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New Optimal Regularity Results for Infinite Dimensional Elliptic Equations.

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Sunto. – *In questo articolo si ottengono stime di Schauder di tipo nuovo per equazioni ellittiche infinito-dimensionali del secondo ordine con coefficienti Hölderiani a valori nello spazio degli operatori Hilbert-Schmidt. In particolare si mostra che la derivata seconda delle soluzioni è Hilbert-Schmidt.*

1. – Introduction and setting of the problem.

In this paper we are concerned with the infinite dimensional elliptic equation

$$\lambda u(x) - \frac{1}{2}\operatorname{Tr}[Q(x)\, D^2 u(x)] = f(x), \qquad x \in H,\ \lambda > 0, \tag{1.1}$$

where H is a real separable Hilbert space and f, $u : H \mapsto \mathbb{R}$ belong to $C_b(H)$, the space of all real bounded uniformly continuous functions.

Elliptic equations with infinitely many variables have applications in several domains as Field Theory, Dirichlet Forms and Statistical Mechanics (see Ma and Röckner [14], Stroock [19], Berezansky and Kondratiev [1]). A further motivation to study equation (1.1) comes from the well known connection with the stochastic differential equation

$$dX(t) = Q^{1/2}(X(t))\, dW(t). \tag{1.2}$$

Equations like (1.2) can be treated by usual techniques if $Q(x)$ is Lipschitz continuous with respect to x, see for instance [6]. However, solving directly equation (1.1), allows to establish existence and uniqueness in law for solutions of (1.2) also when the coefficients are only Hölder continuous (we refer to [23], [24] for details).

Equation (1.1) has been studied by Gross [8] and Dalecky (see [4]) in case when $Q(x) = Q$, Q being a positive self-adjoint trace-class operator. They proved existence and uniqueness of solutions by probabilistic arguments. Later A. Piech (see [15]) has constructed a fundamental solution in the case:

$$Q(x) = Q^{1/2}(I + F(x))\,Q^{1/2}, \qquad x \in H\,, \tag{1.3}$$

where $F(x)$ is a family of trace-class operators, satisfying strong smoothness assumptions. Let us remark that existence and uniqueness of viscosity solutions for equation (1.1) can be established (see [9], [12], [20]).

Cannarsa and Da Prato (see [2] and [3]) have studied equation (1.1) when F is Hölder-continuous from H with values in the space $\mathcal{L}_1(H)$ of trace class operators. They show that when $f \in C_Q^\theta(H)$ (the set of all functions that are $\theta-$Hölder continuous in the directions of $Q^{1/2}H$, $\theta \in]0,1[$), the solution u of (1.1) belongs to $\mathcal{C}_Q^2(H)$ (see below for a precise definition) and its second Q-derivative, $D_Q^2 u$, is a Q-Hölder continuous map with values in the space $\mathcal{L}(H)$ of all bounded linear operators in H. However they give no informations about a typical regularity problem arising in infinite dimensions: whether, for a solution u of (1.1), the bounded linear operator $D_Q^2 u(x)$, $x \in H$, is compact, or of Hilbert-Schmidt type, or of trace class, etc. Because of this lack, in [2], very restrictive hypotheses on F are required.

In this paper we prove that $D_Q^2 u(x)$ is in fact of Hilbert-Schmidt type. Note that in light of the Gross results (see [8]), this seems to be the best possible regularity result for $D_Q^2 u(x)$ even when $F = 0$. Using this result we are able to relax the hypotheses on the coefficients F of (1.1) obtaining again existence and uniqueness theorems for solutions.

Another important phenomenon, typical of the infinite dimensions, is the difficulty of characterizing the domain of the generator $\mathcal{A}$ of the heat semigroup in $\mathcal{C}_b(H)$ and its interpolation spaces $(C_b(H), D(\mathcal{A}))_{\theta/2,\infty}$. This problem arises in the study of the spatial regularity for solutions of elliptic equations like (1.1). When $H = \mathbb{R}^n$, it is well known that the following interpolatory result holds

$$(C_b(\mathbb{R}^n), D(\mathcal{A}))_{\theta/2,\infty} = C_b^\theta(\mathbb{R}^n)\,, \tag{1.4}$$

for $\theta \in]0,1[$. We stress that (1.4) is a key step in the modern treatment of Schauder estimates for (1.1) (see for instance Lunardi [13] and Triebel [22]).

On the contrary, in infinite dimensions, we have the *strict* inclusion

$$(C_b(H), D(\mathcal{A}))_{\theta/2,\infty} \subset C_Q^\theta(H) \tag{1.5}$$

and it is a long standing problem the characterization of $(C_b(H), D(\mathcal{A}))_{\theta/2,\infty}$.

In the case of the equation with constant coefficients

$$\lambda\psi(x) - \frac{1}{2}\,\mathrm{Tr}\,[QD^2\psi(x)] = f(x)\,, \qquad x \in H,\ \lambda > 0\,, \tag{1.6}$$

we prove that, for all $N \in \mathcal{L}_2(H)$, $\mathrm{Tr}\,[ND_Q^2\psi] \in (C_b(H), D(\mathcal{A}))_{\theta/2,\,\infty}$ and

$$\sup_{\|N\|_{\mathcal{L}_2} \leq 1} \|\mathrm{Tr}\,[ND_Q^2\psi]\|_{(C_b(H),\, D(\mathcal{A}))_{\theta/2,\,\infty}} \leq C\|f\|_{\theta,\,Q}, \tag{1.7}$$

where $C = C(\lambda, \theta, Q)$. It is a deep fact that this *sharp* form of Schauder estimates allows to obtain for the general equation (1.1) that $\mathrm{Tr}\,[ND_Q^2 u] \in C_Q^\theta(H)$ and

$$\sup_{\|N\|_{\mathcal{L}_2} \leq 1} \|\mathrm{Tr}\,[ND_Q^2 u]\|_{C_Q^\theta(H)} \leq C\|f\|_{\theta,\,Q} \tag{1.8}$$

which is weaker than (1.7) but nonetheless sufficient in order to prove existence of solutions for (1.1).

It seems that our considerations are a new and consistent contribution to the difficult problem of studying the regularity of domains of differential operators in infinite dimensions.

The paper is organized as follows. In Section 2 we study some regularity properties of the heat semigroup in $C_b(H)$. In Section 3 we present the main results. The first one asserts that the inclusion in (1.5) is strict (see Theorem 3.1). This clarifies that condition (1.7) is stronger than (1.8). The proof uses a recent result by van Neerven and Zabczyk (see [21]). In our second theorem we prove the optimal regularity (1.7) for solutions of equation (1.6) (see Theorem 3.3). To this purpose we use only analytic tools: estimates on the heat semigroup (see Proposition 2.2) and Interpolation Theory (as in [2] and [5]).

Using this result, in Section 4, we are able to treat equations (1.1) when F is only a Q-Hölder-continuous map with values in the space $\mathcal{L}_2(H)$ of Hilbert-Schmidt operators in H. We prove the a priori estimates (1.8) for solutions of (1.1) (see Theorem 4.2). The proof of this result requires a new method and relies on a non standard interpolation lemma of independent interest (see Lemma 4.3), involving Hilbert-Schmidt norms of second derivatives of mappings.

Once we have proved the a priori estimates, by adapting the Maximum Principle and the Continuity Method used in [2], we obtain a theorem of existence, uniqueness and optimal regularity for solutions u of (1.1) (see Theorem 4.6).

