Bollettino
Unione Matematica Italiana

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*Bollettino dell'Unione Matematica Italiana, Serie 8*, Vol. **4-B** (2001), n.2, p. 441–456.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2001_8_4B_2_441_0>

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Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2001.

Bollettino U. M. I.
(8) 4-B (2001), 441-456

# Generalized Maximum Principle and Evaluation of the First Eigenvalue for Heisenberg-type Operators with Discontinuous Coefficients.

M. Chicco - M. R. Lancia

**Sunto.** – *Precedenti risultati riguardanti il principio di massimo generalizzato e la valutazione del primo autovalore per operatori uniformemente ellittici di tipo variazionale vengono estesi agli operatori subellittici di tipo Heisenberg non simmetrici e a coefficienti discontinui.*

## Introduction.

In the mathematical literature the strong maximum principle for degenerate operators with nonnegative characteristic form and regular coefficients has been studied by Bony [4]; for a complete discussion see also [20]. In his results the usual assumption on the sign of the coefficient $c$ in the operator $L$ is adopted. Following this philosophy, maximum principles can be also achieved for Heisenberg-type operators [17].

In this paper we prove a generalized maximum principle and we compute the first eigenvalue for linear second order subelliptic operators whose principal part is in divergence form with respect to the Heisenberg vector fields with discontinuous coefficients.

The *generalized* maximum principle differs from the ordinary one as follows. In the *ordinary* maximum principle usually one states that if a subsolution (of the elliptic equation) is $\leqslant m$ on $\partial\Omega$, then it is also $\leqslant \max(0, m)$ in the interior of $\Omega$. This property is valid provided one assumes the coefficient $c$ of $L$ to be non negative a. e. in $\Omega$.

In the present work we want to prove a *generalized maximum principle*: if a subsolution $u$ is non-positive on $\partial\Omega$, then it is non-positive also in the interior of $\Omega$. This property, as proved in Theorem 3.1, is valid iff there exists (at least) a subsolution $w$ such that $w \leqslant 0$ in $\Omega$ and $a(w, v) < 0$ for some test function $v \geqslant 0$ in $\Omega$ (for the definition of the bilinear form $a(\cdot, \cdot)$, associated with the subelliptic operator $L$, see (2.1)). These results were proved in [6] for uniformly elliptic divergence form equations, and they are extended in the present note to the case of Heisenberg type subelliptic operators.

We consider a local solution of the equation

$$Lu = 0 \qquad \text{in } \Omega$$

where $\Omega$ is an open bounded connected set of $\mathbb{R}^{2n+1}$, with smooth boundary. The operator $L$ is given by

$$Lu = -\sum_{j=1}^{2n} X_j^* \left[ \sum_{i=1}^{2n} a_{ij}(x)\, X_i u + d_j(x)\, u \right] + \sum_{i=1}^{2n} b_i(x)\, X_i u + c(x)\, u$$

where $X_j$ are the Heisenberg vector fields in $\mathbb{R}^{2n+1}$, $X_j^*$ is the $L^2$-adjoint of $X_j$. The operator $L$ is assumed to be uniformly subelliptic. More precisely we shall assume the following:

(A) $a_{ij}$ are measurable functions on $\mathbb{R}^{2n+1}$, and there exist positive constants $\mu$, $M$ such that $\mu \leqslant M$ and

$$\mu |\xi|^2 \leqslant \sum_{i,j=1}^{2n} a_{ij}(x)\, \xi_i \xi_j \leqslant M |\xi|^2$$

for all $x \in \mathbb{R}^{2n+1}$ and $\xi \in \mathbb{R}^{2n}$

(B) there exists $q > 2n+2$ such that

$$b_i \in L^q(\Omega),\ d_i \in L^q(\Omega),\ c \in L^{q/2}(\Omega)$$

for $i = 1, \ldots, 2n$.

Under these assumptions we can prove the following

THEOREM 3.1. – *Let $B_R \subset\subset \Omega$ with $192R < \overline{R}$. Every non positive subsolution $u$ on $\partial B_R$ satisfies a generalized maximum principle iff there exists a negative subsolution in $B_R$.*

Here by $B_R$ we denote the intrinsic balls defined in (1.7) and $\overline{R}$ is a suitable value of the radius (see proposition 2.1). The lower bound on $\overline{R}$ is motivated by the use of Harnack inequality (proved in [18]) in lemma 2.5 and corollary 2.6. The previous results enable us to give in theorem 4.2 a characterization of the first eigenvalue having largest real part:

THEOREM 4.2. – *Let $\lambda_1$ denote the eigenvalue of Problem* (4.1) *having the largest real part. Then $\lambda_1$ is real and it turns out*

$$\lambda_1 = -\sup \left\{ \inf_{v \in \overset{\circ}{S}{}^2(B_R),\, v > 0} \frac{a(w, v)}{(w, v)_{L^2(B_R)}} : w \in S^2(B_R),\ w < 0 \ \ in \ B_R \right\}$$

*where* $S^2(B_R) = \{u \in L^2(B_R) : X_i u \in L^2(B_R),\ i = 1, \ldots, 2n\}$ (*see Definition* (1.2) *below*).

