Bollettino Unione Matematica Italiana

F. Flamini, C. Madonna

Geometric linear normality for nodal curves on some projective surfaces

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **4-B** (2001), n.1, p. 269–283.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2001_8_4B_1_269_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2001.

Bollettino U. M. I.
(8) 4-B (2001), 269-283

Geometric Linear Normality for Nodal Curves on Some Projective Surfaces.

F. FLAMINI (*) C. MADONNA

Sunto. – *In questo lavoro si generalizzano alcuni risultati di [3] riguardanti la proprietà di alcune curve nodali, su superficie non-singolari in $\mathbb{P}^r$, di essere «geometricamente linearmente normali» (concetto che estende la ben nota proprietà di essere linearmente normale). Precisamente, per una data curva C, irriducibile e dotata di soli punti nodali come uniche singolarità, che giace su una superfice S proiettiva, non-singolare e linearmente normale, si determina un limite superiore «sharp» sul numero dei nodi di C, $\delta = \delta(C, S)$, di modo che C è geometricamente linearmente normale se il numero dei suoi nodi è minore di δ. Trattiamo alcuni esempi di superficie che sono elementi di una componente del luogo di Noether-Lefschetz delle superficie in $\mathbb{P}^3$ oppure scoppiamenti di alcune superficie proiettive cui il nostro risultato numerico si può applicare facilmente. Infine, per dimostrare che il nostro bound è ottimale, nel paragrafo 3 vengono considerati inoltre esempi di superficie «canoniche» intersezioni complete.*

Introduction.

It is well-known that projective, non-singular complete intersection varieties are *linearly normal*, i.e. they are not isomorphic projection of non-degenerate varieties in higher dimensional projective spaces. From the cohomological point of view, a projective variety $X \subset \mathbb{P}^r$ is linearly normal if and only if $h^1(X, \mathcal{I}_X(1)) = 0$, i.e. the linear series $|\mathcal{O}_X(1)|$ cut out by the hyperplanes is complete. This definition makes sense even if X is singular.

However, one can extend this notion by considering the *geometric linear normality property* of singular varieties $X \subset \mathbb{P}^r$, having some restricted type of singularities which can arise from projections. We state the following:

DEFINITION 1. – Let C be any reduced curve in $\mathbb{P}^r$. We say that C is *geometrically linearly normal* if the normalization map $\nu : \widetilde{C} \to C$ cannot be factored into a non-degenerate map $\widetilde{C} \to \mathbb{P}^N$, with $N > r$, followed by a projection.

In [3], conditions for the geometric linear normality property of certain nodal curves on smooth projective surfaces in $\mathbb{P}^3$ have been studied. In this pa-

(*) 1991 *Mathematics Subject Classification*: 14 J 29, 14 H 10.
The first author is a member of GNSAGA-CNR.

per, we generalize Theorem 3.5 in [3] by proving the following main theorem.

THEOREM 2. – *Let S be a smooth, non-degenerate and linearly normal surface in $\mathbb{P}^r$ and let H be the general hyperplane section on S, such that $h^1(S, \mathcal{O}_S(H)) = 0$. Let C be a smooth, irreducible divisor on S. Suppose that*:

i) $CH > H^2$;

ii) $(C-2H)^2 > 0$ *and* $C(C-2H) > 0$;

iii) $\nu(C, H) < 4(C(C-2H)-4)$, *where* $\nu(C, H)$ *is the Hodge number of* C *and* H;

iv) $\delta < (C(C-2H) + \sqrt{C^2(C-2H)^2})/8$.

Then, if $C' \in |C|$ is a reduced, irreducible curve with only δ nodes as singular points and if N denotes the 0*-dimensional scheme of nodes in C', N imposes independent conditions to $|C-H+K_S|$.*

COROLLARY 1. – *In the hypotheses of Theorem* 2, *if C is linearly normal in $\mathbb{P}^r$ then C' is geometrically linearly normal.*

REMARK 1. – Before going into details, we want to spend a few words on the cohomological conditions we gave in the statement of the theorem above. First of all, the linear normality of S means that $h^1(S, \mathcal{I}_S(H)) = 0$ and this is clearly necessary since, otherwise, we can not hope to say too much on C'. On the other hand, as it will be also clear from the proof of Theorem 1, the vanishing condition $h^1(S, \mathcal{O}_S(H)) = 0$ implies that the linear series $|\omega_{\widetilde{C}}(\nu^*(-H))|$ is complete, where $\omega_{\widetilde{C}}$ denotes the canonical sheaf on the smooth curve $\widetilde{C}$ and $\nu : \widetilde{C} \to C$ is the normalization map. More precisely, if $C \subset S$ is a δ-nodal curve and if $\mu : \widetilde{S} \to S$ denotes the blow-up of S along the set of nodes of C, such that $B = \sum_{i=1}^{\delta} E_i$ is the exceptional divisor in $\widetilde{S}$, the map μ induces the normalization map $\nu : \widetilde{C} \to C$. The exact sequence defining $\omega_{\widetilde{C}}$ gives rise to

$$0 \to \mathcal{O}_{\widetilde{S}}(\mu^*(K_S - H) + B) \to \mathcal{O}_{\widetilde{S}}(\mu^*(K_S + C - H) - B) \to$$

$$\to \omega_{\widetilde{C}}(\nu^*(-H)) \to 0\,.$$

We observe that $h^1(\widetilde{S}, \mathcal{O}_{\widetilde{S}}(\mu^*(K_S - H) + B)) = 0$ implies that the map