We point out that arguments of this paper can be used to improve in the same direction the results of [16] and [23].

Let H be a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$. We denote by $\mathcal{L}(H)$ the Banach space of all bounded linear operators on H, endowed with the norm: $\|T\|_{\mathcal{L}(H)} = \sup_{|v| \leq 1} |Tv|$, $T \in \mathcal{L}(H)$. $\mathcal{L}_1(H)$ denotes the subspace of $\mathcal{L}(H)$ of all trace class operators, i.e.

$$\mathcal{L}_1(H) := \Big\{ T \in \mathcal{L}(H) : \exists a_k,\, b_k \in H,\, k \in \boldsymbol{N},\, \sum_{k=1}^{\infty} |a_k|\,|b_k| < \infty\,,$$

$$Tx = \sum_{k=1}^{\infty} a_k < x,\, b_k\rangle,\, x \in H \Big\}$$

If $T \in \mathcal{L}_1(H)$, the trace norm $\|T\|_1$ is the infimum of $\sum_{k=1}^{\infty} |a_k|\,|b_k|$ over all possible representations. $\mathcal{L}_1(H)$ is a Banach space endowed with the trace norm. If $T \in \mathcal{L}_1(H)$, then the trace of T, $\mathrm{Tr}(T)$, is defined by:

$$\mathrm{Tr}(T) := \sum_{k=1}^{\infty} \langle T g_k,\, g_k \rangle$$

where (g_k) is a complete orthonormal basis in H. This definition is independent of the choice of the basis. $\mathcal{L}_2(H)$ denotes the subspace of $\mathcal{L}(H)$ of all Hilbert-Schmidt operators. $\mathcal{L}_2(H)$ is a Banach space endowed with the norm $\|L\|_2 = \left(\sum_{k=1}^{\infty} |L g_k|^2 \right)^{1/2}$, $L \in \mathcal{L}_2(H)$.

Let Q be a strictly positive self-adjoint trace class operator in H. This means, that there exists a complete orthonormal basis of H, $\{e_k\}_{k \geq 1}$ and a sequence $\lambda_k > 0$, such that

$$Qx = \sum_{k=1}^{\infty} \lambda_k \langle x,\, e_k \rangle\, e_k \quad \forall x \in H\,, \qquad \sum_{k=1}^{\infty} \lambda_k < \infty\,.$$

Moreover $\mathcal{N}(x,\, tQ)$ denotes the Gaussian measure in H with mean $x \in H$ and covariance operator tQ (we refer to [6] for definitions and main properties of Gaussian measures in Hilbert spaces).

We introduce some functions spaces. Let $(E,\, \|\cdot\|_E)$ be a real Banach space, $\mathcal{C}_b(H,\, E)$ stands for the Banach space of all uniformly continuous and bounded functions $f : H \to E$, endowed with the sup norm $\|\cdot\|_0$ (i.e. $\|f\|_0 = \sup_{x \in H} \|f(x)\|_E$.) When we want to emphasize E, we will write $\|f\|_{E,\,0}$ instead of $\|f\|_0$, $f \in \mathcal{C}_b(H,\, E)$.

$\mathcal{C}_b^{\theta}(H,\, E)$, $\theta \in (0,\, 1)$, denotes the subspace of $\mathcal{C}_b(H,\, E)$ consisting of all functions which are θ-Hölder continuous from H into E.

Let $f \in \mathcal{C}_b(H,\, E)$, the modulus of continuity of f will be indicated by ω_f. When $E = \boldsymbol{R}$, we set $\mathcal{C}_b(H) = \mathcal{C}_b(H,\, \boldsymbol{R})$. This convention will be used for all functions spaces. We define other functions spaces related to the operator Q.

DEFINITION 1.1. – $\mathcal{C}_Q^1(H)$ is the set of all $f \in \mathcal{C}_b(H)$ such that:

(i) for any $v \in H$, $x \in H$, there exists the derivative of f at x, in the direction $Q^{1/2} v$ that we denote by $D_{Q^{1/2} v} f(x)$;

(ii) for any $x \in H$, there exists $D_Q f(x) \in H$ such that:

$$D_{Q^{1/2}v} f(x) = \langle D_Q f(x), v \rangle, \qquad \forall v \in H;$$

(iii) the mapping $H \to H$, $x \mapsto D_Q f(x)$ belongs to $\mathcal{C}_b(H, H)$.

It is easy to prove that if $f \in \mathcal{C}^1_Q(H)$, defining the *partial derivatives* $D_k f = D_{e_k} f$, $k \geqslant 1$, we have $D_Q f(x) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} D_k f(x) e_k$, $x \in H$.

$\mathcal{C}^2_Q(H)$ is the set of all functions in $\mathcal{C}^1_Q(H)$ such that:

(i) there exists, for any $v \in H$, $x \in H$, the directional derivative

$$D_{Q^{1/2}v}[D_Q f](x) = \lim_{s \to 0^+} \frac{D_Q f(x + sQ^{1/2} v) - D_Q f(x)}{s} \quad \text{in } H;$$

(ii) for any $x \in H$, there exists $D^2_Q f(x) \in \mathcal{L}(H)$, such that

$$D_{Q^{1/2}v}[D_Q f](x) = D^2_Q f(x)(v), \qquad v \in H;$$

(iii) the map $H \to \mathcal{L}(H)$, $x \mapsto D^2_Q f(x)$ belongs to $\mathcal{C}_b(H, \mathcal{L}(H))$.

Setting $D_{e_h}(D_k f) = D_{hk} f$, $h, k \geqslant 1$, we can easily show that

$$\langle D^2_Q f(x)\, u, v \rangle = \sum_{h,k=1}^{\infty} \sqrt{\lambda_h \lambda_k}\, D_{hk} f(x)\, u_k v_h, \; x, u, v \in H, f \in \mathcal{C}^2_Q(H).$$

In a similar way it is possible to define the spaces $\mathcal{C}^n_Q(H)$ and the differential operators D^n_Q. Moreover $\mathcal{C}^\infty_Q(H) = \bigcap_{n \geqslant 1} \mathcal{C}^n_Q(H)$. Every $\mathcal{C}^n_Q(H)$, $n \geqslant 1$, turns out to be a Banach space with respect to the norm

$$\|f\|_{n, Q} = \|f\|_0 + \sum_{j=1}^{n} \|D^j_Q f\|_0, \qquad f \in \mathcal{C}^n_Q(H). \qquad \blacksquare$$

Let now $(E, \|\cdot\|_E)$ be a Banach space. $\mathcal{C}^\theta_Q(H, E)$, $\theta \in (0, 1)$ is the set of all functions $f \in \mathcal{C}_b(H, E)$ such that there exists $M = M(\theta, Q, f) > 0$ and for any $z, w \in H$: $\|f(Q^{1/2} z) - f(Q^{1/2} w)\|_E \leqslant M |z - w|^\theta$.

