$ shows that for each $h \geq 2$ also the homomorphism

$$i_2 \colon \Theta_h^{\mathcal{T}}(S^n, D_-^n, x_0) \to \Theta_h^{\mathcal{T}}(S^n, S^n - p_1, x_0)$$

is an isomorphism.

These two results provide a convenient approach to prove the following

THEOREM 3.1. – *Let $\mathcal{T}$ be a connected, coconnected manifold class. For each $n \geq 2$ the homomorphism $i : \Theta_n^{\mathcal{T}}(D_+^n, S^{n-1}, x_0) \to \Theta_n^{\mathcal{T}}(S^n, D_-^n, x_0)$ induced by inclusion is onto.*

PROOF. – For each $h \geq 2$ we can consider the commutative diagram

$$\begin{array}{ccc} \Theta_h^{\mathcal{T}}(D_+^n, S^{n-1}, x_0) & \xrightarrow{i_1} & \Theta_h^{\mathcal{T}}(S^n - p_2, S^n - (p_1 \cup p_2), x_0) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ \Theta_h^{\mathcal{T}}(S^n, D_-^n, x_0) & \xrightarrow{i_2} & \Theta_h^{\mathcal{T}}(S^n, S^n - p_1, x_0 = \end{array}$$

where j is induced by inclusion. Being i_1 and i_2 isomorphisms, to prove that i is onto it is equivalent to prove that j is onto.

Then, we now show that if $h = n$ the homomorphism

$$j : \Theta_n^{\mathcal{T}}(S^n - p_2, S^n - (p_1 \cup p_2), x_0) \to \Theta_n^{\mathcal{T}}(S^n, S^n - p_1, x_0)$$

is onto.

Let α be any element of $\Theta_n^{\mathcal{T}}(S^n, S^n - p_1, x_0)$ and let (P, Δ, f) be a representative triple of α, that is $f \colon P \to S^n$, $f(\partial P) \subseteq S^n - p_1$, $f(\Delta) = x_0$. Up to a homotopy we can suppose that f is a simplicial map, p_1 is the barycentre of an n-simplex τ of S^n, and also p_2 is the barycentre of an n-simplex σ.

We need a representative element (P', Δ', f') of α such that $f'(P') \subseteq S^n - p_2$, $f'(\partial P') \subseteq S^n - (p_1 \cup p_2)$, $f'(\Delta') = x_0$.

If $f^{-1}(p_2) = \emptyset$, we take $P' = P$, $\Delta' = \Delta$, $f' = f$.

If not, being f a simplicial map, $f^{-1}(p_2)$ consists of a finite number of regular points $b_1, \ldots, b_r$: the barycentres of the top dimensional simplexes $\sigma_1, \ldots, \sigma_r$ of the triangulation of P such that $f(\sigma_1) = \ldots = f(\sigma_r) = \sigma$.

Similary $f^{-1}(p_1)$ consists at most of a finite number of regular points: the barycentres of the n-simplexes $\tau_1, \ldots, \tau_s$ such that $f(\tau_1) = \ldots = f(\tau_s) = \tau$.

Let $M = P - \bigcup_{l=1,\dots,s} \mathring{\tau}'_l(\mathring{\Delta} \cup (\Delta \mathring{\cap} \partial P))$, where $\tau'_l \subsetneq \mathring{\tau}_l$. Since P is a connected polyhedron (see Prop. a), and also $\dot{\tau}'_l$ is connected (because $n \geq 2$), the polyhedron M is a connected $\mathcal{F}$-manifold (Prop. d).

Let b a regular point of $\partial P - \Delta$. Since $f(b) \neq p_1$, then $b \in M \cap \partial P \subsetneq \partial M$.