$$H^0(\widetilde{S}, \mathcal{O}_{\widetilde{S}}(\mu^*(K_S + C - H) - B)) \to H^0(\widetilde{C}, \omega_{\widetilde{C}}(\nu^*(-H)))$$

is surjective. Indeed, observe that by Serre duality on $\widetilde{S}$, $h^1(\widetilde{S}, \mathcal{O}_{\widetilde{S}}(\mu^*(K_S - H) + B)) = h^1(\widetilde{S}, \mathcal{O}_{\widetilde{S}}(K_{\widetilde{S}} - \mu^*(H))) = h^1(\widetilde{S}, \mathcal{O}_{\widetilde{S}}(\mu^*(H)))$, so the vanishing follows from Leray spectral sequence and our assumption on $h^1(S, \mathcal{O}_S(H))$.

We remark that our main result gives purely numerical conditions on the divisors C and H in order to determine the geometric linear normality property for nodal curves on S. As we shall see in the sequel, these conditions can be directly checked in many cases, where other criteria fail.

The paper consists of three sections. In Section 1, we recall some terminology and notation. Section 2 contains the main theorem, whereas Section 3 is devoted to examples, some of which show the sharpness of our results.

Acknowledgments. The authors wish to thank Prof. L. Chiantini and Prof. E. Sernesi for their fundamental remarks and helpful discussions.

1. – Notation and preliminaries.

We work in the category of $\mathbb{C}$-schemes. X is an *algebraic m-fold* if it is a reduced, irreducible and nonsingular scheme of finite type over $\mathbb{C}$ and of dimension m. If $m = 1$, then X is a smooth *curve*; $m = 2$ is the case of a nonsingular *surface*. If $X \subset Y$ are two algebraic schemes, $\mathcal{J}_{X/Y}$ or $\mathcal{J}_X$, denotes the *ideal sheaf* of X in Y. When Y is smooth, K_Y denotes a canonical divisor.

Let X be a m-fold and let $\mathcal{E}$ be a rank r vector bundle on X; $c_i(\mathcal{E})$ will denote the i^{th}-*Chern class* of $\mathcal{E}$, $1 \leq i \leq r$. As usual, $h^i(-) := \dim H^i(-)$.

If C is a curve, $p_a(C) = h^1(\mathcal{O}_C)$ denotes its *arithmetic genus*, whereas $p_g(C)$ denotes its *geometric genus*, the arithmetic genus of its normalization. For a smooth curve C, ω_C shall denote the canonical sheaf, i.e. $\omega_C \cong \mathcal{O}_C(K_C)$.

Let $S \subset \mathbb{P}^r$ be a smooth, non-degenerate linearly normal surface, and H be the hyperplane section on S; then,

$$h^0(S, \mathcal{O}_S(H)) = h^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(H)) = r + 1 \,. \tag{1}$$

DEFINITION 2. – Let S be a smooth projective surface and denote by $Div(S)$ the set of the divisors on S. An element $B \in Div(S)$ is said to be *nef*, if $B \cdot D \geq 0$ for each irreducible curve D on S (where $\cdot$ denotes the *intersection form* on S; in the sequel we will omit $\cdot$). A nef divisor B is said to be *big* if $B^2 > 0$.

REMARK 2. – We recall that, given a smooth surface S, $N(S)^+$ usually denotes the ample divisor cone on S; thus $F \in N(S)^+$ if and only if $F^2 > 0$ and $FA > 0$ for any ample divisor A on S. By Kleiman's criterion (see, for example, [8]), a nef divisor B is in the closure of $N(S)^+$.

DEFINITION 3. – Let S be a smooth surface and $C \in Div(S)$. We denote by $\nu(C, H)$ the *Hodge number* of C and H,

$$\nu(C, H) := (CH)^2 - C^2H^2.$$

By the Index Theorem (see, for example, [1] or [5]) this number is non-negative.

REMARK 3. – Let $S \subset \mathbb{P}^r$ be a smooth, non-degenerate linearly normal surface and H be the hyperplane section on S. Let $C \in Div(S)$ be an effective divisor. Suppose that C is smooth, non-degenerate and such that $C - H$ big and nef. Clearly $h^0(\mathcal{O}_S(H-C)) = 0$, hence we have the following exact sequence

$$0 \to H^0(S, \mathcal{O}_S(H)) \to H^0(C, \mathcal{O}_C(H)) \to H^1(S, \mathcal{O}_S(H-C)) \to \dots$$

By Serre duality, $h^1(S, \mathcal{O}_S(H-C)) = h^1(S, \mathcal{O}_S(K_S + C - H))$ and, by the Kawamata-Viehweg vanishing theorem (see, for example, [11]), this equals 0. Hence, by (1), it follows

$$h^0(C, \mathcal{O}_C(H)) = h^0(S, \mathcal{O}_S(H)) = r+1\,, \tag{2}$$

so we get that C is linearly normal.

We recall the following:

DEFINITION 4. – We say that a linear system on a surface is a *Bertini linear system* if its general element is smooth and irreducible.

DEFINITION 5. – Let S be a smooth projective surface. A rank 2 vector bundle $\mathcal{E}$ on S is said to be *Bogomolov-unstable* if there exist M, $B \in Div(S)$ and a 0-dimensional scheme Z (possibly empty) with the following exact sequence

$$0 \to \mathcal{O}_S(M) \to \mathcal{E} \to \mathcal{I}_Z(B) \to 0 \tag{3}$$

and moreover $(M - B) \in N(S)^+$.