$\mathcal{C}^\theta_Q(H, E)$ is a Banach space endowed with the norm

$$\|f\|_{\theta, Q, E} = \|f\|_0 + [f]_{\theta, Q}, [f]_{\theta, Q} = \sup_{z, w \in H} \frac{\|f(Q^{1/2} z) - f(Q^{1/2} w)\|_E}{|z - w|^\theta},$$

where $f \in \mathcal{C}^\theta_Q(H, E)$. When $E = \boldsymbol{R}$, we set $\mathcal{C}^\theta_Q(H) = \mathcal{C}^\theta_Q(H, \boldsymbol{R})$, $\theta \in (0, 1)$. Finally we define $\mathcal{C}^{1+\theta}_Q(H) = \{f \in \mathcal{C}^1_Q(H) : D_Q f \in \mathcal{C}^\theta_Q(H, H)\}$ that is a Banach space equipped with the norm:

$$\|h\|_{1+\theta, Q} = \|h\|_{1, Q} + \|D_Q h\|_{\theta, Q}, \qquad h \in \mathcal{C}^{1+\theta}_Q(H). \qquad \blacksquare$$

Some comments on Definition 1.1 are in order. The space $\mathcal{C}^1_Q(H)$ was introduced in [3]. The spaces $\mathcal{C}_Q^n(H)$, $n \geqslant 2$ are considered in [16], they are a slight modification of those used in [2].

H can be considered as an abstract Wiener space, i.e. (H_0, H, i) where $H_0 = Q^{1/2}H$ is the reproducing kernel space of the Gaussian measure $\mathcal{N}(0, Q)$ and $i : H_0 \to H$ is the natural embedding (H_0 is a Hilbert space is based on viscosity solutions endowed with the inner product $\langle u, v\rangle_{H_0} \overset{\text{def}}{=} \langle Q^{-1/2}u, Q^{-1/2}v\rangle_H$, $u, v \in H_0$, see [8] and [11] for more details).

It is not difficult to verify that $\mathcal{C}_Q^1(H)$ coincides with the space of all functions $g \in \mathcal{C}_b(H)$, such that g is H_0-differentiable on H in the Gross sense (see [8] § 3) and its H_0-derivative: $D_{H_0} g \in \mathcal{C}_b(H, H_0)$. The same happens for the higher order H_0-derivatives (an analysis of these connections is given in [17] and in Zabczyk [10]). Another equivalent definition for the spaces $\mathcal{C}_Q^n(H)$ is contained in [23]. Finally the space $\mathcal{C}_Q^\theta(H)$ is introduced in [2].

2. – Regularity properties of the heat semigroup.

We denote by O_t the *heat semigroup* on $\mathcal{C}_b(H)$, defined as follows,

$$O_t f(x) = \int_H f(x+y)\, \mathcal{N}(0, tQ)\, dy\,, \qquad f \in \mathcal{C}_b(H),\ x \in H,\ t > 0\,. \tag{2.1}$$

It is well known that O_t, is a strongly continuous semigroup on $\mathcal{C}_b(H)$. The infinitesimal generator of O_t will be denoted by $\mathcal{A}$.

We briefly review the basic Cameron-Martin formula. It asserts that the measures $\mathcal{N}(0, tQ)$ and $\mathcal{N}(x, tQ)$, $t > 0$, $x \in H$, are either equivalent or singular. They are equivalent if and only if $x \in Q^{1/2}H$. Further if $x = Q^{1/2}h$, $h \in H$, the Radon-Nikodym derivative of $\mathcal{N}(Q^{1/2}h, tQ)$ with respect to $\mathcal{N}(0, tQ)$, is given, for any $t > 0$, by the following formula:

$$\frac{d\mathcal{N}(Q^{1/2}h, tQ)}{d\mathcal{N}(0, tQ)}(y) = \exp\left[-\frac{1}{2t}|h|^2 + \frac{1}{\sqrt{t}}\langle (tQ)^{-1/2}y, h\rangle\right], \qquad y \in H,\ \mathcal{N}(0, tQ) - a.e.\,, \tag{2.2}$$

where $\langle (tQ)^{-1/2}(\cdot), h\rangle$ is a Gaussian random variable, i.e. it is normally distributed with mean 0 and covariance $|h|^2$ with respect to $\mathcal{N}(0, tQ)$, $t > 0$. Moreover the map: $H \to L^2(H, \mathcal{N}(0, tQ))$, $h \mapsto \langle (tQ)^{-1/2}(\cdot), h\rangle$ is a linear isometry.

Applying the Cameron-Martin formula as in Proposition 9 of [8] or in [11], § II.6.2, one derives the next result.

PROPOSITION 2.1. – *Let* $f \in \mathcal{C}_b(H)$, *then* $O_t f \in \mathcal{C}_Q^\infty(H)$, $t > 0$, *with the first and second derivatives given by*

$$(2.3)\quad \begin{cases} \langle D_Q O_t f(x), v\rangle = \dfrac{1}{\sqrt{t}} \displaystyle\int_H f(x+y)\langle (tQ)^{-1/2} y, u\rangle\, \mathcal{N}(0, tQ)\, dy\,, \\ \langle D_Q^2 O_t f(x)\, u, v\rangle = \\ \qquad \dfrac{1}{t}\displaystyle\int_H f(x+y)\langle (tQ)^{-1/2} y, u\rangle\langle (tQ)^{-1/2} y, v\rangle\, \mathcal{N}(0, tQ)\, dy - \\ \qquad\qquad \dfrac{1}{t} O_t f(x)\langle u, v\rangle, \qquad u, v, x \in H,\ t > 0\,. \end{cases}$$

Moreover one has

$$(2.4)\quad \|D_Q O_t f\|_0 \leqslant \frac{1}{\sqrt{t}}\|f\|_0, \quad \|D_Q^2 O_t f\|_{0, \mathcal{L}(H)} \leqslant \frac{\sqrt{2}}{t}\|f\|_0, \quad f \in \mathcal{C}_b(H).$$

Notice that for any $g \in \mathcal{C}_Q^1(H)$, there results for $x, u, v \in H$, $t > 0$,

$$(2.5)\quad \langle D_Q^2 O_t g(x)\, u, v\rangle = \frac{1}{\sqrt{t}}\int_H \langle D_Q g(x+y), v\rangle\langle (tQ)^{-1/2} y, u\rangle\, \mathcal{N}(0, tQ)\, dy\,,$$

so that, since $\int_H |\langle (tQ)^{-1/2} y, u\rangle|^2 \mathcal{N}(0, tQ)\, dy = |u|^2$, $u \in H$, we infer

$$\|D_Q^2 O_t g\|_{0, \mathcal{L}(H)} \leqslant \frac{1}{\sqrt{t}}\|g\|_{1, Q}, \qquad g \in \mathcal{C}_Q^1(H),\ t > 0\,.$$

We need the following fact on Hilbert-Schmidt operators (see for instance p. 1098 of [7]). Denote by $\mathcal{F}_1$ the subspace of $\mathcal{L}(H)$ of all finite rank operators N, such that $\|N\|_{\mathcal{L}_2(H)} \leqslant 1$. Let $L \in \mathcal{L}(H)$, then $L \in \mathcal{L}_2(H)$ if and only if

$$(2.6)\qquad \sup_{N \in \mathcal{F}_1} |\operatorname{Tr}(NL)| = c < \infty\,.$$

Moreover if (2.6) holds then $\|L\|_2 = c$. The next result can be deduced by [8]. However we present here a direct and simpler proof (another proof is given in Zabczyk [10]).