In this paper, we extend to the non euclidean context of the Heisenberg group the results proved in [6, 7] for the case of linear second order elliptic partial differential equations in divergence form with discontinuous coefficients. Following the proofs in [6] we use a local Harnack inequality proved in [18], for more general Hörmander vector fields (see also [2]), and we had to adapt some proofs in [6] to our case. Actually , we use some results about connections between Hausdorff measure and the analogue of $p$-capacity with respect to the Heisenberg vector fields (here we will not give the proofs which can be found in [16]).

In section 1 we describe the main tools we shall make use of. We introduce the Heisenberg group and the associated Sobolev spaces. We recall that $\mathbb{R}^{2n+1}$, equipped with the distance intrinsically associated with the Heisenberg vector fields, is a homogeneous space in the sense of abstract harmonic analysis [8] with homogeneous dimension $N = 2n+2$. Then, we state the results about the connections between Hausdorff measure and 2-capacity, where the 2-capacity is defined with respect to the Heisenberg vector fields. In section 2, we recall some preliminary results proved in [18] and we deduce some straightforward consequences which, together with the relation between Hausdorff measure and 2-capacity, will turn out essential to the proof of corollary 2.6. In section 3 we prove the generalized maximum principle, where the technique follows the lines of [6] in the euclidean context. Finally, in section 4 we can give a characterization of the first eigenvalue having largest real part.

## 1. – Notations.

Consider the euclidean space $\mathbb{R}^{2n+1}$, whose elements we denote by $x = (x_1, \ldots, x_n, y_1, \ldots, y_n, t)$, equipped with the multiplication law

$$x \cdot x' = (x_1 + x_1', \ldots, x_n + x_n', y_1 + y_1', \ldots, y_n + y_n', t + t' + 2 \sum (x_i' y_i - x_i y_i')) . \tag{1.1}$$

It is a group whose identity is the origin and where the inverse is given by

$$x^{-1} = (-x_1, \ldots, -x_n, -y_1, \ldots, -y_n, -t)$$

The space $\mathbb{R}^{2n+1}$ with the structure (1.1) is the Heisenberg group denoted by $H^n$ [22]. The non isotropic dilations

$$\delta \circ x = (\delta x_1, \ldots \delta x_n, \delta y_1, \ldots \delta y_n, \delta^2 t) \qquad (\delta \in \mathbb{R}, x \in H^n) \tag{1.2}$$

are automorphisms of $H^n$. The nonnegative function

$$\varrho(x) = \left( \sum_{i=1}^{n} (x_i^2 + y_i^2)^2 + t^2 \right)^{1/4} \tag{1.3}$$

defines a norm for the Heisenberg group, in particular it is homogeneous of degree 1 with respect to the dilations (1.2), i.e.

$$\varrho(\delta \circ x) = |\delta| \varrho(x) \tag{1.4}$$

for every $x \in H^n$ and $\delta \in \mathbb{R}$. Moreover, there exist positive constants $c_1$ and $c_2$ such that

$$c_1 |x| \leq \varrho(x) \leq c_2 |x|^{1/2} \tag{1.5}$$

for every $x$ in a bounded set of $H^n$, where $|x|$ denotes the Euclidean norm in $\mathbb{R}^{2n+1}$.

By (1.4) and (1.5) it follows that the function $d$ defined by

$$d(x, x') = \varrho(x'^{-1} \cdot x) \tag{1.6}$$

is a distance in $H^n$, topologically equivalent to the Euclidean one and left invariant with respect to the law (1.1).

By using the distance $d$ we define the intrinsic balls

$$B(x, R) = B_R(x) = \{x' \in H^n \colon d(x, x') < R\}. \tag{1.7}$$

The Lebesgue measure $dx = dx_1 \ldots dx_n dy_1 \ldots dy_n dt$ is invariant with respect to the translations (1.1) so that for every $x \in H^n$ and $R > 0$ we have

$$|B_R(x)| = |B_R(0)| \tag{1.8}$$

Since the jacobian of the dilations (1.2) is given by

$$J_\delta = \delta^{2n+2} \tag{1.9}$$

from (1.8) and (1.9) it follows that for every $x \in H^n$ and $R > 0$

$$|B_R(x)| = R^{2n+2} |B_1|. \tag{1.10}$$

In our context we need a vector field basis which is invariant with respect to the translations (1.1). Such a basis is given by

$$X_i = \begin{cases} \dfrac{\partial}{\partial x_i} + 2y_i \dfrac{\partial}{\partial t} & \text{for } i = 1, \ldots, n \\[2ex] \dfrac{\partial}{\partial y_{n-i}} - 2x_{i-n} \dfrac{\partial}{\partial t} & \text{for } i = n+1, \ldots, 2n \end{cases} \tag{1.11}$$

$$T = \frac{\partial}{\partial t}.$$

For the vector fields (1.11) we have the commutative law

$$[X_i, X_{i+n}] = -4T \qquad \text{for every } i = 1, \ldots, n \tag{1.12}$$

while the other commutators vanish.