REMARK 4. – We recall that $\mathcal{E}$ is Bogomolov-unstable when $c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) > 0$ (see [2] or [10]).

2. – Geometric linear normality on some projective, non-degenerate and linearly normal surfaces.

In this section we discuss the problem of geometric linear normality for nodal curves on a smooth projective surface, which is linearly normal and satisfies a suitable cohomological condition (see Remark 1). More precisely, we characterize the geometric linear normality of a nodal curve C, in a Bertini linear system, in terms of its set of nodes.

THEOREM 1. – *Let S be a smooth, non-degenerate and linearly normal surface in $\mathbb{P}^r$ such that $h^1(S, \mathcal{O}_S(H)) = 0$. Let $|D|$ be a Bertini linear system on S, whose general element is supposed to be linearly normal in $\mathbb{P}^r$. Let $C \in |D|$ be an irreducible curve with only δ nodes as singular points. Then C is geometrically linearly normal if and only if the set of nodes, N, imposes independent conditions to the linear system $|D + K_S - H|$.*

PROOF. – Let D be the general member of the linear system $|D|$. By the linear normality hypothesis and by Riemann-Roch, we have

$$h^1(D, \mathcal{O}_D(H)) = (r+1) - deg(D) + p_a(D) - 1\,,$$

hence, by Serre duality and by adjunction on S, we get

$$h^0(D, \mathcal{O}_D(D + K_S - H)) = (r+1) - deg(D) + p_a(D) - 1\,. \tag{4}$$

Now, let $C \in |D|$ be a curve with only δ nodes as singularities. Denote by $\mu : \tilde{S} \to S$ the blow-up of S along the set of nodes of C, N, and let $B = \sum_{i=1}^{\delta} E_i$ be the exceptional divisor in $\tilde{S}$. The blow-up induces the normalization map $\nu : \tilde{C} \to C$. By adjunction theory,

$$\omega_{\tilde{C}} = \mathcal{O}_{\tilde{C}}(K_{\tilde{S}} + \tilde{C}) = \mathcal{O}_{\tilde{C}}(\mu^*(K_S + C) - B) = \mathcal{O}_{\tilde{C}}(\nu^*(K_S + C)(-\tilde{N}))\,, \tag{5}$$

where $\mathcal{O}_{\tilde{C}}(\tilde{N}) = \mathcal{O}_{\tilde{C}}(B)$ is a divisor of degree 2δ on $\tilde{C}$, formed by the points which map to the nodes of C. From Riemann-Roch on $\tilde{C}$, it follows that

$$h^1(\tilde{C}, \mathcal{O}_{\tilde{C}}(\nu^*(H))) = h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(\nu^*(H))) - deg(C) + p_a(C) - 1 - \delta\,.$$

By using (5) and the fact that $C \sim D$ on S, we get

$$h^0(\mathcal{O}_{\tilde{C}}(\nu^*(K_S + D - H)(-\tilde{N}))) = h^0(\mathcal{O}_{\tilde{C}}(\nu^*(H))) - deg(C) + p_a(C) - 1 - \delta\,. \tag{6}$$

Observe that $h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(\nu^*(H))) = r+1$ if and only if

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(\nu^*(K_S + D - H)(-\tilde{N}))) = (r+1) - deg(C) + p_a(C) - 1 - \delta\,.$$

By using (4) and the fact that the adjunction on S is independent from the chosen element in $|D|$, we obtain

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(\nu^*(H))) = r+1 \Leftrightarrow h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(\nu^*(K_S + D - H)(-\tilde{N}))) = h^0(C, \mathcal{O}_C(D + K_S - H)) - \delta\,. \tag{7}$$

Now, we use our assumption $h^1(S, \mathcal{O}_S(H)) = 0$. It implies, by duality on S, that

$$h^0(S, \mathcal{O}_S(D + K_S - H)) - h^0(S, \mathcal{O}_S(K_S - H)) = h^0(C, \mathcal{O}_C(D + K_S - H))$$

whereas, on $\tilde{S}$,

$$h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\mu^*(K_S + D - H) - B)) - h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\mu^*(K_S - H) + B) =$$

$$h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(\nu^*(K_S + D - H)(-\tilde{N}))),$$

since, by Leray spectral sequence, $h^1(\mathcal{O}_{\tilde{S}}(\mu^*(K_S - H) + B)) = h^1(\mathcal{O}_S(H)) = 0$. Substituting in (7), it gives

$$h^0(\mathcal{O}_{\tilde{C}}(\nu^*(H))) = r + 1 \Leftrightarrow h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\mu^*(K_S + D - H) - B)) =$$

$$h^0(S, \mathcal{O}_S(D + K_S - H)) - \delta.$$

The claim follows from the fact that $h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(\mu^*(K_S + D - H) - B)) = h^0(S, \mathcal{I}_{N/S}(K_S + D - H))$. ■

For what concerns the geometric linear normality problem, by considering *Bogomolov unstable* vector bundles on S we can obtain an upper-bound δ_u on the number of nodes such that if C has at most $\delta_u - 1$ nodes, then it is geometrically linearly normal. Using the procedure of [4], we can prove the following result.