PROPOSITION 2.2. – *For any* $f \in \mathcal{C}_b(H)$, *we have that* $D_Q^2 O_t f \in$

$\mathcal{C}_b(H, \mathcal{L}_2(H))$, $t > 0$ *and moreover*

$$(i)\ \ \|D_Q^2 O_t f\|_{0, \mathcal{L}_2(H)} \leqslant \frac{2}{t}\|f\|_0, \qquad f \in \mathcal{C}_b(H),$$

$$(ii)\ \ \|D_Q^2 O_t g\|_{0, \mathcal{L}_2(H)} \leqslant \frac{1}{\sqrt{t}}\|g\|_{1, Q}, \qquad g \in \mathcal{C}_Q^1(H).$$

PROOF. – First notice that by setting $O_t f = O_{t/2} O_{t/2} f$ and using formula (2.5), we obtain

$$(2.7)\quad \langle D_Q^2 O_t f(x)\, u, v\rangle =$$

$$\frac{\sqrt{2}}{\sqrt{t}} \int_H \langle D_Q O_{t/2} f(x+y), v\rangle \left\langle \left(\frac{t}{2}Q\right)^{-1/2} y, u \right\rangle \mathcal{N}\left(0, \frac{t}{2}Q\right) dy ,$$

where $u, v, x \in H$. We want to apply (2.6) in order to obtain that $D_Q^2 O_t f(x) \in \mathcal{L}_2(H)$, $x \in H$, $t > 0$. To this end we fix $N \in \mathcal{F}_1$.

In Im N we fix an othonormal basis (l_k), $k = 1, \dots n$. Then we set, for convenience, $(\sqrt{2}/\sqrt{t})\langle ((t/2)\, Q)^{-1/2} y, u\rangle = R_u(y)$, $u \in H$, $y \in H$. Now applying first the Hölder inequality and then the Schwarz inequality we obtain from (2.7):

$$(2.8)\qquad \mathrm{Tr}\,(N D_Q^2 O_t f(x))|^2 = \left| \sum_{k=1}^n \langle D_Q^2 O_t f(x)(l_k), N^* l_k\rangle \right|^2 =$$

$$\left| \sum_{k=1}^n \frac{\sqrt{2}}{\sqrt{t}} \int_H \langle D_Q O_{t/2} f(x+y), l_k\rangle \left\langle \left(\frac{t}{2}Q\right)^{-1/2} y, N^* l_k \right\rangle \mathcal{N}\left(0, \frac{t}{2}Q\right) dy \right|^2 \leqslant$$

$$\frac{2}{t}\int_H \left| \sum_{k=1}^n \langle D_Q O_{t/2} f(x+y), l_k\rangle\, R_{N^* l_k}(y) \right|^2 \mathcal{N}\left(0, \frac{t}{2}Q\right) dy \leqslant$$

$$\frac{2}{t}\int_H \left(\sum_{k=1}^n |\langle D_Q O_{t/2} f(x+y), l_k\rangle|^2 \right) \left(\sum_{k=1}^n |R_{N^* l_k}(y)|^2 \right) \mathcal{N}\left(0, \frac{t}{2}Q\right) dy \leqslant$$

$$\frac{2}{t}\|D_Q O_{t/2} f\|_0^2 \sum_{k=1}^n \int_H |R_{N^* l_k}(y)|^2 \mathcal{N}\left(0, \frac{t}{2}Q\right) dy = \frac{2}{t}\|D_Q O_{t/2} f\|_0^2 \sum_{k=1}^n |N^* l_k|^2 =$$

$$\frac{2}{t}\|D_Q O_{t/2} f\|_0^2 \|N^*\|_2^2 = \frac{2}{t}\|D_Q O_{t/2} f\|_0^2, \qquad x \in H,\ t > 0 .$$

Now using formula (2.4) it follows

$$|\operatorname{Tr}(ND_Q^2 O_t f(x))| \leq \frac{\sqrt{2}}{\sqrt{t}}\frac{\sqrt{2}}{\sqrt{t}}\|f\|_0 = \frac{2}{t}\|f\|_0,$$

so that by (2.6) we have $D_Q^2 O_t f(x) \in \mathcal{L}_2(H)$ and $\|D_Q^2 O_t f\|_{0, \mathcal{L}_2(H)} \leq (2/t)\|f\|_0$.

To verify the uniform continuity of $D_Q^2 O_t f$, we proceed as in (2.8) in order to obtain, for any $x, z \in H$, $N \in \mathcal{F}_1$,

$$|\operatorname{Tr}(N[D_Q^2 O_t f(x) - D_Q^2 O_t f(z)])| \leq \frac{2}{t}\omega_f(|x-z|), \qquad x, z \in H.$$

Invoking (2.6) we find

$$\|D_Q^2 O_t f(x) - D_Q^2 O_t f(z)\|_{\mathcal{L}_2(H)} \leq \frac{2}{t}\omega_f(|x-z|)$$

and the uniform continuity follows. To deduce (ii), we start from

$$\langle D_Q^2 O_t g(x)\, u, v\rangle = \frac{1}{\sqrt{t}}\int_H \langle D_Q g(x+y), v\rangle\langle (tQ)^{-1/2} y, u\rangle \mathcal{N}(0, tQ)\, dy,$$

where $g \in \mathcal{C}_Q^1(H)$, and proceed as in (2.8). The proof is complete. ■

3. – Optimal regularity results: constant coefficients.

In this section we are dealing with the following equation

$$\lambda u(x) - \frac{1}{2}\operatorname{Tr}[QD^2 u(x)] = f(x), \qquad x \in H,\ \lambda > 0,$$

that we write as

$$\lambda u - \mathcal{A}u = f, \tag{3.1}$$

where $\mathcal{A}$ is the generator of the heat semigroup O_t on $\mathcal{C}_b(H)$ and $f \in \mathcal{C}_Q^\theta(H)$, $\theta \in (0, 1)$.

We briefly review the real interpolation spaces which will be used (see [22] for details).

Let $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ be Banach spaces, such that F is continuously embedded in E. For any $\theta \in (0, 1)$,

$$(E, F)_{\theta, \infty} \overset{def}{=} \{x \in E \text{ such that } [x]_{\theta, \infty} = \sup_{t>0} t^{-\theta} K(t, x) < \infty\}, \tag{3.2}$$

where $K(t, x) = \inf\{\|a\|_E + t\|b\|_F : x = a + b,\ a \in E,\ b \in F\}$. $(E, F)_{\theta,\infty}$ is a Banach space endowed with the norm $\|\cdot\|_{\theta,\infty} = \|\cdot\|_E + [\cdot]_{\theta,\infty}$.

We use the following result, proved in [2], § 5.1,

$$\mathcal{C}_Q^\theta(H) = (\mathcal{C}_b(H), \mathcal{C}_Q^1(H))_{\theta,\infty}, \qquad \theta \in (0,1). \tag{3.3}$$

Moreover we define $\mathcal{D}_{\mathcal{A}}(\theta, \infty) \overset{\text{def}}{=} (\mathcal{C}_b(H), D(\mathcal{A}))_{\theta,\infty}$, $\theta \in (0,1)$, where $\mathcal{A}$ is the generator of O_t.

It is well known that: $f \in \mathcal{D}_{\mathcal{A}}(\theta, \infty)$ if and only if $[f]_{\theta,\mathcal{A}} = \sup_{t \in (0,1]} \|O_t f - f\|_0 t^{-\theta} < \infty$. Moreover in $\mathcal{D}_{\mathcal{A}}(\theta, \infty)$ a norm equivalent to $\|\cdot\|_{\theta,\infty}$ is the following:

$$\|\cdot\|_{\theta,\mathcal{A}} = \|\cdot\|_0 + [\cdot]_{\theta,\mathcal{A}}.$$

The next result is proved in [3], § 5.1:

$$\mathcal{D}_{\mathcal{A}}(\theta/2, \infty) \hookrightarrow \mathcal{C}_Q^\theta(H),\ \theta \in (0,1), \quad \text{with a continuous embedding}. \tag{3.4}$$

Our next result shows that the inclusion in (3.4) is strict.

