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On the Onset of Convection in Porous Media: Temperature Depending Viscosity.

F. CAPONE (*)

Sunto. – *Si considera l'insorgere della convezione naturale in un mezzo poroso (Horton-Rogers-Lapwood problem), assumendo che la viscosità del fluido dipenda dalla temperatura. Adoperando il metodo diretto di Liapunov, si effettua l'analisi della stabilitá non lineare della soluzione di conduzione per i modelli di Darcy e di Forchheimer.*

1. – Introduction.

As it is well known, the convective flow in porous media, has a notable relevance in many geophysical and industrial applications: the predictions of groundwater movement in aquifer, nuclear engeneering etc. For this reason — in the past as nowadays — the aforesaid problem has attracted the attention of many writers (see, for instance, [1, 9, 10, 13]).

There has been a difference of opinion as to what is the appropriate form of the equation of motion. In several papers the flow in a porous medium is governed by the Darcy's law, i.e.

$$\nabla p = -\frac{\mu}{k}\boldsymbol{v}\,. \tag{1.1}$$

But, when the motion is not slow we have to consider inertial effects, too. According to [9, 10], the appropriate modification to Darcy's law is:

$$\nabla p = -\frac{\mu}{k}\boldsymbol{v} - c_F k^{-1/2} \varrho_f |\boldsymbol{v}| v\,, \tag{1.2}$$

to which we refer as the Forchheimer equation.

Another thing one has to consider is that for a lot of real fluids the viscosity varies strongly with temperature and hence is not realistic to consider the viscosity as a constant (in the setting of porous medium see [11]). In particular,

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liquids, generally, has a viscosity exponentially depending on the temperature [2, 3, 4, 5].

The aim of this article is to perform a nonlinear stability analysis of natural convection in a porous medium, via the Liapunov direct method, when the viscosity of the fluid exhibits the following constitutive law [3]:

$$\mu = \mu_0 f[\gamma(T - T_0)], \qquad (\mu_0 = \text{const.} > 0) \tag{1.3}$$

with μ_0 the dynamic viscosity at the reference temperature T_0 and f a convex, nonincreasing, positive function of the temperature T. In particular we shall analyze the case in which

$$f[\gamma(T - T_0)] = \exp[-\gamma(T - T_0)] \tag{1.4}$$

where γ is a positive constant [2, 5].

The scheme of the paper is as follows. In Section 2 we discuss the various forms of the momentum equation related to the theory of flow in porous media. Then, in Section 3, we focus the energy equation that is valid in a porous material. In Section 4, on considering both the Darcy and Forchheimer models, we write the perturbation equations to the nonconvective stationary solution. Finally in Sections 5 and 6, we study the nonlinear stability of the motionless state — via the Liapunov direct method — for Darcy and Forchheimer models, respectively.

2. – Momentum equation.

In this section we discuss some of the various forms of the momentum equation related to the theory of flow in porous materials (1). The theory of porous flow is longely based on a generalization of the empirical Darcy law which express a proportionality between the flow rate and the applied pressure difference. The Darcy law for a slow flow in homogeneous isotropic material, under the gravity action, leads to the following equation:

$$\nabla p = -\frac{\mu}{k}\boldsymbol{v} + \varrho \boldsymbol{g} \tag{2.1}$$

where μ is the viscosity of the fluid, p is the pressure, k is the permeability of the medium, $\boldsymbol{v}$ is the seepage velocity (2), and $\varrho \boldsymbol{g}$ is the gravity.

(1) Porous material means a material consisting of a solid matrix with interconnected void spaces (pores), through which one or more fluids flow.

(2) The seepage velocity $\boldsymbol{v}$ means the fluid velocity measured relative to axes fixed in porous solid. We shall assume that the seepage velocity $\boldsymbol{v}$ is related to the pore-average velocity $\boldsymbol{V}$ by the Dupuit-Forchheimer equation [9, 10]

$$\boldsymbol{v} = \Phi \boldsymbol{V} \tag{2.2}$$

in which Φ is the (constant) porosity, with $\Phi \in (0, 1)$.