THEOREM 2. – *Let S be a smooth, non-degenerate and linearly normal surface in $\mathbb{P}^r$ such that $h^1(\mathcal{O}_S(H)) = 0$. Let C be a smooth, irreducible divisor on S. Suppose that*:

i) $CH > H^2$;

ii) $(C - 2H)^2 > 0$ *and* $C(C - 2H) > 0$;

iii) $\nu(C, H) < 4(C(C - 2H) - 4)$, *where $\nu(C, H)$ is the Hodge number of C and H*;

iv) $\delta < (C(C - 2H) + \sqrt{C^2(C - 2H)^2})/8$.

If $C' \in |C|$ is a reduced, irreducible curve with only δ nodes as singular points and if N denotes the 0-dimensional scheme of nodes of C', then N imposes independent conditions to $|C - H + K_S|$.

PROOF. – By contradiction, assume that N does not impose independent conditions to $|C - H + K_S|$. Let $N_0 \subset N$ be a minimal 0-dimensional subscheme of N for which this property holds and let $\delta_0 = |N_0|$. This means that $h^1(S, \mathcal{I}_{N_0}(C - H + K_S)) \neq 0$ and that N_0 satisfies the Cayley-Bacharach condition (see, for example [7]). Therefore, a non-zero element of $H^1(\mathcal{I}_{N_0}(C - H + K_S))$ gives rise to a non-trivial rank 2 vector bundle $\mathcal{E} \in Ext^1(\mathcal{I}_{N_0}(C - H), \mathcal{O}_S)$ fitting in the following exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{I}_{N_0}(C - H) \to 0, \tag{8}$$

with $c_1(\mathcal{E}) = C - H$ and $c_2(\mathcal{E}) = \delta_0$ hence

$$c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) = (C-H)^2 - 4\delta_0. \tag{9}$$

By iv)

$$(C-H)^2 - 4\delta_0 \geqslant (C-H)^2 - 4\delta = C^2 - 2CH + H^2 - 4\delta > H^2 > 0\,,$$

since $\delta_0 \leqslant \delta$ thus $\mathcal{E}$ is Bogomolov-unstable (see Definition 5 and Remark 4), hence $h^0(\mathcal{E}(-M)) \neq 0$. Twisting (8) by $-M$, we obtain

$$0 \to \mathcal{O}_S(-M) \to \mathcal{E}(-M) \to \mathcal{J}_{N_0}(C-H-M) \to 0\,. \tag{10}$$

We claim that $h^0(\mathcal{O}_S(-M)) = 0$; otherwise, $-M$ would be an effective divisor, therefore $-MA > 0$, for each ample divisor A. From (3), it follows that $c_1(\mathcal{E}) = M + B$, so, by (3) and (8),

$$M - B = 2M - C + H \in N(S)^+. \tag{11}$$

Thus

$$MH > \frac{(C-H)H}{2}\,; \tag{12}$$

next by i) it follows that $H(C-H) > 0$, hence $-MH < 0$.

The cohomological exact sequence associated to (10) allows us to deduce that there exists a divisor $\Delta \in |C-H-M|$ s.t. $N_0 \subset \Delta$ and s.t. the irreducible nodal curve $C' \in |C|$, whose set of nodes is N, is not a component of Δ. Otherwise, $-M-H$ would be an effective divisor, whereas, by (12), we get

$$H(-M-H) = -H^2 - HM < -H^2 - \frac{(C-H)\,H}{2} = -\frac{(C+H)\,H}{2} < 0\,,$$

since $H(C+H) = (C-H)\,H + 2H^2 > 0$.

Next, by Bezout's theorem

$$C'\Delta = C'(C-H-M) \geqslant 2\delta_0. \tag{13}$$

On the other hand, taking M maximal, we may further assume that the general section of $\mathcal{E}(-M)$ vanishes in codimension 2. Denote by Z this vanishing-locus, thus, $c_2(\mathcal{E}(-M)) = deg(Z) \geqslant 0$; moreover,

$$c_2(\mathcal{E}(-M)) = c_2(\mathcal{E}) + M^2 + c_1(\mathcal{E})(-M) = \delta_0 + M^2 - M(C-H)\,,$$

which implies

$$\delta_0 \geqslant M(C-H-M)\,. \tag{14}$$

Applying the Index theorem to the divisor pair $(C, 2M - C + H)$, we get

$$C^2(2M - C + H)^2 \leqslant (C(C-H) - 2C(C-H-M))^2. \tag{15}$$

Note now that, from hypothesis i) and the second one of ii) it follows that $C(C-H) > 0$, since $C(C-2H) > 0$ hence $C^2 - HC > HC > 0$. In the same way we find $C^2 > 0$. Since C is irreducible, this also implies that C is a nef divisor. From (13) and from the positivity of $C(C-H)$, it follows that

$$C(C-H) - 2C(C-H-M) \leqslant C(C-H) - 4\delta_0. \tag{16}$$

We observe that the left side member of (16) is non-negative, since $C(C-H) - 2C(C-H-M) = C(2M-C+H)$, where C is effective and, by (11), $2M - C + H \in N(S)^+$. Squaring both sides of (16), together with (15), we find

$$C^2(2M - C + H)^2 \leqslant (C(C-H) - 4\delta_0)^2. \tag{17}$$

On the other hand, by (14), we get

$$(2M - C + H)^2 = 4\left(M - \frac{(C-H)}{2}\right)^2 =$$

$$(C-H)^2 - 4(C-H-M)\,M \geqslant (C-H)^2 - 4\delta_0,$$

i.e

$$(2M - C + H)^2 \geqslant (C-H)^2 - 4\delta_0. \tag{18}$$

Next, we define

$$F(\delta_0) := 16\delta_0^2 - 4C(C-2H)\,\delta_0 + (CH)^2 - C^2H^2. \tag{19}$$

Putting together (17) and (18), it follows that $F(\delta_0) \geqslant 0$. We will show that, with our numerical hypotheses, one has $F(\delta_0) < 0$, proving the statement.