THEOREM 3.1. – *For any* $\theta \in (0,1)$, *we have*: $\mathcal{D}_{\mathcal{A}}(\theta/2, \infty) \neq \mathcal{C}_Q^\theta(H)$.

PROOF. – Assume, by contraddiction, that there exists a $\hat\theta \in (0,1)$ such that

$$\mathcal{D}_{\mathcal{A}}(\hat\theta/2, \infty) = \mathcal{C}_Q^{\hat\theta}(H). \tag{3.5}$$

By (3.4) and (3.5), applying the Open Mapping Theorem, we obtain that the norms $\|\cdot\|_{\hat\theta/2,\mathcal{A}}$ and $\|\cdot\|_{\hat\theta,Q}$ are equivalent.

Now we will use the following recent result, proved in [21],

$$\|O_{t+h} - O_t\|_{\mathcal{L}(\mathcal{C}_b(H))} = 2, \qquad t \geq 0,\ h > 0. \tag{3.6}$$

Fix any $t > 0$. By (3.6), for any $h > 0$, there exists a map $f_h \in \mathcal{C}_b(H)$ such that $\|f_h\|_{\mathcal{C}_b(H)} \leq 1$ and moreover

$$2 - h < \|O_{t+h} f_h - O_t f_h\|_0 = \|O_h O_t f_h - O_t f_h\|_0 \leq [O_t f_h]_{\hat\theta/2,\mathcal{A}} h^{\hat\theta/2}. \tag{3.7}$$

Therefore once we have proved that

$$\sup_{h>0} [O_t f_h]_{\hat\theta/2,\mathcal{A}} < \infty, \tag{3.8}$$

we will obtain a contradiction, letting $h \to 0^+$ in (3.7). Now we check (3.8).

Using the fact that $\|\cdot\|_{\hat{\theta}/2,\mathcal{A}}$ is equivalent to $\|\cdot\|_{\hat{\theta},Q}$ and Proposition 2.1, we infer

$$\begin{aligned}\|O_t f_h\|_{\hat{\theta}/2,\mathcal{A}} &\leq C_1 \|O_t f_h\|_{\hat{\theta},Q}\\ &\leq C\|O_t f_h\|_{1,Q} \leq \frac{C}{\sqrt{t}}\|f_h\|_0 \leq \frac{C}{\sqrt{t}}, \quad h>0 .\end{aligned} \tag{3.9}$$

Thus (3.8) is verified and the assertion follows. ∎

Now we prove a preliminary non optimal regularity result for (3.1).

PROPOSITION 3.2. – *Consider* $u = R(\lambda, \mathcal{A}) f$, $f \in \mathcal{C}_Q^\theta(H)$ $\lambda > 0$, $\theta \in (0,1)$. *Then* $u \in \mathcal{C}_Q^2(H)$ *and* $D_Q^2 u \in \mathcal{C}_b(H, \mathcal{L}_2(H))$. *Moreover there exists a constant* $c = c(\lambda, Q, \theta) > 0$, *such that:*

$$\|u\|_{2,Q} + \|D_Q^2 u\|_{0,\mathcal{L}_2(H)} \leq c\|f\|_{\theta,Q}. \tag{3.10}$$

PROOF. – We have, by the Hille-Yosida Theorem,

$$u = \int_0^\infty e^{-\lambda t} O_t f\, dt . \tag{3.11}$$

By the first estimate of (2.4), differentiating under the integral sign in (3.11) and taking into account that

$$\begin{aligned}|\langle D_Q O_t f(x) - D_Q O_t f(z), u\rangle|^2 &\leq\\ &\frac{1}{t}\omega_f(|x-z|)^2 \int_H |\langle (tQ)^{-1/2} y, u\rangle|^2 \mathcal{N}(0, tQ)\, dy \leq\\ &\frac{1}{t}\omega_f(|x-z|)^2 |u|^2, \quad x, z, u \in H, t > 0 ,\end{aligned} \tag{3.12}$$

we deduce easily that $u \in \mathcal{C}_Q^1(H)$. To get more regularity for u, consider that from Proposition 2.2, there results for $t > 0$:

$$\|D_Q^2 O_t h\|_{0,\mathcal{L}_2(H)} \leq \frac{2}{t}\|h\|_0, \quad \|D_Q^2 O_t g\|_{0,\mathcal{L}_2(H)} \leq \frac{1}{\sqrt{t}}\|g\|_{1,Q}, \quad h \in \mathcal{C}_b(H), g \in \mathcal{C}_Q^1(H).$$

Interpolating between these estimates, since $f \in (\mathcal{C}_b(H), \mathcal{C}_Q^1(H))_{\theta,\infty}$, one has

$$\|D_Q^2 O_t f\|_{0,\mathcal{L}_2(H)} \leq c_\theta t^{\theta/2-1}\|f\|_{\theta,Q}, \quad t > 0 . \tag{3.13}$$

Using this estimate, we can readily derive that there exists $D_Q^2 u(x) \in \mathcal{L}(H)$ for any $x \in H$ and

$$\langle D_Q^2 u(x)(u), v\rangle = \int_0^\infty e^{-\lambda t} \langle D_Q^2 O_t f(x)(u), v\rangle\, dt , \quad x, u, v \in H .$$

To get that $D_Q^2 u(x) \in \mathcal{L}_2(H)$, we use formula (2.6). Let $N \in \mathcal{F}_1$, where $\mathcal{F}_1$ denotes the subspace of $\mathcal{L}(H)$ of all finite rank operators, such that $\|N\|_{\mathcal{L}_2(H)} \leqslant 1$, there results

$$(3.14) \qquad |\operatorname{Tr}(N D_Q^2 u(x))| \leqslant \int_0^\infty e^{-\lambda t} |\operatorname{Tr}(N D_Q^2 O_t f(x))| \, dt \leqslant$$

$$\leqslant c_\theta \|f\|_{\theta, Q} \int_0^\infty e^{-\lambda t} t^{\theta/2 - 1} \, dt = C_{\theta, \lambda} \|f\|_{\theta, Q},$$

so that $D_Q^2 u(x) \in \mathcal{L}_2(H)$, $x \in H$ and moreover $\|D_Q^2 u\|_{0, \mathcal{L}_2(H)} \leqslant C_{\theta, \lambda} \|f\|_{\theta, Q}$.