We recall that a commutator of two vector fields $V_1$ and $V_2$ is the new vector field given by

$$[V_1, V_2] = V_1 V_2 - V_2 V_1 \tag{1.13}$$

therefore, $X_1, \ldots, X_{2n}$ are a basis for the Lie algebra of the vector fields invariant with respect to (1.1). Moreover they are homogeneous of degree 1 with respect to the dilations (1.2), whereas, by (1.12), $T$ is homogeneous of degree 2 i.e.

$$X_i(u(\delta \circ x)) = \delta((X_i u)(\delta \circ x))(i = 1, \ldots, 2n) \tag{1.14}$$

and

$$T(u(\delta \circ x)) = \delta^2((Tu)(\delta \circ x)). \tag{1.15}$$

The family of the intrinsic balls $B_R(x)$ reflects the nonisotropic nature of the Heisenberg vector fields.

The space $\mathbb{R}^{2n+1}$, equipped with the distance $d$, acquires the structure of a space of homogeneous type of dimension $N = 2n + 2$ in the sense of Coifman and Weiss [8].

Let $\Omega$ be a bounded open subset of $\mathbb{R}^{2n+1}$. The following covering lemma holds (see [8] ch. 3 lemma 1.1)

LEMMA 1.1. – *For every $\varepsilon \in (0, 1)$, the ball $B(x, r)$, $x \in \Omega$, $0 < r < r_0$, can be covered by the union of balls $B(y_i, \varepsilon r)$ with $y_i \in B(x, r)$, for $i = 1, 2, \ldots, l$ such that $l < c\varepsilon^{\alpha}$ for suitable constants $c > 0$ and $\alpha > 0$. The constants $c$ and $\alpha$ are independent on $x$, $r$ and $\varepsilon$ and depend only on $c_0$.*

This dimension will play a crucial role in view of the close connection between capacity and Hausdorff dimension which we shall describe later. Following [10, 22] we give the

DEFINITION 1.2. – *We denote by $S^2(\Omega)$ the Sobolev-type space of the functions $u \in L^2(\Omega)$, such that the distribution derivatives $X_i u$ belong to $L^2(\Omega)$ for $i = 1, \ldots, 2n$.*

The norm in $S^2(\Omega)$ is given by

$$\|u\|^2 = \int_{\Omega} \left( |u|^2 + \sum_{i=1}^{2n} |X_i u|^2 \right) dx. \tag{1.16}$$

The closure of $C_0^\infty(\Omega)$ in the above norm is denoted by $\overset{\circ}{S}{}^2(\Omega)$. By $S^2_{\mathrm{loc}}(\Omega)$ we mean the set of functions $u$ which belong to $S^2(\Omega')$ for every $\Omega' \subset\subset \Omega$.

In the following, we set for brevity

$$\|Xu\|^2_{L^2(\Omega)} := \int_\Omega \sum_{i=1}^{2n} |X_i u|^2 .$$

We recall the following properties of the above Sobolev spaces:

*Poincaré inequality* (see [12]). For every $B_R(x) \subset \mathbb{R}^{2n+1}$ and $u \in \overset{\circ}{S}{}^2(B_R(x))$ we have

$$\int_{B_R(x)} |u|^2 \, dx \leqslant cR^2 \int_{B_R(x)} \sum_{j=1}^{2n} |X_j u|^2 \, dx . \tag{1.17}$$

*Sobolev inequality* (see [13]). There exists a constant $S > 0$ such that for every $u \in S^2(\mathbb{R}^{2n+1})$ we have

$$\left( \int_{\mathbb{R}^{2n+1}} |u|^{2^*} \, dx \right)^{2/2^*} \leqslant S \int_{\mathbb{R}^{2n+1}} \sum_{j=1}^{2n} |X_j u|^2 \, dx , \tag{1.18}$$

where

$$2^* = \frac{2N}{N-2} \left( = \frac{2n+2}{n} \right).$$

*Compact embedding* (see [10]). For every bounded domain $\Omega$ the Sobolev space $\overset{\circ}{S}{}^2(\Omega)$ is compactly embedded into $L^p(\Omega)$, for every $p < 2^*$.

Let $E \subset \mathbb{R}^{2n+1}$ a compact set. We define the capacity of $E$ as

$$\operatorname{cap}_2 E = \inf \left\{ \int_{\mathbb{R}^{2n+1}} \sum_{j=1}^{2n} |X_j f|^2 \, dx : f \in C_0^\infty(\mathbb{R}^{2n+1}); \ f \geqslant 1 \ \text{ on } E \right\} \tag{1.19}$$

For the reader's convenience we now recall some results whose proofs can be found in [16].

In the sequel, we denote by $H^s(\cdot)$ the s-dimensional Hausdorff measure in the metric space $(\mathbb{R}^{2n+1}, d)$.

PROPOSITION 1.3. – *Let $A$ denote a compact set in $H^n$ and $B(x, r) \subset \overline{\Omega}$ then the following properties hold*:

i) $\operatorname{cap}_2(B(x, r)) = r^{N-2} |B_1|$;

ii) $\operatorname{cap}_2(A) \leqslant cH^{N-2}(A)$;

*where $c$ is a costant depending only on $N$.*

THEOREM 1.4. – *Let $A$ denote a compact set in $H^n$, if $H^{N-2}(A) < \infty$ then* $\mathrm{cap}_2(A) = 0$.

Proof. See [16].

THEOREM 1.5. – *Let $A \subset H^n$ be a compact subset. If* $\mathrm{cap}_2(A) = 0$ *then $H^s(A) = 0$ for every $s > N-2$.*

Proof. See [16].