Let us observe that the Darcy law (2.1) is linear in the seepage veolocity $\boldsymbol{v}$. This law is valid when $\boldsymbol{v}$ is «sufficiently small», but when $\boldsymbol{v}$ increases, according to [9, 10], the appropriate modification to Darcy's equation is the following

$$\nabla p = -\frac{\mu}{k}\boldsymbol{v} + \varrho \boldsymbol{g} - c_F k^{-1/2} \varrho_f |\boldsymbol{v}| \boldsymbol{v} \tag{2.3}$$

in which c_F is a dimensionless form-drag constant. We shall refer to the last term of (2.3) as the Forchheimer term and to the equation (2.3) as the Forchheimer model.

Many Authors (see, for instance, [9, 10]) as extension of the Darcy law, instead of (2.3), have used the following equation

$$\varrho_f \left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\, \boldsymbol{V} \right] = -\nabla p - \frac{\mu}{k}\boldsymbol{v} + \varrho \boldsymbol{g} \tag{2.4}$$

which, on using Dupuit-Forchheimer relationship (2.2), becomes

$$\varrho_f \left[\frac{\Phi^{-1}\partial \boldsymbol{v}}{\partial t} + \Phi^{-2}(\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v} \right] = -\nabla p - \frac{\mu}{k}\boldsymbol{v} + \varrho \boldsymbol{g}\,. \tag{2.5}$$

The equation (2.5) holds unless the porosity Φ is very large (close to unity), otherwise one has to drop the convective term

$$\Phi^{-2}(\boldsymbol{v} \cdot \nabla)\, \boldsymbol{v}$$

and has to consider the following equation

$$\frac{\varrho_f}{\Phi}\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla p - \frac{\mu}{k}\boldsymbol{v} + \varrho \boldsymbol{g}\,.$$

Finally, an alternative to the Darcy's equation is the well-known Brinkman's law. With inertial terms omitted, the Brinkman's law is the following

$$\nabla p = -\frac{\mu}{k}\boldsymbol{v} + \tilde{\mu}\, \Delta \boldsymbol{v} + \varrho \boldsymbol{g}$$

in which $\tilde{\mu}$ is the effective viscosity (3).

3. – Energy equation.

In this section we focus the equation expressing the first law of thermodynamics in a porous medium [9, 10]. In particular, we shall consider the sim-

(3) Brinkman set μ and $\tilde{\mu}$ equal to each other, but in general they are only approximately equal.

ple situation of an isotropic medium, in which radiative effects, viscous dissipation and the work done by pressure changes are negligible. Further we shall assume that

(i)

$$T_s = T_f = T \tag{3.1}$$

being T_s and T_f the temperature of the solid and fluid phases, respectively, i.e. we shall suppose the presence of a local equilibrium;

(ii) there is not net heat transfer from one phase to the other.

Under the assumptions (i) and (ii), the energy equation for the solid phase and for the fluid phase are respectively:

$$(1-\Phi)(\varrho c)_s \frac{\partial T}{\partial t} = (1-\Phi)\, \nabla \cdot (k_s \nabla T) + (1-\Phi)\, q_s' \tag{3.2}$$

$$\Phi(\varrho c_p)_f \frac{\partial T}{\partial t} + (\varrho c_p)_f \boldsymbol{v} \cdot \nabla T = \Phi \nabla \cdot (k_f \nabla T) + \Phi q_f' \tag{3.3}$$

in which: Φ is the porosity of the medium; the subscript s and f refer to the solid and to the fluid phases, respectively; c is the specific heat of the solid; c_p is the specific heat of the fluid at constant pressure; k is the thermal conductivity; q' is the heat production per unit volume; $\boldsymbol{v}$ is the seepage velocity. On adding the equations (3.2) and (3.3), we obtain

$$(\varrho c)_m \frac{\partial T}{\partial t} + (\varrho c_p)_f \boldsymbol{v} \cdot \nabla T = \nabla \cdot (k_m \nabla T) + q_m' \tag{3.4}$$

with

$$(\varrho c)_m = (1-\Phi)(\varrho c)_s + \Phi(\varrho c_p)_f$$

$$k_m = (1-\Phi)\, k_s + \Phi k_f$$

$$q_m' = (1-\Phi)\, q_s' + \Phi q_f'$$

that are, respectively, the overall heat capacity per unit volume, the overall thermal conductivity and the overall heat production per unit volume of the medium.