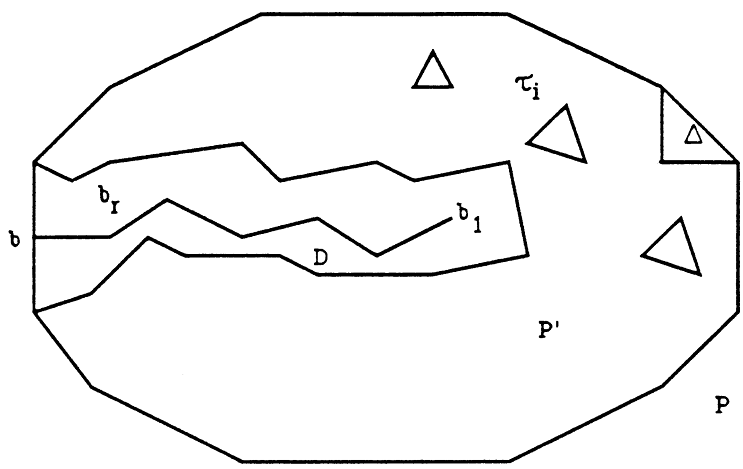
By lemma 2.1 and relative remark, there exists a regular and collapsible polygonal path X joining the points $b_1, \dots, b_r, b$ such that $X - b \subsetneq M - \partial M$, and hence $f(X) \subsetneq S^n - p_1$.

Standard arguments of PL-topology assure that an ε-neighbourhood of X is a PL-disc D, and there exists $\varepsilon > 0$ sufficiently small so that such a disc satisfies

$$f(D) \subsetneq S^n - p_1.$$

Obviously $f(\partial D) \subsetneq S^n - (p_1 \cup p_2)$.

$D \cap \partial M = D \cap \partial P$ is a PL-disc D', the star of b in ∂P, then (Prop. b) the polyhedron $P - (\mathring{D} \cup \mathring{D}')$ is an $\mathcal{F}$-pseudodisc P',



and we have $f(P') \subsetneq S^n - p_2$ (because $D \supsetneq X \supsetneq f^{-1}(p_2)$).

$$f(\partial P') = f(\partial P \cup \partial D - \mathring{D}') \subsetneq S^n - (p_1 \cup p_2).$$

Finally, being $\Delta \subset P'$, the triple $(P', \Delta, f_{/P'})$ determines an element β of $\Theta_n^{\mathcal{F}}(S^n - p_2, S^n - (p_1 \cup p_2), x_0)$.

In order to prove that $j(\beta) = \alpha$ it remains to construct an $\mathcal{F}$-cobordism (V, V', G) between (P, Δ, f) and $(P', \Delta, f_{/P'})$.

Let $V = P \times I$, $V' = \Delta \times I$ and $G = f \times id$. Being $g(\partial V - (P \cup P') \subsetneq S^n -$

P_1, by Prop. c it is straightforward to verify that the triple $(P \times I, \Delta \times I, f \times id)$ satisfies the required conditions. ■

Now we are able to prove the following

THEOREM 3.2. – *Let $\mathcal{F}$ be a connected, coconnected manifold class. The forgetful homomorphism $\psi_{\mathcal{PL},\mathcal{F}}\colon \pi_n(S^n, x_0) \to \Theta_n^{\mathcal{F}}(S^n, x_0)$ is an isomorphism for each $n \geqslant 1$. Hence $\Theta_n^{\mathcal{F}}(S^n, x_0)$ is isomorphic to $\boldsymbol{Z}$.*

PROOF. – Consider the commutative diagram (see introduction)

$$\begin{tikzcd} \pi_n(S^n, x_0) \arrow[rr, "\psi_{\mathcal{PL},\mathcal{C}}"] \arrow[dr, "\psi_{\mathcal{PL},\mathcal{F}}"'] & & H_n(S^n, x_0) \\ & \Theta_n^{\mathcal{F}}(S^n, x_0) \arrow[ur, "\psi_{\mathcal{F},\mathcal{C}}"'] & \end{tikzcd}$$

In this case the Hurewicz homomorphism $\psi_{\mathcal{PL},\mathcal{C}}$ is an isomorphism for each $n \geqslant 1$, and hence $\psi_{\mathcal{PL},\mathcal{F}}$ is injective.