DEFINITION 1.6. – *Let $\Omega$ denote a bounded connected set and $E \subset \overline{\Omega}$. Let $u \in S^2(\Omega)$. The function $u$ is nonnegative in the sense of $S^2(\Omega)$, or briefly $u \geq 0$ on $E$ in $S^2(\Omega)$ if there exists a sequence $\{u_n\} \in C^\infty(\overline{\Omega})$ such that*

$$\begin{cases} u_n(x) \geq 0, \ \forall x \in E \\ u_n(x) \to u \ \text{ in } \ S^2(\Omega). \end{cases}$$

REMARK 1. – *If $u \in S^2(\Omega)$ and $v \in S^2(\Omega)$ we say that $u \geq v$ on $E$ in $S^2(\Omega)$ if $u - v \geq 0$ on $E$ in $S^2(\Omega)$.*

The following proposition can be easily proved [11]:

PROPOSITION 1.7. – *If $u \in S^2(\Omega)$ and $u \leq k$ a.e. in $\Omega$ then $u \leq k$ on $K$ in the sense of $S^2(\Omega)$ for every compact set $K \subset \Omega$.*

REMARK 2. – *It is easy to see, as in the euclidean context, that if $u \in S^2(\Omega)$ and $E \subset \Omega$ is a compact subset and $u = 0$ in the sense of $S^2(\Omega)$ then $u = 0$ in $E$ in the sense of the capacity defined in* (1.19).

## 2. – Preliminary results.

In this section we recall, in the case of interest for us, some results proved in [18] and we deduce some consequence which will turn out essential to our purposes.

2.1. *Variational formulation of the problem.*

Let $\Omega$ be a bounded domain in $\mathbb{R}^{2n+1}$. Let us consider the following differential operator

$$Lu = -\sum_{j=1}^{2n} X_j^* \left[ \sum_{i=1}^{2n} a_{ij}(x) X_i u + d_j(x) u \right] + \sum_{i=1}^{2n} b_i(x) X_i u + c(x) u$$

where $X_1, X_2, \ldots, X_{2n}$ are the Heisenberg vector fields and $X_j^*$ is the $L^2$-adjoint of $X_j$.

We make the following assumptions:

A) $a_{ij}$ $(i, j = 1, \ldots, 2n)$ are $2n \times 2n$ measurable functions on $\Omega$ that satisfy the following conditions:

i) there exists $M > 0$ such that $|a_{ij}(x)| \leq M$ for a.e. $x \in \Omega$;

ii) there exists $\mu > 0$ such that

$$\mu|\xi|^2 \leq \sum_{i, j=1}^{2n} a_{ij}\xi_i\xi_j$$

for almost every $x \in \Omega$ and every $\xi \in \mathbb{R}^m$;

B) $b_i \in L^N(\Omega)$, $i = 1, \ldots, 2n$;

C) $c \in L^{N/2}(\Omega)$ and $d_j \in L^N(\Omega)$, $j = 1, \ldots, 2n$ where $N = 2n + 2$.

Let $a$ be the bilinear form defined by:

$$a(u, v) = \int_\Omega \left(\sum_{i,j} (a_{ij}X_i u X_j v + b_i(X_i u)\, v + d_j u X_j v + cuv\right) dx \tag{2.1}$$

$u \in S^2_{\text{loc}}(\Omega)$, $v \in \overset{\circ}{S}{}^2(\Omega)$.

In [18] it was proved the following:

Proposition 2.1. – *Let $B_R = B_R(x) \subset \Omega$, $R \leq R_0/2$. Then*:

i) *$a$ is continuous in $\overset{\circ}{S}{}^2(\Omega) \times \overset{\circ}{S}{}^2(\Omega)$*

ii) *there exists $\overline{R}$ with $0 \leq \overline{R} \leq R_0/2$ and $\nu > 0$ depending on the structural costants and on $S, M, \mu, b_i, d_i, c$ but not depending on $x \in \Omega$, such that $a$ is coercive on $\overset{\circ}{S}{}^2(B_R)$ for every $R \leq \overline{R}$*:

$$a(u, u) \geq \nu\|u\|_{\overset{\circ}{S}{}^2(B_R)} \text{ for every } u \in \overset{\circ}{S}{}^2(B_R). \tag{2.2}$$

(Here $R_0$ is a suitable constant defined in [18]).

As pointed out in [18] (see remark 2.2), Proposition 2.1 still holds when the hypothesis $C$) is replaced by

$$C')\ d_j \in L^q(\Omega)\ (j = 1, 2, \ldots, 2n),\ c \in L^{q/2}(\Omega),\ q > N$$

which cannot be removed in order to prove Harnack inequality and Theorem 2.3. Therefore from now on we will assume $C'$) to hold.

Definition 2.2. – *We say that $u \in S^2_{\text{loc}}(\Omega)$ is a local solution of the equation*

$$Lu = 0 \tag{2.3}$$

*if for every* $B_R \subseteq \Omega$ *and for every* $v \in S^2(B_R)$ *it turns out*

$$a(u, v) = \int_{B_R} \Big( \sum_{i,j} (a_{ij} X_i u X_j v + b_i (X_i u) v + d_j u X_j v + cuv \Big) \, dx = 0 \,. \tag{2.4}$$

*If in addition* $u \in S^2(\Omega)$ *we say that* $u$ *is a solution.*

*We say that* $u$ *is a local subsolution if*

$$a(u, \varphi) \leqslant 0 \quad \forall \varphi \in \overset{\circ}{S}{}^2(B_R), \ \varphi \geqslant 0 \,. \tag{2.5}$$

In [18] a Harnack inequality is proved.