It remains to establish the uniform continuity of $D_Q^2 u$. This is equivalent to show that for any sequence $(z_n) \subset H$ such that $z_n \to 0$ as $n \to \infty$, we have:

$$(3.15) \qquad \lim_{n \to \infty} \sup_{x \in H} \|D_Q^2 u(x + z_n) - D_Q^2 u(x)\|_{\mathcal{L}_2(H)} = 0 .$$

Let us fix a countable dense subset L of H. Since $\mathcal{L}_2(H)$ is separable, we choose also a countable dense subset $\mathcal{M}$ of $\mathcal{F}_1$. Now using that for any $N \in \mathcal{F}_1$ the linear map: $\mathcal{L}_2(H) \to \boldsymbol{R}$, $A \mapsto \operatorname{Tr}(NA)$ is continuous, we obtain

$$(3.16) \qquad \|T\|_2 = \sup_{N \in \mathcal{F}_1} |\operatorname{Tr}(NT)| = \sup_{N \in \mathcal{M}} |\operatorname{Tr}(NT)|, \qquad T \in \mathcal{L}_2(H).$$

From this formula it follows readly that the maps $\gamma_n :]0, \infty[\to \boldsymbol{R}$,

$$\begin{aligned} \gamma_n(t) &= \sup_{x \in H} \|D_Q^2 O_t f(x + z_n) - D_Q^2 O_t f(x)\|_{\mathcal{L}_2(H)} \\ &= \sup_{x \in L,\, N \in \mathcal{M}} |\operatorname{Tr}(N[D_Q^2 O_t f(x + z_n) - D_Q^2 O_t f(x)])|, \qquad t > 0, \end{aligned}$$

are Borel for any $n \geqslant 1$. Thus we can write

$$\sup_{x \in H} \|D_Q^2 u(x + z_n) - D_Q^2 u(x)\|_{\mathcal{L}_2(H)} \leqslant \int_0^\infty e^{-\lambda t} \gamma_n(t) \, dt .$$

Now $\lim_{n \to \infty} \gamma_n(t) = 0$, $t > 0$, by Proposition 2.2. Hence letting $n \to \infty$ in right-hand side of the last formula, we find (3.15) by the Dominated Convergence Theorem. This completes the proof. ■

In the next result we present Schauder estimates for (3.1) and improve Theorem 5.1 of [2]. To this end we will use the space $\mathcal{F}_1$, introduced in (2.6).

THEOREM 3.3. – *Consider* $u = R(\lambda, \mathcal{A}) f$, $f \in \mathcal{C}_Q^\theta(H)$ $\lambda > 0$, $\theta \in (0, 1)$. *Then* $u \in \mathcal{C}_Q^2(H)$ *and* $D_Q^2 u \in \mathcal{C}_b(H, \mathcal{L}_2(H))$. *Moreover for any* $N \in \mathcal{F}_1$, *one has that*

$\mathrm{Tr}\,(ND_Q^2 u) \in \mathcal{D}_{\mathcal{A}}(\theta/2, \infty)$ *and there exists a constant* $c = c(\lambda, Q, \theta) > 0$, *such that:*

$$\|u\|_{2, Q} + \sup_{N \in \mathcal{T}_1} \|\mathrm{Tr}\,(ND_Q^2 u)\|_{\theta/2, \mathcal{A}} \leq c\|f\|_{\theta, Q}. \tag{3.17}$$

In particular (3.17) *implies that* $D_Q^2 u \in C_Q^\theta(H, \mathcal{L}_2(H))$ *and it holds:*

$$\|u\|_{2, Q} + \|D_Q^2 u\|_{\theta, Q, \mathcal{L}_2} \leq \hat{c}\|f\|_{\theta, Q}. \tag{3.18}$$

PROOF. – We take $N \in \mathcal{T}_1$ and prove that $\mathrm{Tr}\,(ND_Q^2 u) \in D_{\mathcal{A}}(\theta/2, \infty)$ with norm independent of N. For any function $h \in \mathcal{C}_Q^2(H)$, we set:

$$Uh(x) = \mathrm{Tr}\,(ND_Q^2 h)(x), \quad x \in H. \tag{3.18}$$

Thus for any for $\xi \in [0, 1]$, we have to estimate $I_\xi = \sup_{x \in H} |O_\xi Uu(x) - Uu(x)|$, $\xi \in [0, 1]$. Remark that, differentiating under the integral sign in (2.1), we find:

$$UO_t h(x) = O_t Uh(x)\; h \in \mathcal{C}_Q^2(H), \quad x \in H,\; t \geq 0.$$

This yields, applying (3.13),

$$\begin{aligned} I_\xi &= \sup_{x \in H} \left| \int_0^\infty e^{-\lambda t}(UO_{t+\xi} f(x) - UO_t f(x))\,dt \right| \\ &= \sup_{x \in H} \left| (e^{\lambda \xi} - 1) \int_0^\infty e^{-\lambda t}\, UO_t f(x)\,dt - e^{\lambda \xi} \int_0^\xi e^{-\lambda t}\, UO_t f(x)\,dt \right| \\ &\leq c\|f\|_{\theta, Q} \left[(e^{\lambda \xi} - 1) \int_0^\infty e^{-\lambda t} t^{\theta/2 - 1}\,dt + e^{\lambda \xi} \int_0^\xi e^{-\lambda t} t^{\theta/2 - 1}\,dt \right] \\ &\leq \widehat{C}\|f\|_{\theta, Q}\, \xi^{\theta/2}, \quad \xi \in [0, 1], \end{aligned} \tag{3.19}$$

where $\widehat{C} = \widehat{C}(\lambda, Q, \theta)$.

Hence by (3.19) and (3.4) we obtain that

$$\mathrm{Tr}\,(ND_Q^2 u) \in D_{\mathcal{A}}(\theta/2, \infty) \subset \mathcal{C}_Q^\theta(H)$$

and

$$\|\mathrm{Tr}\,(ND_Q^2 u)\|_{\mathcal{C}_Q^\theta(H)} \leq C_1 \|\mathrm{Tr}\,(ND_Q^2 u)\|_{D_{\mathcal{A}}(\theta/2, \infty)} \leq C_2 \|f\|_{\theta, Q}$$

where C_1 and C_2 do not depend on N. Then, taking the supremum over

$N \in \mathcal{F}_1$, we infer

$$\|D_Q^2 u\|_{\theta,\,Q,\,\mathcal{L}_2} \leqslant 2C_1 \sup_{N \in \mathcal{F}_1} \|\mathrm{Tr}\,(N D_Q^2 u)\|_{\theta/2,\,\mathcal{A}} \leqslant 2C_2 \|f\|_{\theta,\,Q}. \tag{3.20}$$

Combining (3.20) with Proposition 3.2, the thesis follows. ■

4. – Elliptic equations with variable coefficients.

We consider now the following elliptic equation

$$\lambda u(x) - \mathcal{A} u(x) - \frac{1}{2}\,\mathrm{Tr}\,(F(x)\, D_Q^2 u(x)) = f(x), \qquad x \in H,\ \lambda > 0\,, \tag{4.1}$$

where $f \in \mathcal{C}_Q^{\theta}(H)$, $\theta \in (0,1)$ and F satisfies the following assumptions:

HYPOTHESIS 1.

(i) $F : H \to \mathcal{L}_2(H)$,

(ii) $F(x)$ *is self-adjoint and non negative,* $x \in H$,

(iii) $F \in \mathcal{C}_Q^{\theta}(H, \mathcal{L}_2(H))$.

A *solution of* (4.1) is, by definition, a map $u \in D(\mathcal{A}) \cap \mathcal{C}_Q^2(H)$, such that $D_Q^2 u \in \mathcal{C}_Q^{\theta}(H, \mathcal{L}_2(H))$ and in addition satisfies (4.1).

We first have the following Maximum Principle for (4.1), which can be proved as in ([2], Theorem A.1):

THEOREM 4.1. – *Let* $\lambda > 0$, $f \in C_b(H)$ *and* $u \in D(\mathcal{A}) \cap C_Q^2(H)$ *be a solution of equation* (4.1), *where* F *fulfills Hypothesis* 1. *Then*:

$$\|u\|_0 \leqslant \frac{1}{\lambda}\|f\|_0. \tag{4.2}$$

A priori estimates for (4.1) are proved in the next result, that improves Theorem 6.2 of [2].