Then, in order to state the assert, we only need to show that $\psi_{\mathcal{PL},\mathcal{F}}$ is onto. That is: for any element $a \in \Theta_n^{\mathcal{F}}(S^n, x_0)$ there exists a representative triple of the form (S^n, Δ, f).

This is trivial if $n = 1$, because by Prop. a it follows $\mathcal{F}^1 = \{S^1\}$.

Now we suppose $n \geqslant 2$. The theorem 3.1 assures that the suspension homomorphism

$$s = j^{-1} \circ i \circ \partial^{-1}\colon \Theta_{n-1}^{\mathcal{F}}(S^{n-1}, x_0) \to \Theta_n^{\mathcal{F}}(S^n, x_0)$$

is onto (j, ∂ are isomorphisms).

On the other hand we can observe that s may be geometrically regarded as follows

$$s([\Sigma, \Delta, f]) = [s\Sigma, \Delta', f']$$

($s\Sigma$ stands for suspension of Σ), where f' is the map (homotopic to sf) which coincides with $f \times id$ on a bicollar C of Σ in $s\Sigma$ and with sf elsewhere, and Δ' is a top dimensional simplex of C contained in $\Delta \times I$.

Then the assert follows by induction. ■

Our approach to problem has the incidental avantage of proving the following

COROLLARY 3.3. – *The suspension homomorphism $s : \Theta_{n-1}^{\mathcal{F}}(S^{n-1}, x_0) \to \Theta_n^{\mathcal{F}}(S^n, x_0)$ is an isomorphism.*

PROOF. – A homomorphism of $\boldsymbol{Z}$ onto itself is necessary an isomorphism. ∎

REFERENCES

[1] G. A. ANDERSON, *Resolutions of generalized polyhedral manifolds*, Tôhoku Mat. Journ., **31** (1979), 495-517.
[2] S. DRAGOTTI - R. ESPOSITO - G. MAGRO, *$\mathcal{F}$-varietà e funzioni dual cellulari*, Ric. Mat., **33** (1990), 21-33.
[3] S. DRAGOTTI - R. ESPOSITO - G. MAGRO, *Construction of functors connecting homology and homotopy theories*, Proc. of Amer. Math. Soc., **120** (1994), pp. 635-646.
[4] S. DRAGOTTI - R. ESPOSITO - G. MAGRO - L. PARLATO, *Geometric representation of homology functor*, Ricerche di Matematica, **45** (1996), 3-12.
[5] S. DRAGOTTI - G. MAGRO, *A representation of homotopy theory by homotopy spheres*, Ricerche di Matematica, **46** (1997), 13-22.
[6] R. ESPOSITO - L. PARLATO, *I teoremi di Hurewicz e Whitehead per funtori omotopici associati a manifold classes*, Ricerche di Matematica, **44** (1995), 359-368.
[7] C. R. F. MAUNDER, *Algebraic topology*, Van Nostrand, London, 1970.
[8] C. P. ROURKE - B. J. SANDERSON, *Introduction to PL topology*, Ergeb. Mat. Band. 69, Springer-Verlag, Berlin and New York, 1972.
[9] S. BUONCRISTIANO - C. P. ROURKE - B. J. SANDERSON, *A geometric approach to homology theory*, Cambridge University Press - London - New York, 1976.
[10] E. H. SPANIER, *Algebraic Topology*, McGraw-Hill, New York, 1966.
[11] J. H. C. WHITEHEAD, *Note on suspension*, Quart. J. Math. Oxford (2), **1** (1950), 9-22.

Dipartimento di Matematica e Applicazioni Renato Caccioppoli
Via Cintia, 80126 Napoli, Italia

Pervenuta in Redazione
il 13 novembre 1997