THEOREM 2.3. – *Let* $u \in S^2_{\mathrm{loc}}(\Omega)$ *be a positive local solution of* (2.3). *There exists* $C > 0$, *such that for every* $B_R$ *with* $B_{192R} \subset \Omega$ *and* $192R < \overline{R}$ *we have*

$$\sup_{B_R} u \leqslant C \inf_{B_R} u \,. \tag{2.6}$$

From Harnack inequality and theorem (3.2) in [18] the following result can be deduced:

COROLLARY 2.4. – *Let* $u$ *be a local positive solution in* $\Omega$ *of* $Lu = -\sum_{i=1}^{2n} X_i^\star f_i$ *with* $f_i \in L^p(\Omega)$, $p > N$ *then*

$$\sup_{B(x, r)} u \leqslant K \Big( \inf_{B(x, r)} u + \sum \|f_i\|_{L^p(B_R)} |R|^{1 - N/p} \Big),$$

*where* $K$ *does not depend on* $B_R$.

From the above corollary it follows

LEMMA 2.5. – *Let* $u \in S^2(\Omega)$, $u \leqslant 0$ *in* $\Omega$, $u$ *non identically zero in* $\Omega$, $a(u, v) = 0$ *for every* $v \in \overset{\circ}{S}{}^2(\Omega)$. *Then for every* $B_R \subset\subset \Omega$ *with* $192R < \overline{R}$

$$\sup_{B_R} u < 0 \,. \tag{2.7}$$

Proof: see the proof of corollary (8.1) in [21].

COROLLARY 2.6. – *Let* $w \in S^2(\Omega)$, $w \leqslant 0$ *in* $\Omega$, $w$ *not identically zero in* $\Omega$, $a(w, v) \leqslant 0$ *for every* $v \in \overset{\circ}{S}{}^2(\Omega)$, $v \geqslant 0$.

*Then, for every* $B_R \subset\subset \Omega$, *with* $192R < \overline{R}$ *we have*

$$\sup_{B_R} w < 0 \tag{2.8}$$

Proof. Let us consider the following subsets of $\Omega$:

$B_1 := \{x \in \Omega :$ there exists a neighborhood $U$ of $x$ s.t. $w = 0$ a.e. in $U\}$

$B_2 := \{x \in \Omega : \text{there exists a neighborhood } U \text{ of } x \text{ s.t. } \operatorname{ess\,sup}_U w < 0\}$

$B_3 := \Omega \backslash (B_1 \cup B_2) = \{x \in \Omega$: for every neighborhood of $x$ there exist a subset $A_1$ s.t. $|A_1| > 0$ where $w = 0$ a.e. and a subset $A_2$ with $|A_2| > 0$ where $w < 0$ a.e.$\}$

It turns out that our hypothesis implies that $B_1$ cannot coincide with $\Omega$. If $B_2 \equiv \Omega$, by compactness arguments we have that for every compact set $D \subset \Omega$ it follows $\operatorname{ess\,sup}_D w < 0$, whence the thesis. Therefore, if both $B_1$ and $B_2$ are not empty, then $B_3$ also is not empty, due to the fact that $\Omega$ is a connected set. Finally we can restrict ourselves to consider only the last case.

Let $x_0 \in B_3$. Then there exists a ball $B(x_0, r)$ with radius $r$ small enough such that $w \notin \overset{\circ}{S}{}^2(B(x_0, r))$. In fact, if this were not the case, by contradiction there would exist a $r_0 > 0$ s.t. $w \in \overset{\circ}{S}{}^2(B(x_0, r))$ for every $r \in (0, r_0)$. Thus, by virtue of remark 2, $w = 0$ in the capacity sense on the boundary $\partial B(x_0, r)$ of every ball $B(x_0, r)$ for $r \in (0, r_0)$. Theorem 1.5 yields that if $\operatorname{cap}_2(\partial B(x_0, r)) = 0$ then $H^{N-1}(\partial B(x_0, r)) = 0$ for every $r \in (r, r_0)$. By means of the intrinsic coarea formula (see (17) in [9], or [5]) we have that

$$\int_{B(x_0, r_0)} w(x)\, |X\varrho|\, dx = 0$$

where $|X\varrho| = \left( \sum_{i=1}^{2n} (X_i \varrho)^2 \right)^{1/2}$. Taking into account that $|X\varrho| > 0$ a.e. we deduce that $w(x) = 0$ a.e. in $B(x_0, r_0)$, a contradiction. Therefore we can choose $r$ s.t. $w \notin \overset{\circ}{S}{}^2(B(x_0, r))$ and s.t. the form $a(\cdot, \cdot)$ is coercive in $\overset{\circ}{S}{}^2(B(x_0, r))$. (As pointed out in [18] this happens whenever the measure of $B(x_0, r)$ is suitably small). Then if we consider the solution $u$ of the Dirichlet problem

$$\begin{cases} a(u, v) = 0 \quad \forall v \in \overset{\circ}{S}{}^2(B(x_0, r)) \\ u - w \in \overset{\circ}{S}{}^2(B(x_0, r)) \end{cases}$$