On setting

$$A = \frac{(\varrho c)_m}{(\varrho c_p)_f} (> 0) \tag{3.5}$$

$$k_1 = \frac{k_m}{(\varrho c_p)_f} \tag{3.6}$$

if $q'_m = 0$, i.e. if there is no heat production, the equation (3.4) becomes

$$A\frac{\partial T}{\partial t} + \boldsymbol{v}\cdot\nabla T = \nabla\cdot(k_1 \nabla T). \tag{3.7}$$

4. – Statement of the problem.

Let us consider an infinite horizontal layer saturated with a homogeneus fluid under the action of a vertical gravity field $\boldsymbol{g} = -g\boldsymbol{k}$ in which an adverse temperature gradient is mantained. The fluid is contained in a porous medium between the planes $z = 0$ and $z = d$ with assigned temperatures $T(0) = T_1 + \Delta\mathcal{T}$ and $T(d) = T_1$, with $\Delta\mathcal{T} > 0$.

Taking into account the law (1.3) for the dynamic viscosity, with $f \in C^2(\mathbb{R})$ and:

$$f > 0, \qquad f' \leqslant 0, \qquad f'' \geqslant 0, \tag{4.1}$$

applying the Oberbeque-Boussinesq approximation [6, 7, 9] and on taking into account the equation (3.7) with constant overall thermal conductivity k_1, the Darcy-Oberbeque-Boussinesq equations (DOB) [5] and Forchheimer-Oberbeque-Boussinesq equations (FOB) [4] are, respectively

DOB Equations

$$\begin{cases} \nabla p = -\dfrac{\mu_0}{k} f[\gamma(T - T_0)]\,\boldsymbol{v} + g\varrho_0 \alpha(T - T_0)\,\boldsymbol{k} \\ \nabla\cdot\boldsymbol{v} = 0 \\ AT_t + \boldsymbol{v}\cdot\nabla T = k_1 \Delta T, \end{cases} \tag{4.2}$$

FOB Equations

$$\begin{cases} \nabla p = -\dfrac{\mu_0}{k} f[\gamma(T - T_0)]\,\boldsymbol{v} + g\varrho_0 \alpha(T - T_0)\,\boldsymbol{k} - c_F \varrho_0 k^{-1/2}\,|\boldsymbol{v}|\boldsymbol{v} \\ \nabla\cdot\boldsymbol{v} = 0 \\ AT_t + \boldsymbol{v}\cdot\nabla T = k_1 \Delta T, \end{cases} \tag{4.3}$$

where: z is the upward vertical, $\boldsymbol{v}$ is the seepage velocity, ϱ_0 is the density of the fluid at temperature T_0, p is the reduced pressure, μ_0 is a reference dynamic viscosity, g is the gravity, α is the thermal expansion coefficient, k is the permeability, c_F is a dimensionless form-drag constant, k_1 is the overall thermal conductivity.

To the systems (4.2) and (4.3) we add the boundary conditions

$$T(z=0)=T_1+\Delta\mathcal{T}\,,\qquad T(z=d)=T_1 \tag{4.4}$$

with $\Delta\mathcal{T}>0$.