Indeed, the discriminant of the equation $F(\delta_0) = 0$ is $16C^2(C-2H)^2$, and it is a positive number, since $(C-2H)^2 > 0$, by the first one of ii), and $C^2 > 0$. We remark that $F(\delta_0) < 0$ iff $\delta_0 \in (\alpha(C,H), \beta(C,H))$, where

$$\alpha(C,H) = \frac{C(C-2H) - \sqrt{C^2(C-2H)^2}}{8}$$

and

$$\beta(C,H) = \frac{C(C-2H) + \sqrt{C^2(C-2H)^2}}{8};$$

so we have to show that, $\delta_0 \in (\alpha(C,H), \beta(C,H))$.

From iv), it follows that $\delta_0 < \beta(C, H)$. Note that $\alpha(C, H) \geq 0$. Indeed, if $\alpha(C, H) < 0$ then $C(C-2H) < \sqrt{C^2(C-2H)^2}$, which contradicts the Index Theorem, since $C(C-2H) > 0$. In order to simplify the notation, we put $t := C(C-2H)$. Thus, $\alpha(C, H) < 1$ if and only if $t - 8 < \sqrt{t^2 - 4\nu(C, H)}$.

If $t - 8 < 0$, the previous inequality trivially holds, so $\delta_0 > \alpha(C, H)$. Note also that, by iii), $4\nu(C, H) < 16t - 64$, so that $\beta(C, H) > 1$, which ensures there exists at least a positive integral value for the number of nodes.

If $t - 8 \geq 0$, $\alpha(C, H) < 1$ directly follows from iii), whereas $\beta(C, H) > 1$ holds since it is equivalent to $t - 8 > -\sqrt{t^2 - 4\nu(C, H)}$.

In conclusion, our numerical hypotheses contradict $F(\delta_0) \geq 0$, therefore the assumption $h^1(\mathfrak{I}_N(D - H + K_S)) \neq 0$ leads to a contradiction. ■

COROLLARY 1. – *In the hypotheses of previous theorem, if C is linearly normal in $\mathbb{P}^r$ then C' is geometrically linearly normal.*

REMARK 5. – Observe that, if $t - 8 \geq 0$, then

$$\frac{C(C-2H) + C(C-2H) - 8}{8} < \frac{C(C-2H) + \sqrt{C^2(C-2H)^2}}{8},$$

therefore we may change the bound $\delta < \beta(C, H)$ with the more «readable» one $\delta \leq (C(C-2H)/4) - 1$.

Indeed,

$$\frac{C(C-2H) + C(C-2H) - 8}{8} < \frac{C(C-2H) + \sqrt{C^2(C-2H)^2}}{8} \leq \frac{C(C-2H)}{4}.$$

3. – Examples.

This section will be devoted to the study of some examples, which also show the sharpness of our bound in Theorem 2.

First of all, assume that S is a smooth, projective, non-degenerate and linearly normal surface, with Picard group $\mathbb{Z}$-generated by the hyperplane section H. Suppose also that $h^1(S, \mathcal{O}_S(H)) = 0$; then our results easily apply to the cases of nodal curves $C \sim nH$ on S, such that $n \geq 3$ and $deg(S) > (4/n(n-2))$. Indeed, condition ii) in Theorem 2 implies that $n > 2$, whereas condition iii) gives that $\nu(nH, H) = 0$, so $C(C-2H) - 4 = n(n-2)H^2 - 4 > 0$ if and only if $H^2 > (4/n(n-2))$; this means that the degree of S must be greater than or equal to 2, but with the further condition that $S \subset \mathbb{P}^r$ is non-degenerate.

In particular, if we go back to the case of a general surface $S \subset \mathbb{P}^3$, such that

$deg\,(S) \geqslant 2$, the bound on the number of nodes is

$$\delta < \frac{n(n-2)}{4} deg\,(S),$$

which generalizes Theorem 3.5 in [3], where the cases in which K_S is an ample divisor on S are considered.

We can also state the following generalization

PROPOSITION 1. – *Let S be a smooth, non-degenerate complete intersection surface of type $(a_1, \ldots, a_{r-2})$ in $\mathbb{P}^r$, $r \geqslant 4$, and let $C \in |nH|$ with only δ nodes as singular points. Suppose that $n \geqslant 3$ and $deg\,(S) \geqslant 4$; then, if*

$$\delta < \frac{n(n-2)}{4} deg\,(S), \tag{20}$$

hence C is geometrically linearly normal.