(the existence easily follows as in thm. (3.3) of [21]), we can prove that

$$w \leqslant u \text{ a.e., on } B(x_0, r). \tag{2.9}$$

In fact, if we set $g = \max(w - u, 0)$, the function $g$ turns out to be in $\overset{\circ}{S}(B(x_0, r))$, $g \geqslant 0$ a.e. and $a(g, g) \leqslant 0$ whence $g \equiv 0$ a.e. in $B(x_0, r)$. In the same way we can prove that $u \leqslant 0$ on $B(x_0, r)$ and that $u$ is not identically zero on $B(x_0, r)$ because $u - w \in \overset{\circ}{S}{}^2(B(x_0, r))$ and $w \notin \overset{\circ}{S}{}^2(B(x_0, r))$. Therefore by lemma 2.5 we have

$$\sup_{B(x,\tilde{r})} u < 0 \text{ for every } B(x,\tilde{r}) \subset B(x_0, r).$$

If by chance $x_0 \in B(x,\tilde{r})$ then it would follow that $\sup\limits_{B(x,\tilde{r})} w = 0$, thus contradicting (2.9).

## 3. – The generalized maximum principle.

In this section we prove that every non positive subsolution $u$ on $\partial B_R$ satisfies a generalized maximum principle iff there exists a negative subsolution in $B_R$.

THEOREM 3.1. – *The two following statements are equivalent*:

a) *for every* $u \in S^2_{\text{loc}}(\Omega)$ *such that* $u \leqslant 0$ *on* $\partial B_R$, $B_R \subset \Omega$, $192R < \overline{R}$ *and* $a(u, v) \leqslant 0$ $\forall v \in \overset{\circ}{S}{}^2(B_R)$, $v \geqslant 0$, *then* $u \leqslant 0$ *in* $B_R$.

b) *There exists a function* $w \in S^2_{\text{loc}}(\Omega)$ *such that* $w \leqslant 0$ *in* $B_R \subset \Omega$, $a(w, v) \leqslant 0$ $\forall v \in \overset{\circ}{S}{}^2(B_R)$, $v \geqslant 0$ *in* $B_R$, *and* $a(w, v) < 0$ *at least for one function* $v \in \overset{\circ}{S}{}^2(B_R)$, $v \geqslant 0$ *a.e. in* $B_R$.

PROOF. – We prove that a) $\Rightarrow$ b). For the sake of simplicity, let $f \in C^0(\overline{\Omega})$ $f < 0$. Let us consider the boundary value problem

$$\begin{cases} a(w, v) = \displaystyle\int_\Omega fv dx \ \ \forall v \in \overset{\circ}{S}{}^2(B_R) \\ w \in \overset{\circ}{S}{}^2(B_R) \end{cases}$$

there exists a unique solution $w \in \overset{\circ}{S}{}^2(B_R)$ for every $B_R \subset \Omega$ with $R < \overline{R}$.

Let $w$ denote the solution, then it satisfies b). In fact, condition a) and the hypothesis on $f$ yield

$$w \leqslant 0 \text{ in } B_R,\ a(u, w) < 0 \text{ if } v \in \overset{\circ}{S}{}^2(B_R),\ v > 0 \text{ in } B_R.$$

Let us now prove that b) $\Rightarrow$ a).

Let $u \in S^2_{\text{loc}}(\Omega)$, $u \leqslant 0$ on $\partial B_R$, $a(u, v) \leqslant 0$ for every $v \in \overset{\circ}{S}{}^2(B_R)$, $v \geqslant 0$; we want to show that $u \leqslant 0$ in $B_R$. Let us consider the function $w_k(x) = \max(u + kw, 0)$, where $k \in \mathbb{R}$ and $w$ is the function which satisfies condition b). Let

$$\begin{aligned} &A_R(k) = \{x \in B_R \colon w_k(x) > 0\} \\ &k_0 = \inf\{k : w_k(x) = 0 \text{ a.e. in } B_R\}; \end{aligned} \tag{3.1}$$

if we show that $k_0 \leqslant 0$ then we get our result.

We proceed by contradiction. Let us assume that $k_0 > 0$. Let $B_R(x)$, $192R < \overline{R}$, so that by corollary (2.6) we claim that $\sup\limits_{B_R} w < 0$; moreover it can be shown (as in [21] sect. 5) that $\sup\limits_{B_R} u < +\infty$. Therefore there exists a $h \in \mathbb{R}$ such that

$$w_k = 0 \text{ in } B_R \text{ for } k \geqslant h$$

whence obviously

$$\lim_{k \to +\infty} |A_R(k)| = 0 . \tag{3.2}$$

We now prove that

$$\lim_{k \to k_0} |A_R(k)| = 0 \tag{3.3}$$

even though $k_0$ is finite. If (3.3) did not hold, there would exist a set $H \subset B_R$, $|H| > 0$ such that $u + k_0 w = 0$ in $H$ and $u + k_0 w \leqslant 0$ in $B_R$. Then let $B_r \subset B_R$ such that $|B_r \cap H| > 0$, Corollary 2.6 applied to the function $u + k_0 w$ yields $u + k_0 w = 0$ in $B_r$, so that by covering arguments (see [8]), $u + k_0 w = 0$ in $B_R$, i.e. $u = -k_0 w$. Hence $a(w, v) = 0$ $\forall v \in \overset{\circ}{S}{}^2(B_R)$ which contradicts condition b) and proves (3.3). On the other hand, as the function $R \ni k \to |A_R(k)|$ is decreasing, from (3.3) we get $|A_R(k_0)| = 0$ whence $w_{k_0} = 0$ a.e. in $B_R$. Therefore arguing as before we have a contradiction and then $k_0 \leqslant 0$.