On introducing the following dimensionless quantities

$$t=t^*\,\frac{d^2}{k_1}\,,\qquad p=p^*\,\frac{\mu_0 k_1}{k}\,,\qquad \boldsymbol{x}=\boldsymbol{x}^*\,d\,,\qquad \boldsymbol{v}=\boldsymbol{v}^*\,\frac{k_1}{d}\,,\qquad T=T^*\,\Delta\mathcal{T}$$

$$R^2=\frac{\alpha g\Delta\mathcal{T}\,dk\varrho_0}{k_1\mu_0}\,,\quad \Gamma=\gamma\Delta\mathcal{T}\,,\quad T_0=T_0^*\,\Delta\mathcal{T}\,,\quad T_1=T_1^*\,\Delta\mathcal{T}\,,\quad J=\frac{c_F\,\varrho_0 k_1 k^{1/2}}{\mu_0 d}$$

where, in particular, R is the Rayleigh number, J and Γ are physical parameters, dropping all asterisks, the dimensionless form of the systems (4.2) and (4.3) in the strip $\mathbb{R}^2\times[0,1]$ are, respectively

DOB Equations

$$\begin{cases}\nabla p=-f[\Gamma(T-T_0)]\,\boldsymbol{v}+R^2(T-T_0)\,\boldsymbol{k}\\ \nabla\cdot\boldsymbol{v}=0\\ AT_t+\boldsymbol{v}\cdot\nabla T=\Delta T\,,\end{cases} \tag{4.5}$$

FOB Equations

$$\begin{cases}\nabla p=-f[\Gamma(T-T_0)]\,\boldsymbol{v}+R^2(T-T_0)\,\boldsymbol{k}-J|\boldsymbol{v}|\boldsymbol{v}\\ \nabla\cdot\boldsymbol{v}=0\\ AT_t+\boldsymbol{v}\cdot\nabla T=\Delta T\,.\end{cases} \tag{4.6}$$

To the systems (4.5) and (4.6) we append the boundary conditions

$$T(z=0)=T_1+1\,,\qquad T(z=d)=T_1. \tag{4.7}$$

The problems (4.5), (4.7) and (4.6), (4.7) admit the following nonconvecting stationary solution $m_0(\overline{\boldsymbol{v}}=\boldsymbol{0},\ \overline{T}=-z+T_1+1,\ \overline{p}(z))$, where $\overline{p}(z)$ is such that

$$\frac{d\overline{p}(z)}{dz}=R^2(\overline{T}(z)-T_0)\,.$$

Indicating by $R\boldsymbol{u}=(Ru,Rv,Rw)$, θ, $R\pi$ respectively the perturbation to the velocity field, the temperature field and the pressure field, the equations governing the perturbation in the strip $\mathbb{R}^2 \times [0, 1]$ are:

DOB Equations

$$\text{(4.8)} \qquad \begin{cases} \nabla\pi = -f[\Gamma(\xi - z) + \Gamma\theta]\,\boldsymbol{u} + R\theta\boldsymbol{k} \\ \nabla\cdot\boldsymbol{u} = 0 \\ A\theta_t = \Delta\theta + Rw - R\boldsymbol{u}\cdot\nabla\theta \end{cases}$$

FOB Equations

$$\text{(4.9)} \qquad \begin{cases} \nabla\pi = -f[\Gamma(\xi - z) + \Gamma\theta]\,\boldsymbol{u} + R\theta\boldsymbol{k} - JR|\mathbf{u}|\mathbf{u} \\ \nabla\cdot\mathbf{u} = 0 \\ A\theta_t = \Delta\theta + Rw - R\boldsymbol{u}\cdot\nabla\theta \end{cases}$$

with

$$\xi = T_1 + 1 - T_0.$$

To the previous system we append the following initial conditions

$$\text{(4.10)} \qquad \boldsymbol{u}(P, 0) = \boldsymbol{u}_0(P), \qquad \theta(P, 0) = \theta_0(P)$$

and boundary conditions:

$$\text{(4.11)} \qquad w = \theta = 0 \quad \text{on } z = 0, 1\,.$$

In the sequel we assume that the perturbation fields $\boldsymbol{u}$, θ and π are sufficiently smooth, that they are periodic functions of x and y, of periods $2\pi/a_1$, $2\pi/a_2$. We shall denote by $\Omega = [0, 2\pi/a_1] \times [0, 2\pi/a_2] \times [0, 1]$ the periodicity cell, and $a = (a_1^2 + a_2^2)^{1/2}$ the wave number, by $\langle\cdot\rangle$ and $\|\cdot\|$ respectively the integral and the L^2-norm on Ω. Finally, taking into account the fact that the stability of m_0 makes sense only in a class of solutions of (4.8), (4.10), (4.11) and (4.9)-(4.11) in which m_0 is unique, we exclude any other rigid solutions on requiring that

$$\langle u\rangle = \langle v\rangle = 0\,.$$

5. – Nonlinear stability of m_0 for the Darcy model.

In this section we study the nonlinear stability of m_0 for the i.b.v.p. (4.8), (4.10), (4.11) in which we consider the law (1.3)-(1.4) for the fluid viscosity [5]. By following the Liapunov direct method [7, 9, 13], we choose the following Liapunov functional

$$\text{(5.1)} \qquad V = \frac{A}{2}\|\theta\|^2$$

and we evaluate the time derivative of $V(t)$ along the solution of (4.8), (4.10), (4.11). Then on taking into account the boundary conditions (4.11) and on applying the divergence theorem, the integration on the periodicity cell Ω leads to:

$$\frac{dV}{dt} = 2R\langle \theta w \rangle - \langle \exp[\Gamma(z-\xi) - \Gamma\theta]\,|\boldsymbol{u}|^2\rangle - \|\nabla\theta\|^2. \tag{5.2}$$

Then, on taking into account the a-priori estimate for temperature perturbations proved in [5] and on setting

$$\boldsymbol{u}_1 = \exp[-\Gamma(\overline{\theta}_0 + 1)/2]\,\boldsymbol{u}$$

from (5.2) one has:

$$\frac{dV}{dt} \leqslant 2R\exp[\Gamma(\overline{\theta}_0+1)/2]\langle \theta w_1\rangle - \langle \exp[\Gamma(z-\xi)]\,|\boldsymbol{u}_1|^2\rangle - \|\nabla\theta\|^2, \tag{5.3}$$

in which $\boldsymbol{u}_1 = (u_1, v_1, w_1)$, $\overline{\theta}_0 = \operatorname{ess}\sup_{\Omega}[(\theta_0(P) + \overline{T}(z) - T_1 - 1)_+](<\infty)$ (4).
Now, by following the standard energy method, we set

$$\frac{1}{R_L} = \max_{\mathcal{H}} \frac{I_1}{D_1}, \tag{5.4}$$

where

$$I_1 = 2\langle \theta w_1\rangle, \qquad D_1 = \langle \exp[\Gamma(z-\xi)]\,|\boldsymbol{u}_1|^2 > + \|\nabla\theta\|^2$$

and $\mathcal{H}$ is the class of *admissible kinematic perturbations*, i.e.

$$\mathcal{H} = \{\boldsymbol{u}_1, \theta : \nabla\cdot\boldsymbol{u}_1 = 0;\ \boldsymbol{u}_1, \theta \text{ are regular in } x \text{ and } y \text{ directions, of period } 2\pi/a_1, 2\pi/a_2, \text{ satisfying (4.11) and such that } D_1 < \infty\}.$$

The maximum (5.4) exists by virtue of the Rionero's theorem [12].

From (5.3), by virtue of (5.4), we obtain

$$\frac{dV}{dt} \leqslant \left(\frac{R\exp[\Gamma(\overline{\theta}_0+1)/2] - R_L}{R_L}\right) D_1. \tag{5.5}$$

The following nonlinear stability result holds true

(4) The function $(\theta_0(P) + \overline{T}(z) - T_1 - 1)_+$ means the following truncated function

$$(\theta_0(P) + \overline{T}(z) - T_1 - 1)_+ = \begin{cases} \theta_0(P) + \overline{T}(z) - T_1 - 1 & \theta_0(P) > T_1 + 1 - \overline{T}(z) \\ 0 & \theta_0(P) \leqslant T_1 + 1 - \overline{T}(z). \end{cases}$$