THEOREM 4.2. – *Assume that* F *satisfies Hypothesis* 1 *and* $f \in \mathcal{C}_Q^{\theta}(H)$. *Let* u *be a solution of* (4.1). *Then there exists a constant* $c = c(\lambda, Q, \theta, \|F\|_{\theta,\,Q}) > 0$, *such that*:

$$\lambda\|u\|_{\theta,\,Q} + \|\mathcal{A} u\|_{\theta,\,Q} + \|u\|_{2,\,Q} + \|D_Q^2 u\|_{\theta,\,Q,\,\mathcal{L}_2(H)} \leqslant c\|f\|_{\theta,\,Q}.$$

We need two preliminary Lemmas. The first one is a non standard interpolation estimate.

LEMMA 4.3. – *Let* $v \in C_Q^2(H)$ *such that* $D_Q^2 v \in \mathcal{C}_b(H, \mathcal{L}_2(H))$. *Assume that* *for any* $N \in \mathcal{F}_1$, $\mathrm{Tr}\,[ND_Q^2 v] \in D_{\mathcal{A}}(\theta/2,\ \infty)$ *and*

$$\|D_Q^2 v\|_{\theta/2,\,\mathcal{A},\,\mathcal{L}_2} \stackrel{\mathrm{def}}{=} \sup_{N \in \mathcal{F}_1} \|\mathrm{Tr}\,[ND_Q^2 v]\|_{\theta/2,\,\mathcal{A}} < +\infty\ . \tag{4.3}$$

Then, for any $t > 0$, *the following interpolatory inequality holds*:

$$\|D_Q^2 v\|_{0,\,\mathcal{L}_2} \leq C_\theta \|v\|_0^{\theta/(2+\theta)} \|D_Q^2 v\|_{\theta/2,\,\mathcal{A},\,\mathcal{L}_2}^{2/(2+\theta)}. \tag{4.4}$$

PROOF. – First notice that

$$\|D_Q^2 v\|_{0,\,\mathcal{L}_2} = \sup_{x \in H}\ \sup_{N \in \mathcal{F}_1} |\,\mathrm{Tr}\,[ND_Q^2 v](x)\,|\ .$$

Then for any $N \in \mathcal{F}_1$ and $t > 0$ we have:

$$\begin{aligned}
\|\mathrm{Tr}\,[ND_Q^2 v]\|_0 &\leq \|O_t(\mathrm{Tr}\,[ND_Q^2 v]) - \mathrm{Tr}\,[ND_Q^2 v]\|_0 + \|O_t(\mathrm{Tr}\,[ND_Q^2 v])\|_0 \\
&\leq t^{\theta/2} \|\mathrm{Tr}\,[ND_Q^2 v]\|_{\theta/2,\,\mathcal{A}} + \|\mathrm{Tr}\,[ND_Q^2 O_t v]\|_0 \\
&\leq t^{\theta/2} \sup_{N \in \mathcal{F}_1} \|\mathrm{Tr}\,[ND_Q^2 v]\|_{\theta/2,\,\mathcal{A}} + \|D_Q^2 v\|_{0,\,\mathcal{L}_2} \\
&\leq t^{\theta/2} \|D_Q^2 v\|_{\theta/2,\,\mathcal{A},\,\mathcal{L}_2} + \frac{1}{t}\|v\|_0.
\end{aligned}$$

In the last passage we have used Proposition 2.2. Taking the infimum over $t > 0$ in the last term, we obtain the thesis. ■

Let $F \in \mathcal{L}_2(H)$, $F = F^*$, F nonnegative, and let $S := Q^{1/2}(I + F)\,Q^{1/2}$. By Proposition 5.1 in [2], we obtain that $\mathcal{C}_Q^\theta(H) = \mathcal{C}_S^\theta(H)$ and $\mathcal{C}_Q^k(H) = \mathcal{C}_S^k(H)$, $k \geq 1$, $\theta \in (0,\,1)$, with equivalence of norms. Denote by O_t^S the heat semigroup in $\mathcal{C}_b(H)$ associated with S, obtained replacing Q by S in (2.1), and by $(\mathcal{A}^S,\ D(\mathcal{A}^S))$ its infinitesimal generator. Using the core $D(\mathcal{A}_0)$ given in [18], it is not difficult to verify that

$$\{f \in D(\mathcal{A}) : \mathcal{A}f \in C_Q^\theta(H)\} = \{f \in D(\mathcal{A}^S) : \mathcal{A}^S f \in C_Q^\theta(H)\}$$

and on this space $\mathcal{A}^S = \mathcal{A} + (1/2)\,\mathrm{Tr}\,(FD_Q^2)$. Moreover we have:

LEMMA 4.4. – *Let* $S := Q^{1/2}(I + F)\,Q^{1/2}$, *where* $F \in \mathcal{L}_2(H)$, $F = F^*$ *and* F *is nonnegative. Let* v *satisfy the hypotheses of Lemma* 4.3. *Then for any* $N \in \mathcal{F}_1$, *we have* $\mathrm{Tr}\,[ND_S^2 v] \in D_{\mathcal{A}}(\theta/2,\ \infty)$ *and*

$$\|D_S^2 v\|_{\theta/2,\,\mathcal{A},\,\mathcal{L}_2} \stackrel{\mathrm{def}}{=} \sup_{N \in \mathcal{F}_1} \|\mathrm{Tr}\,[ND_S^2 v]\|_{\theta/2,\,\mathcal{A}} \leq C \|D_Q^2 v\|_{\theta/2,\,\mathcal{A},\,\mathcal{L}_2} \tag{4.5}$$

where $C = C(S, Q, \theta)$. *Moreover*:

$$\|D_S^2 v\|_{0, \mathcal{L}_2} \le C_\theta \|v\|_0^{\theta/(2+\theta)} \|D_S^2 v\|^{2}_{\theta/2, \mathcal{A}, \mathcal{L}_2} {}^{\cdot}\, 2 + \theta \qquad \blacksquare \tag{4.6}$$

For any $x \in H$, $r > 0$ we denote by $\varrho_{x,r}$ a function in $C_b^\infty(H)$ [1] such that

$$0 \le \varrho_{x,r} \le 1 \qquad \varrho_{x,r}(z) = \begin{cases} 1 & \text{if } z \in B(x, r) \\ 0 & \text{if } z \notin B(x, 2r). \end{cases}$$

It is easy to prove that

LEMMA 4.5. – *Let* $u \in D(\mathcal{A})$. *Then* $\varrho_{x,r} u \in D(\mathcal{A})$, *for any* $x \in H$, $r > 0$. *Moreover*

$$\mathcal{A}(\varrho_{x,r} u) = \varrho_{x,r} \mathcal{A} u + \langle D_Q u, Q^{1/2} D\varrho_{x,r} \rangle + \frac{1}{2} \operatorname{Tr}[Q D^2 \varrho_{x,r}]. \tag{4.7}$$