COROLLARY 3.2. – *Let us assume condition* a) *of Theorem* 3.1 *to hold. Let* $\mu > 0$ *and let* $z$ *be in* $S^2(\Omega)$, $z \leqslant 0$ *on* $\partial B_R$, *for every* $B_R \subset \Omega$ *such that*

$$a(z, v) + \mu(z, v)_{L^2(B_R)} \leqslant 0 \quad \forall v \in \overset{\circ}{S}{}^2(B_R), \qquad v \geqslant 0 .$$

*Then* $z$ *is* $\leqslant 0$ *in* $B_R$.

PROOF. – By virtue of theorem 3.1 there exists a function $w \in \overset{\circ}{S}{}^2(B_R)$ s.t. $w \leqslant 0$ in $B_R \subset \Omega$, $a(w, v) < 0$ $\forall v \in \overset{\circ}{S}{}^2(B_R)$, $v > 0$ in $B_R$. Then $a(w, v) + \mu(w, v)_{L^2(B_R)} < 0$, $\forall v \in \overset{\circ}{S}{}^2(B_R)$, $v > 0$, so that Theorem 3.1 applied to the form $a(w, v) + \mu(w, v)$ gives the result.

## 4. – Characterization of the eigenvalues.

In this section, we give a characterization of the eigenvalues of the problem

$$\begin{cases} a(w, v) + \lambda(w, v)_{L^2(\Omega)} = 0, \ \forall v \in \overset{\circ}{S}{}^2(\Omega) \\ u \in \overset{\circ}{S}{}^2(\Omega) . \end{cases} \tag{4.1}$$

Before stating our result, we recall a theorem which will be useful later (see [14]), for the sake of clarity.

THEOREM 4.1 (Kreĭn and Rutman). – *Let $X$ be a Banach space and $T$ denote a completely continuous operator from $X$ into $X$ such that $T$ maps a cone $K$ into $K$ and $T$ has at least a non-zero eigenvalue. Then $T$ has a positive eigenvalue $\lambda_1$ such that $\lambda_1 \geq |\lambda|$ for every $\lambda$ eigenvalue of $T$.*

THEOREM 4.2. – *Let $\lambda_1$ denote the eigenvalue of problem* (4.1) *having the largest real part. Then $\lambda_1$ is real and it turns out*

$$\lambda_1 = -\sup\left\{ \inf_{v \in \overset{\circ}{S}{}^2(B_R),\, v>0} \frac{a(w,v)}{(w,v)_{L^2(B_R)}} : w \in S^2(B_R),\ w<0 \ \ in\ \ B_R \right\} \tag{4.2}$$

PROOF. – The proof will take four steps.

*Step 1*). We prove that $\lambda_1 \in \mathbb{R}$. Suppose that $\mu$ does not belong to the spectrum of (4.1) and consider the operator

$$G_\mu : L^2(B_R) \to \overset{\circ}{S}{}^2(B_R)$$

defined as:

$$a(G_\mu u, v) + \mu (G_\mu u, v)_{L^2(B_R)} = (u, v)_{L^2(B_R)} \quad \forall v \in \overset{\circ}{S}{}^2(B_R). \tag{4.3}$$

Taking into account that $\overset{\circ}{S}{}^2(B_R)$ is compactly embedded in $L^2(B_R)$ (see [3, 12]) for every $B_R \subset \Omega$ such that $R < \overline{R}$, the operator $G_\mu$ is compact in $L^2(B_R)$. Let $\mu$ be large enough such that there exists a positive constant $\mu_0$ with

$$a(z,z) + \mu \|z\|^2_{L^2(B_R)} \geq \mu_0 \|z\|^2_{\overset{\circ}{S}{}^2(B_R)}\ \forall z \in \overset{\circ}{S}{}^2(B_R) \tag{4.4}$$

(see [18] and [21]). If (4.4) holds, $\mu$ does not belong to the spectrum; moreover if $u \in L^2(B_R)$, $u \leq 0$ in $B_R$ then $G_\mu u \leq 0$ in $B_R$. In fact, set $\tilde{u} = \max(G_\mu u, 0)$, $v = \tilde{u}$ in (4.3); taking (4.4) into account we get

$$\mu_0 \|\tilde{u}\|^2_{\overset{\circ}{S}{}^2(B_R)} \leq a(\tilde{u}, \tilde{u}) + \mu \|\tilde{u}\|_{L^2(B_R)} = (u, \tilde{u})_{L^2(B_R)} \leq 0\,.$$