THEOREM 1. – *In the class of perturbations* $\theta(P, t)$ *such that*

$$\overline{\theta}_0 = \operatorname{ess}\sup_{\Omega} [(\theta_0(P) + \overline{T}(z) - T_1 - 1)_+] < M \tag{5.6}$$

where M *is a positive constant, the condition*

$$R < R_L \exp[-\Gamma(M+1)/2] \tag{5.7}$$

ensures the asymptotic, exponential, nonlinear stability of the conduction solution m_0 *with respect to the V-norm, according to the following inequality*

$$V(t) \leq V(0) \exp\left[\left(\frac{R\exp[\Gamma(M+1)/2] - R_L}{R_L}\right) t\right], \qquad t \geq 0\,. \tag{5.8}$$

Now, in order to determine the numbers R_L involved in the stability condition (5.7), we have to solve the variational problem (5.4). To this end let us write the Euler-Lagrange equations that solve the problem (5.4), i.e.

$$\begin{cases} \nabla\omega = -\exp[\Gamma(z-\xi)]\,\boldsymbol{u}_1 + R_L\theta\boldsymbol{k} \\ 0 = \Delta\theta + R_L w_1, \end{cases} \tag{5.9}$$

with ω a Lagrange multiplier, to which we add the boundary condition (4.11).

REMARK 1. – *Let us notice that, the Euler-Lagrange equations* (5.9) *coincide with the linear version of the equations* (4.8), *under the hypothesis that the principle of exchange of stabilities holds true. But, the validity of the principle of exchange of stabilities immediately follows from the symmetry of the linear operator* L *of the problem* (4.8) *with* $f[\Gamma(\xi - z)] = \exp[\Gamma(z-\xi)]$ (5) [7].

(5) Let us consider the expression of the linear operator L of the system (4.8), i.e.

$$L \stackrel{\text{def}}{=} \begin{bmatrix} -\exp[\Gamma(z-\xi)] & 0 & 0 & 0 & -\partial_x \\ 0 & -\exp[\Gamma(z-\xi)] & 0 & 0 & -\partial_y \\ 0 & 0 & -\exp[\Gamma(z-\xi)] & R & -\partial_z \\ 0 & 0 & R & \Delta & 0 \\ -\partial_x & -\partial_y & -\partial_z & 0 & 0 \end{bmatrix}. \tag{5.10}$$

By inspection of the matrix (5.10), it immediately follows the symmetry of L with respect to the L^2-scalar product.

The coincidence between the Euler-Lagrange equations (5.9) *and the linear version of the system* (4.8) *ensure that*

$$R \leqslant R_L$$

is a necessary *and* sufficient *condition for the linear stability* [6, 7].

REMARK 2. – *On denoting by R_E the critical value of nonlinear stability, let us notice that from* (5.7) *we have*

$$R_E = R_L \exp[-\Gamma(M+1)/2]$$

and hence

$$R_E < R_L,$$

with R_L the threshold of linear stability. However, let us underline that in our result, when $R \to R_E$, is not request that the initial data become vanishingly small.

The numerical values of R_L in terms of

$$R_{CL}^2 = \min_{a^2} R_L^2$$

are evaluated by using the compound matrix method and the golden section search [13] and are listed in Table 1 [5].

TABLE 1. – *Critical Rayleigh numbers against Γ, with $\xi = 0.5$*

Γ	R_{CL}^2
0	39.4784
0.5	39.4154
1	39.2203
1.5	38.8761
2	38.3574
2.5	37.6347
3	36.6803

The choice $\xi = 1/2$ means to choose $T_0 = (2T_1 + 1)/2$ as reference temperature.

6. – Nonlinear stability of m_0 for Forchheimer model.