PROOF. – Observe, first, that this result obviously generalizes the bounds for general smooth surfaces in $\mathbb{P}^3$, of degree $d \geqslant 2$, mentioned above and the ones of Theorem 3.5, in [3], to the cases of non-degenerate, complete intersections in higher dimensional projective spaces. The proof is a straightforward application of Theorem 2. Indeed, the cohomological condition trivially holds, for a 2-dimensional complete intersection; moreover, the hypotheses on n and on $deg\,(S)$ ensure that conditions i), ii) and iii) of Theorem 2 hold. In the statement of the proposition we considered the bound $deg\,(S) \geqslant 4$ instead of the one obtained by numerical computations, i.e. $deg\,(S) \geqslant 2$, since complete intersections of degree 2 and 3 are obviously degenerate if $r \geqslant 4$. The bound on δ is condition iv). ■

In [3] it is proved the sharpness of the bound on δ for a general quintic surface in $\mathbb{P}^3$. In particular, since in this case the Neron-Severi group of S is such that $NS(S) \cong \mathbb{Z}[K_S]$, then, when $C \sim nH$ on S, with n an odd integer, the bound on the number of nodes is $\delta < (5(n-1)^2/4)$ instead of $5n(n-2)/4$. We shall see that the same occurs in some other cases of general complete intersections. Indeed, we will show the sharpness of bound iv), in Theorem 2, by considering nodal curves $C \sim nH$ on general «canonical» complete intersection surfaces. Since in these cases the Hodge number is zero, this bound reduces to (20); moreover, when n is an odd integer, (20) can be replaced by

$$\delta < \frac{(n-1)^2}{4} deg\,(S),$$

as it follows from Theorem 2.2 in [3]. Applying the same procedure of

[3] we will show that these bounds are almost sharp for a sestic surface in $\mathbb{P}^3$.

To do this, we want to recall that the geometric linear normality property is equivalent, in some particular cases, to another important aspect of families of nodal curves on a projective surface.

REMARK 6. – Let S be a nonsingular projective surface, which is non-degenerate and linearly normal, for which $K_S \sim H$. In such a case, the fundamental condition $h^1(S, \mathcal{O}_S(H)) = 0$, used in the proof of Theorem 1, implies that S is a regular surface. Therefore, Theorem 2 determines purely numerical conditions on the nodal curve C ensuring that its set of nodes imposes independent conditions to the linear system which C belongs to or, similarly (see [3]), that C corresponds to a smooth point of the Severi variety $V_{|D|, \delta}$, $C \in |D|$. We recall that, given a Bertini linear system $|D|$ on a surface S, $V_{|D|, \delta}$ denotes the locally closed subscheme of $|D|$ parametrizing irreducible nodal curves with δ nodes in $|D|$. With abuse of language, it is called the *Severi variety* of δ-nodal curves in $|D|$. The fact that C corresponds to a smooth point of such a varity means that the nodes of C can be independently smoothed. In [3] this problem is studied when K_S is an ample divisor on S and C is a divisor which is numerically equivalent to pK_S, where p is a rational number greater than 2. A first improvement of this result is given in [6], where the authors weakened the assumptions of K_S being ample and considered the cases in which C, $C - K_S$ are ample divisors and $C^2 \geqslant K_S^2$. In [4], purely numerical conditions are given in order to generalize these results on the regularity of the Severi variety $V_{|D|, \delta}$.

Examples of projective, regular, non-degenerate and linearly normal surfaces, such that $K_S \sim H$, are given by general complete intersections in $\mathbb{P}^r$ of type $(a_1, \ldots, a_{r-2})$, such that $\left(\sum_{i=1}^{r-2} a_i\right) = r+2$ (see [11]); therefore, only few cases may occur. More precisely, we have a general quintic surface in $\mathbb{P}^3$, surfaces of type $(2,4)$ and $(3,3)$ in $\mathbb{P}^4$, the surface of type $(2,2,3)$ in $\mathbb{P}^5$, whereas in $\mathbb{P}^6$ we have the case $(2,2,2,2)$. In $\mathbb{P}^r$, for $r \geqslant 7$, no non-degenerate case can occur.

In the following example we consider the case of a general complete intersection of type $(2,4)$ in $\mathbb{P}^4$. The construction can be obviously generalized to the other cases in the list above.

EXAMPLE 1. – Let F_2, F_4 be two general hypersurfaces in $\mathbb{P}^4$ of degree 2 and 4, respectively; let S be the surface of degree 8, which is the complete intersection of F_2 and F_4. Denote by W_2 and W_4 the cones in $\mathbb{P}^5$, over F_2 and F_4 respectively, with the same vertex $P \in \mathbb{P}^5$. Let V_2 and V_m be two general 4-folds in $\mathbb{P}^5$ of degree 2 and m, respectively, where m is a positive integer greater than or equal to 3. Let T be the complete intersection 3-fold of V_2 and V_m and denote by π_P the projection $\pi_P \colon T \to T'$ from the vertex P of T onto the variety T' of dimension 3. It is classically known that the degree of T' is $2m$ and that T' contains a double surface G and in order to compute its degree, we use the technique of «hyperplane sections». Indeed, let us denote by E the curve obtained on T taking two consecutive hyperplane sections; hence E is a complete intersection of type $(2, m, 1, 1)$ in $\mathbb{P}^5$ and so $p_g(E) = m(m-2)+1$. Using the same procedure for $T' \in \mathbb{P}^4$, we obtain a plane curve E' of degree $2m$; therefore, its arithmetic genus is $p_a(E') = 2m^3 - 3m + 1$. Hence, $deg(G) = m^2 - m$.