PROOF OF THEOREM 4.2. – Let $f \in C_Q^\theta(H)$ and let u be a solution of equation (4.1). Fix $x_0 \in H$, $r > 0$ and set $v = \varrho_{x_0, r} u = \varrho u$. We shall denote by C_i, $i \in \boldsymbol{N}$, constants depending only on λ, Q, θ, F. By Lemma 4.5 we have

$$\lambda v - \mathcal{A} v - \frac{1}{2} \operatorname{Tr}\,(F(x_0)\, D_Q^2 v) = f_1 + f_2 + f_3,$$

where

$$f_1(x) = \varrho(x)\, f(x), \qquad f_2(x) = \frac{1}{2} \operatorname{Tr}[(F(x) - F(x_0))\, D_Q^2 v(x)]$$

$$f_3(x) = -\langle (I + F(x))\, D_Q u(x), D_Q \varrho(x) \rangle - \frac{1}{2} \operatorname{Tr}[(I + F(x))\, D_Q^2 \varrho(x)].$$

By Theorem 3.3 and Lemma 4.4 we deduce, setting $S := Q^{1/2}(I + F(x_0))Q^{1/2}$,

$$\|v\|_{2, Q} + \|D_Q^2 v\|_{\theta/2, \mathcal{A}^S, \mathcal{L}_2} \le C(\|f_1\|_{\theta, Q} + \|f_2\|_{\theta, Q} + \|f_3\|_{\theta, Q}). \tag{4.8}$$

Notice that

$$\|f_1\|_{\theta, Q} \le K_r \|f\|_{\theta, Q}. \tag{4.9}$$

[1] $\mathcal{C}_b^\infty(H)$ denotes the subspace of $\mathcal{C}_b(H)$ of all functions having uniformly continuous and bounded Fréchet derivatives of any order.

Let us estimate $\|f_2\|_{\theta, Q}$. First we have

$$\|f_2\|_0 \leqslant C_1 \|D_Q^2 v\|_{0, \mathcal{L}_2(H)}.$$

Then, denoting by ω_F the modulus of continuity of F, there results

$$[f_2]_{\theta, Q} \leqslant C_2 \left(\sup_{x \in B(x_0, 2r)} \|F(x) - F(x_0)\|_{\mathcal{L}_2} [D_Q^2 v]_{\theta, Q, \mathcal{L}_2} + M \|D_Q^2 v\|_{0, \mathcal{L}_2} \right)$$

$$\leqslant C_3 (\omega_F(2r)[D_Q^2 v]_{\theta, Q, \mathcal{L}_2} + M \|D_Q^2 v\|_{0, \mathcal{L}_2}).$$

By Lemma 4.3 and by (3.20) it follows that

$$\begin{aligned} \|f_2\|_{\theta, Q} &\leqslant C_4 (\omega_F(2r)[D_Q^2 v]_{\theta, Q, \mathcal{L}_2} + \|D_Q^2 v\|_{0, \mathcal{L}_2}) \\ &\leqslant C_{41} (\omega_F(2r)[D_Q^2 v]_{\theta, Q, \mathcal{L}_2} + \|v\|_0^{\theta/(2+\theta)} \|D_Q^2 v\|_{\theta/2, \mathcal{A}^S, \mathcal{L}_2}^{2/(2+\theta)}) \\ &\leqslant C_5 \left((\omega_F(2r) + r^{\theta/2}) \|D_Q^2 v\|_{\theta/2, \mathcal{A}^S, \mathcal{L}_2} + \frac{1}{r} \|v\|_0 \right). \end{aligned}$$

Using the Maximum Principle we obtain

$$\|f_2\|_{\theta, Q} \leqslant C_6 \left((\omega_F(2r) + r^{\theta/2}) \|D_Q^2 v\|_{\theta/2, \mathcal{A}^S \mathcal{L}_2} + \frac{1}{r} \|f\|_0 \right). \tag{4.10}$$

As for $\|f_3\|_{\theta, Q}$, we easily obtain the following estimate:

$$\|f_3\|_{\theta, Q} \leqslant C_7 (\|f\|_{\theta, Q} + E_r \|u\|_{1+\theta, Q}). \tag{4.11}$$

Collecting (4.8)—(4.11) we deduce

$$\|v\|_{2, Q} + \|D_Q^2 v\|_{\theta/2, \mathcal{A}^S, \mathcal{L}_2} \leqslant C(\|f_1\|_{\theta, Q} + \|f_2\|_{\theta, Q} + \|f_3\|_{\theta, Q}) \leqslant$$

$$C_8 \left((\omega_F(2r) + r^{\theta/2}) \|D_Q^2 v\|_{\theta/2, \mathcal{A}^S, \mathcal{L}_2} + \left(1 + \frac{1}{r}\right) \|f\|_{\theta, Q} + E_r \|u\|_{1+\theta, Q} \right).$$

Now we choose $r > 0$ such that $C_8 (\omega_F(2r) + r^{\theta/2}) < 1/2$. This way, by using also (3.20), we infer

$$\begin{aligned} \|v\|_{2, Q} + \|D_Q^2 v\|_{\theta, Q, \mathcal{L}_2} &\leqslant \|v\|_{2, Q} + 2 \|D_Q^2 v\|_{\theta/2, \mathcal{A}^S, \mathcal{L}_2} \\ &\leqslant C_9 (\|f\|_{\theta, Q} + \|u\|_{1+\theta, Q}). \end{aligned}$$

Since $v = \varrho u$, we obtain

$$\begin{aligned} \|u\|_{C_Q^2(B(x_0, r))} + \|D_Q^2 u\|_{\theta, Q, \mathcal{L}_2(B(x_0, r))} &\leqslant \|v\|_{2, Q} + \|D_Q^2 v\|_{\theta, Q, \mathcal{L}_2(H)} \\ &\leqslant C_9 (\|f\|_{\theta, Q} + \|u\|_{1+\theta, Q}). \end{aligned}$$

Since C_9 is independent of x_0, it follows that

$$\|u\|_{2,Q} + \|D_Q^2 u\|_{\theta,Q,\mathcal{L}_2} \leq C_9(\|f\|_{\theta,Q} + \|u\|_{1+\theta,Q}).$$

Now notice that, in a standard way, one proves that

$$\begin{aligned}\|u\|_{1+\theta,Q} &\leq C_{11}\|u\|_0^{1/(2+\theta)}(\|u\|_{2,Q} + \|D_Q^2 u\|_{\theta,Q,\mathcal{L}(H)})^{(1+\theta)/(2+\theta)}\\ &\leq C_{11}\|u\|_0^{1/(2+\theta)}(\|u\|_{2,Q} + \|D_Q^2 u\|_{\theta,Q,\mathcal{L}_2(H)})^{(1+\theta)/(2+\theta)},\end{aligned}$$

from which it results

$$\|u\|_{2,Q} + \|D_Q^2 u\|_{\theta,Q,\mathcal{L}_2} \leq C_{12}(\|f\|_{\theta,Q} + K_\varepsilon\|u\|_0 + \varepsilon(\|u\|_{2,Q} + \|D_Q^2 u\|_{\theta,Q,\mathcal{L}_2(H)}))\,.$$

Choosing ε small enough and using again the Maximum Principle, we finally get

$$\|u\|_{2,Q} + \|D_Q^2 u\|_{\theta,Q,\mathcal{L}_2} \leq C_{13}\|f\|_{\theta,Q}$$

and the thesis is proved. ■

From Theorem 4.2 we can deduce our final result:

THEOREM 4.6. – *Assume that F fulfills Hypothesis* 1 *and let $f \in C_Q^\theta(H)$. Then there exists a unique solution of equation* (4.1).

PROOF. – One can adapt, without difficulties, the classical Continuity Method, used in Theorem 6.2 of [2]. ■

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