Then $\tilde{u} = 0$ in $B_R$ and $G_\mu u \leq 0$ in $B_R$. Thus the operator $G_\mu$ leaves invariant the cone $K$ defined by

$$K = \{u \in L^2(B_R) : u \leq 0 \ \text{ in } \ B_R\}\,.$$

The operator $G_\mu$, under assumption (4.4), satisfies the hypothesis of Kreĭn-

Rutman Theorem, therefore there exists a real eigenvalue $t_1$ of $G_\mu$ such that

$$|t| \leqslant t_1 \text{ for every eigenvalue } t \text{ of } G_\mu. \tag{4.5}$$

Moreover, it is well known that if $\lambda$ is an eigenvalue of the problem (4.1) the number $t = (\mu - \lambda)$ is an eigenvalue of the operator $G_\mu$ and viceversa. If we set $t_1 = (\mu - \lambda_1)^{-1}$, (4.5) yields

$$|\mu - \lambda| \geqslant \mu - \lambda_1 \tag{4.6}$$

for every $\lambda$ eigenvalue of (4.1). Then taking into account that (4.6) holds under the hypothesis that $\mu$ is large enough, we let $\mu$ diverge at $+\infty$, thus obtaining:

$$\operatorname{Re}(\lambda) \leqslant \lambda_1 \tag{4.7}$$

for every $\lambda$ eigenvalue of (4.1). This guarantees the existence of an eigenvalue $\lambda_1$ having maximal real part. We are now ready to make the second step.

*Step 2*). We set

$$\lambda' = -\sup\left\{ \inf_{v \in \overset{\circ}{S}{}^2(B_R),\, v>0} \frac{a(w, v)}{(w, v)_{L^2(B_R)}} : w \in S^2(B_R),\ w<0 \text{ in } B_R \right\}$$

and we remark that $\lambda' > -\infty$. By contradiction, if $\lambda' = -\infty$ take any real $p$; then there exists a function $w \in S^2(B_R)$, $w<0$ in $B_R$, such that

$$\inf_{v \in \overset{\circ}{S}{}^2(B_R),\, v>0} \frac{a(w, v)}{(w, v)_{L^2(B_R)}} > -p$$

whence $a(w, v) + p(w, v)_{L^2(B_R)} < 0$, $\forall v \in \overset{\circ}{S}{}^2(B_R)$, $v>0$.

From Theorem 3.1 it follows that neither $p$ is an eigenvalue of problem (4.1) nor any number $t>p$. But problem (4.1), like the compact operator $G_\mu$ defined in (4.3), has at least an eigenvalue: this is a contradiction, so $\lambda'$ is real.

We prove now that $\lambda'$ is an eigenvalue of problem (4.1). By contradiction, if $\lambda'$ were not an eigenvalue, there would exist the operator $G_{\lambda'}$, which satisfies the property $G_{\lambda'} u \leqslant 0$ in $B_R$ for every $u \in L^2(B_R)$, $u \leqslant 0$ in $B_R$.

In fact, by the definition of $\lambda'$ and Theorem (3.1), it can be shown that

$$\lim_{\mu \to \lambda'_+} \|G_\mu u - G_{\lambda'} u\|_{L^2(B_R)} = 0$$

and $G_\mu u \leqslant 0$ in $B_R$ if $\mu > \lambda'$. If $\lambda'$ were not an eigenvalue and if $0 < \lambda' - \mu < \|G_{\lambda'}\|^{-1}$ then $\mu$ would not be an eigenvalue so that we would have

$$G_\mu = G_{\lambda'}[I - (\lambda' - \mu)\, G_{\lambda'}]^{-1} = \sum_{j=0}^{\infty} (\lambda' - \mu)^j G_{\lambda'}^{j+1}.$$

Hence, with this choice of $\mu$ it turns out that $G_\mu u \leqslant 0$ in $B_R$ whenever $u \leqslant 0$ in

$B_R$. If $w = G_\mu u$ with $u \in L^2(B_R)$, $u < 0$ it turns out that $a(w, v) + \mu(w, v)_{L^2(B_R)} < 0$, $\forall v \in \overset{\circ}{S}{}^2(B_R)$, $v > 0$ in $B_R$ so that $w < 0$ in $B_R$, by virtue of Corollary 2.6. It follows that

$$\inf_{v \in \overset{\circ}{S}{}^2(B_R),\, v > 0} \frac{a(w, v)}{(w, v)_{L^2(B_R)}} \geq -\mu$$

and this is absurd because $\mu < \lambda'$, therefore $\lambda'$ is an eigenvalue.

*Step 3*). As $\lambda_1$ is the eigenvalue having maximal real part it follows that $\lambda' \leq \lambda_1$.

*Step 4*). We prove that $\lambda_1 \leq \lambda'$. Take a number $p > \lambda'$ and prove that $p > \lambda_1$. If $p > \lambda'$, arguing as in step 2, we deduce that neither $p$ is an eigenvalue of problem (4.1) nor any number $t > p$. Therefore $\lambda_1 \leq p$, and since $p$ is any number greater than $\lambda'$, we conclude $\lambda_1 \leq \lambda'$.

*Acknowledgments.* The authors wish to thank Prof. Marco Biroli and Prof. Umberto Mosco for the useful discussions during the preparation of this paper and the referee for his valuable comments. They are also grateful to dr. Laura Servidei for correcting English style.

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*Pervenuta in Redazione*
*l'8 aprile 2000*