Let $\widetilde{C}$ be the complete intersection curve in $\mathbb{P}^5$ determined by

$$\widetilde{C} := V_2 \cap V_m \cap W_2 \cap W_4 .$$

$\widetilde{C}$ is a smooth curve of degree $16m$, which lies on the cone of dimension 3, $\widetilde{S} := W_2 \cap W_4$. Denote by C the projection of $\widetilde{C}$ from P; C has degree $16m$ and it is complete intersection of S and T' in $\mathbb{P}^4$. Therefore, $C \in |2mH|$ on S and its singularities coincide with the zero-dimensional scheme of $S \cap G$; thus C has a set N of $\delta = 8m^2 - 8m$ nodes and no other singularities. By construction, $\widetilde{C}$ is the normalization of C which, therefore, cannot be geometrically linearly normal. Observe that, the bound in (20) becomes, in this case, $\delta < 8m^2 - 8m$, hence it is sharp.

REMARK 7. – The above construction shows that our result is sharp for «canonical» complete intersection surfaces. Furthermore, from Theorem 1 it follows that, in this example, N cannot impose independent condition to $|C|$, so that the Severi variety $V_{|2mH|,\, 8m^2-8m}$ is not smooth of the expected dimension, i.e. $dim(|2mH|) - 8m^2 - 8m$, in a neighbourhood of C.

PROPOSITION 2. – *The curve C constructed above is a singular point of $V_{|2mH|,\, 8m^2-8m}$, which is generically smooth, of the expected dimension.*

PROOF. – The previous construction, together with Theorem 3, shows that the tangent space of $V_{|2mH|,\, 8m^2-8m}$ at C has codimension $8m^2 - 8m - 1$ in the tangent space of $|2mH|$ at C (see [3] for details). Hence, $h^1(S, \mathfrak{I}_N(2mH)) = 1$, since C is the projection of a smooth, complete intersection in $\mathbb{P}^5$.

Let C' be a curve in a neighbourhood of C in $V_{|2mH|,\, 8m^2-8m}$, for which the set of nodes N' does not impose independent conditions to $|2mH|$. Then, by semicontinuity, $h^1(S, \mathfrak{I}_{N'}(2mH)) = 1$; therefore, also C' is the projection of a curve $\widetilde{C}'$ in $\mathbb{P}^5$ which «lives» in a neighbourhood of $\widetilde{C}$ in the Hilbert scheme of $\mathbb{P}^5$. It follows that also $\widetilde{C}'$ must be a smooth, complete intersection of the cone $\widetilde{S}$ with some complete intersection 3-fold of type $(2, m)$. If we denote by $\mathcal{M}$ the subvariety of $V_{|2mH|,\, 8m^2-8m}$, formed by these projected curves, we can find an upper-bound for $dim(\mathcal{M})$. By keeping the cones W_2 and W_4 fixed, the normalizations of the elements of $\mathcal{M}$ fill a variety of dimension at most

$$h^0(\tilde{C}, \mathcal{N}_{\tilde{C}/\tilde{S}}) = h^0(\tilde{C}, \mathcal{O}_{\tilde{C}}(2) \oplus \mathcal{O}_{\tilde{C}}(m)) = 8m^2 - 16m + 38 .$$

If we let also the vertex P vary in $\mathbb{P}^5$, we get a variety of dimension at most $8m^2 - 16m + 43$. On the other hand, $V_{|2mH|, 8m^2-8m}$ has dimension at least

$$h^0(S, \mathcal{O}_S(2m)) - 1 - 8m^2 + 8m = 8m^2 + 5 .$$

Since $m \geqslant 3$, then $8m^2 + 5 > 8m^2 - 16m + 43$, which means that the general element of the Severi variety does not arise from this construction and is a smooth point of $V_{|2mH|, 8m^2-8m}$. ■

REMARK 8. – We remark that there exist non-canonical surfaces for which the bound is not sharp. Indeed, let us consider a nonsingular sextic surface S in $\boldsymbol{P}^3$. Let C be a curve on S equivalent to nH, with n an even integer greater than 4. Arguing with cones as in the previous example, we can prove that C has $(3/2)(n^2 - 2n)$ nodes, while the bound in this case is given by the number $(3/2)\,n(n-4)$, and C is the projection of a curve in $\mathbb{P}^4$. It remains to understand what happens in the range $[(3/2)\,n(n-4), (3/2)\,n(n-2) - 1]$.

We end this section by considering some examples of surfaces to which our numerical criterion can be easily applied, whereas other criteria fail. We shall focus on blown-up surfaces or surfaces of $\mathbb{P}^3$ which contain a line L.

1) Let $S \subset \mathbb{P}^3$ be a general smooth quartic. We have therefore, $\mathcal{O}_S(K_S) \cong \mathcal{O}_S$. Let H be the plane section of S. If $\pi : \tilde{S} \to S$ denotes the blow-up in a point $p \in S$ and E the π-exceptional divisor, then $K_{\tilde{S}} \sim E$ is not ample. Moreover, given $C \sim m\pi^*(H)$ on $\tilde{S}$, where m a positive integer, it cannot be an ample divisor since $CK_{\tilde{S}} = 0$. Thus, both the results in [3] and [6] cannot be applied.