In this section, by using the Liapunov direct method, we study the nonlinear stability of the conduction solution m_0 for the Forchheimer model on taking into account the viscosity variation (1.3) [4]. To this end we introduce the

Liapunov functional:

$$V = \frac{A}{2}\|\theta\|^2 \tag{6.1}$$

and we evaluate the time derivative of $V(t)$ along the solution of (4.9)-(4.11). Then, by virtue of the boundary conditions (4.11), on applying the divergence theorem and the Taylor's formula, we find

$$A\frac{dV}{dt} \leqslant RI - D - \Gamma\langle f'[\Gamma(\xi - z)]\,\theta|\boldsymbol{u}|^2\rangle - JR\langle|\mathbf{u}|^3\rangle, \tag{6.2}$$

where:

$$I \overset{\text{def}}{=} 2\langle\theta w\rangle, \qquad D \overset{\text{def}}{=} \langle f[\Gamma(\xi - z)]\,|\boldsymbol{u}|^2\rangle + \|\nabla\theta\|^2 . \tag{6.3}$$

By following the standard energy method we set:

$$\frac{1}{R_E^*} = \max_{\mathcal{H}} \frac{I}{D}, \tag{6.4}$$

where

$$\mathcal{H} = \{\boldsymbol{u}, \theta : \nabla\cdot\boldsymbol{u} = 0;\ \boldsymbol{u}, \theta \text{ are regular in } x \text{ and } y \text{ directions, of period } 2\pi/a_1, 2\pi/a_2, \text{ satisfying (4.11) and such that } D < \infty\}$$

and the maximum (6.4) exists by virtue of the Rionero's theorem [12].

From (6.2) and (6.4) it turns out:

$$A\frac{dV}{dt} \leqslant -D\left(1 - \frac{R}{R_E^*}\right) + \Gamma m\langle|\theta|\,|\mathbf{u}|^2\rangle - JR\langle|\mathbf{u}|^3\rangle, \tag{6.5}$$

where $m \overset{\text{def}}{=} -f'[T(\xi - 1)] \geqslant 0$. Now by means of Holder and Young inequalities, and the well known imbedding inequality

$$\left(\int_\Omega \varphi^4\, d\Omega\right)^{1/4} \leqslant \|\nabla\varphi\|,$$

we obtain:

$$\Gamma m\langle|\theta|\,|\boldsymbol{u}|^2\rangle \leqslant (\Gamma m)^3 \frac{4}{27J^2R^2}\|\theta\|\|\nabla\theta\|^2 + JR\langle|\boldsymbol{u}|^3\rangle. \tag{6.6}$$

Further, by using (6.1), $(6.3)_2$, (6.5) and (6.6) we have:

$$A\frac{dV}{dt} \leqslant -(\Gamma m)^3 \frac{4\sqrt{2}}{27J^2R^2} D(t)[\delta - V^{1/2}(t)], \tag{6.7}$$

where

$$\delta = \left(1 - \frac{R}{R_E^*}\right)\left[(\Gamma m)^3 \frac{4\sqrt{2}}{27J^2R^2}\right]^{-1}.$$

Finally, by virtue of Poincaré inequality and (6.7), applying a recursive argument, the following theorem holds true:

THEOREM 2. – *If $R < R_E^*$ and $V(0) < \delta^2$, then the basic motion is nonlinerly exponentially asimptotically stable and there exists a positive constant η such that:*

$$V(t) \leqslant V(0)\exp\{-\eta[\delta - V(0)^{1/2}]\,t\}, \qquad t \geqslant 0\,.$$

THEOREM 3. – *If R_L^* is the critical Rayleigh number of linear instability, then*

$$R_L^* = R_E^*\,, \tag{6.8}$$

and hence the condition $R < R_L^*$ is necessary and sufficient for linear stability.