However, consider $\tilde{H} = 2\pi^*(H) - E$, which is a very ample divisor, since the linear system $|\tilde{H}|$ trivially separates points and tangent vectors on $\tilde{S}$. If we consider the embedding of $\tilde{S}$ via the complete linear system $|\tilde{H}|$, then $\tilde{S}$ is linearly normal. Furthermore,

$$h^1(\mathcal{O}_{\tilde{S}}(\tilde{H})) = h^1(\mathcal{I}_{\{p\}/S}(2H)) = 0 ,$$

since $\{p\}$ imposes independent conditions to $|2H|$ on $S \subset \mathbb{P}^3$. Observe also that the general element of $|m\pi^*(H)|$ is smooth and irreducible. Furthermore, $C - 2\tilde{H} \sim (m-4)\,\pi^*(H) - 2E$; so that the numerical conditions in Theorem 2 become $(C - 2\tilde{H})^2 = 4(m^2 - 8m + 15) > 0$, $C(C - 2\tilde{H}) = 4m(m-4) > 0$, $\nu(C, \tilde{H}) = 4m^2 < 4(C(C - 2\tilde{H}) - 4) = 4(4m^2 - 16m - 4) > 0$. These simultaneously hold as soon as $m \geqslant 6$. Moreover,

$$\delta < \frac{m(m-4) + m\sqrt{(m-4)^2 - 1}}{2}.$$

From Remark 5, we know that, since $t - 8 = C(C - 2\widetilde{H}) - 8 = m^2 - 4m - 2$ is positive for $m \geqslant 6$, then we may change the bound above with the more «readable» one $\delta \leqslant (C(C - 2\widetilde{H})/4) - 1 = m(m-4) - 1 = m^2 - 4m - 1$.

Thus, if there exists a nodal curve $C \in |m\pi^* H|$, $m \geqslant 6$, such that the number of nodes is

$$\delta \leqslant m^2 - 4m - 1,$$

then C is geometrically linearly normal by Theorem 2.

2) Let S be a smooth quintic surface in $\mathbb{P}^3$ which contains a line L. Denote by $\Gamma \subset S$ a plane quartic which is coplanar to L, so that $\Gamma \sim H - L$. Thus,

$$H^2 = 5,\ HL = 1,\ L^2 = -3,\ H\Gamma = 4,\ \Gamma^2 = 0 \ \textit{and}\ \Gamma L = 4.$$

Choose $C \sim 3H + L$, so that $|C|$ contains curves which are residue to Γ in the complete intersection of S with the smooth quartic surfaces of $\mathbb{P}^3$ containing Γ. $|3H + L|$ is base-point-free and not composed with a pencil, since $(3H + L)L = 0$ and $3H$ is an ample divisor. By Bertini's theorems, its general member is smooth and irreducible; but C and $C - K_S$ cannot be both either ample or, even, nef divisors. In fact, $CL = 0$ and $(C - K_S)L = (2H + L)L = -1$. Moreover C is not numerically equivalent to a rational multiple of $K_S \sim H$. Therefore, the results in [3] and in [6] cannot be applied.

Neverthless, S is trivially linearly normal with $h^1(\mathcal{O}_S(H)) = 0$; furthermore, $CH = C(C - 2H) = \nu(C, H) = 16$, $(C - 2H)^2 = 4$, $H^2 = 5$, $4(C(C - 2H) - 4) = 48$; we then obtain $\delta < \frac{16}{4} = 4$. Thus, if $|3H + L|$ contains some nodal, irreducible curves, then, if $\delta \leqslant 3$, this singular curve is geometrically linearly normal; since $K_S \sim H$, this is equivalent to saying that such a curve corresponds to a smooth point of $V_{|3H+L|,\delta}$, which will be everywhere smooth of the expected dimension.

REFERENCES

[1] W. Barth - C. Peters - A. Van de Ven, *Compact Complex Surfaces*, Springer, Berlin, 1984.

[2] F. Bogomolov, *Holomorphic tensors and vector bundles on projective varities*, Math. USSR Isvestija, **13** (1979), 499-555.

[3] L. CHIANTINI - E. SERNESI, *Nodal curves on surfaces of general type*, Math. Ann., **307** (1997), 41-56.
[4] F. FLAMINI, *Families of nodal curves on projective surfaces*, Ph.D thesis (1999).
[5] R. FRIEDMAN, *Algebraic surfaces and holomorphic vector bundles* (UTX), Springer-Verlag, New York, 1998.
[6] G. M. GREUEL - C. LOSSEN - E. SHUSTIN, *New asymptotics in the geometry of equisingular families of curves*, Internat. Math. Res. Notices, **13** (1997), 595-611.
[7] P. GRIFFITHS - J. HARRIS, *Residues and 0-cycles on algebraic varities*, Ann. Math., **108** (1978), 461-505.
[8] R. HARTSHORNE, *Ample Subvarieties of Algebraic Varieties*, Springer, 1970.
[9] R. HARTSHORNE, *Algebraic Geometry* (GTM, No. 52), Springer-Verlag, New York-Heidelberg, 1977.
[10] M. REID, *Bogomolov's theorem* $c_1^2 \leq 4c_2$, Proc. Internat. Symposium on Alg. Geom., Kyoto (1977), 633-643.
[11] B. SHIFFMAN - A. J. SOMMESE, *Vanishing Theorems on Complex Manifolds*, Progress in Mathematics-vol. 56, Boston-Basel-Stuttgart, Birkhäuser, 1985.

Flaminio Flamini: Dipartimento di Matematica, Università degli Studi di Roma - «Roma Tre»
Largo San Leonardo Murialdo 1, 00146 Roma, Italy.
E-mail address: flamini@matrm3.mat.uniroma3.it

C. Madonna: Dipartimento di Matematica, Università degli Studi di Roma - «Tor Vergata»
Viale della Ricerca Scientifica, 00133 Roma, Italy.
E-mail address: madonna@mat.uniroma2.it

Pervenuta in Redazione
il 25 ottobre 1999