PROOF. – In order to prove (6.8) we observe that the linear operator L of (4.9), i.e.

$$L \overset{\text{def}}{=} \begin{bmatrix} \Delta & 0 & 0 & 0 & R \\ 0 & 0 & \partial_x & \partial_y & \partial_z \\ 0 & \partial_x & -f[\Gamma(\xi - z)] & 0 & 0 \\ 0 & \partial_y & 0 & -f[\Gamma(\xi - z)] & 0 \\ R & \partial_z & 0 & 0 & -f[\Gamma(\xi - z)] \end{bmatrix} \tag{6.9}$$

is symmetric with respect to the L^2-scalar product. By virtue of this symmetry, the coincidence (6.8) immediately follows [7].

In order to determine the critical Rayleigh number R_L^*, we have to solve the variational problem (6.4). To this end, let us consider the Euler-Lagrange equations that solve the variational problem (6.4), i.e.

$$\begin{cases} 2f[\Gamma(\xi - z)] - 2\Gamma f'[\Gamma(\xi - z)]\,w_z = R_E^*\,\Delta_1\theta \\ 2\Delta\theta = -R_E^*\,w\,, \end{cases} \tag{6.10}$$

to which we append the boundary conditions (4.11). For many real liquids a good approximation for the dynamic viscosity is (1.3) with f given by (1.4). In this case, by using the compound matrix method and the golden section search [13], we found the numerical values in terms of $R_{CL}^{*2} = \min_{a^2} R_E^{*2}$ and they are listed in Table 2.

TABLE 2. – *Critical Rayleigh numbers against Γ, with $\xi = 0.5$.*

Γ	R_{CL}^{*2}
0	39.4784
1	39.2203
2	38.3574
3	36.6803
4	34.0054
8	16.8315
8.1	16.4123
8.2	15.9988
8.3	15.5912
8.4	15.1897

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REFERENCES

[1] G. I. BARENBLATT - V. M. ENTOV - V. M. RYZHIK: *Theory of fluid flows through natural rocks*, Kluwer Academic Publishers (1990).

[2] F. CAPONE - M. GENTILE, *Nonlinear stability analysis of convection for fluids with exponentially temperature-dependent viscosity*, Acta Mechanica, **107** (1994), 53.

[3] F. CAPONE - M. GENTILE, *Nonlinear stability analysis of the Bénard problem for fluids with a convex nonincreasing temperature depending viscosity*, Continuum Mech. Thermodyn., **7** (1995), 297-309.

[4] F. CAPONE - M. GENTILE, *On the influence of the Forchheimer term in convective instabilities in porous media for fluids with temperature depending viscosity*, Rend. Circolo Mat. Palermo, Serie II, Suppl. **57** (1998), 91-95.

[5] F. CAPONE - S. RIONERO, *Temperature dependent viscosity and its influence on the onset of convetion in a porous medium*, Rend. Acc. Sc. fis. mat. Napoli, vol. **LXVI** (1999), 159-172.

[6] S. CHANDRASEKHAR, *Hydrodynamic and hydromagnetic stability*, New York, Dover (1961).

[7] J. FLAVIN - S. RIONERO, *Qualitative estimates for partial differential equations. An introduction*, Boca Raton, Florida: CRC Press (1996).

[8] D. D. JOSEPH - D. A. NIELD - G. PAPANICOLAU, *Nonlinear equation governing flow in a saturated porous medium*, Water Resources Res., **18**, 1049-1052 and **19** (1982), 591.

[9] D. D. JOSEPH, *Stability of fluid motions I-II*, Springer Tracts in Natural Philosophy, vols. **27-28** (1976).

[10] D. A. NIELD - A. BEJAN, *Convection in porous media*, Berlin Heidelberg New York: Springer-Verlag (1992).
[11] Y. QIN - J. CHADAM, *Nonlinear convective stability in a porous medium*, Studies In Appl. Math. (1996), 273-288.
[12] S. RIONERO, *Metodi variazionali per la stabilità asintotica in media in magnetoidrodinamica*, Ann. Mat. Pura Appl., **78** (1968), 339-364.
[13] B. STRAUGHAN, *The energy method, stability and nonlinear convection*, Berlin Heidelberg New York Tokyo: Springer (1992).

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