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Decay Estimates for Solutions of a Class of Parabolic Problems Arising in Filtration Through Porous Media (*)

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Sunto. – *In questo lavoro si studia un problema di valori al contorno parabolico non lineare che si incontra nello studio dell' infiltrazione di un gas in un mezzo poroso. Si stabiliscono condizioni sui dati che determinano un comportamento di tipo esponenziale decrescente nel tempo per la soluzione e il suo gradiente. Si costruiscono inoltre stime esplicite.*

1. – Introduction.

This paper deals with the following initial-boundary value problem

$$[G(u)]_{xx} + f(u) = u_t, \qquad x \in (-L, L), \qquad t > 0, \tag{1.1}$$

$$u(\pm L, t) = 0, \qquad t > 0, \tag{1.2}$$

$$u(x, 0) = h(x), \qquad x \in (-L, L). \tag{1.3}$$

The differential equation (1.1) arises e.g. in filtration of ground water. With $G(u) := u^m$, $m = \text{const.} > 1$, we have a filtration model for the density of a gas in a porous medium [12].

The following assumptions on the data will be made throughout the paper: f and h are nonnegative C^1-functions and G is a positive nondecreasing C^3-function. Moreover we assume throughout that

$$s(fg)' - fg \geq 0 \ \ \forall s > 0, \qquad f(0) = 0, \tag{1.4}$$

where $g := G'$ is a bounded positive function such that

$$0 < g_m \leq g \leq g_M. \tag{1.5}$$

It is well known that under certain conditions the unique solution (≥ 0) of (1.1)-(1.3) may blow-up at some point in space-time. Our goal is to determine data restrictions sufficient to insure that the solution $u(x, t)$ of (1.1)-(1.3) will remain

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bounded for all time. Moreover we show then that $u(x, t)$ decays exponentially in time. To this end we introduce the following combination of u and u_x:

$$\Psi_\alpha(x, t) := [g^2(u)\, u_x^2 + \alpha u^2 + 2F(u)]\, e^{2\alpha\beta t}. \tag{1.6}$$

In (1.6) α is a nonnegative parameter to be specified later, β is a parameter to be selected such that

$$0 \leqslant \beta \leqslant \frac{1}{g_M}, \tag{1.7}$$

and $F(u)$ is defined as

$$F(u) := \int_0^u f(s)\, g(s)\, ds\,. \tag{1.8}$$

In Section 2 we establish a maximum principle for the auxiliary function Ψ_α. This maximum principle is then applied in Section 3 and 4 to derive explicit decay bounds for the solution u and its first order derivative $|u_x|$. Similar arguments have already been used by the authors in [9, 10], and by L. E. Payne and G. A. Philippin in [7, 8]. We refer to [1, 2, 13, 14] for further informations on parabolic problems.

2. – A maximum principle.

The main result of this Section is formulated in the next theorem:

THEOREM 1. – *Under the assumptions of the first section,the auxiliary function Ψ_α defined in* (1.6) *takes its maximum value either at an interior critical point $(\bar{x}, \bar{t})$ of u or initially at $t = 0$. This may be formulated as follows*

$$\Psi_\alpha(x, t) \leqslant \max \begin{cases} (i) & \Psi_\alpha(\bar{x}, \bar{t}) \;\; \textit{with} \;\; u_x(\bar{x}, \bar{t}) = 0\,, \\ (ii) & \max\limits_{x \in (-L, L)} \Psi_\alpha(x, 0)\,. \end{cases} \tag{2.1}$$

For the proof of Theorem 1 we show that Ψ_α satisfies some parabolic inequality. For convenience the index α will be dropped in the following computation. We compute

$$\Psi_x = \{2G_x G_{xx} + 2\alpha u u_x + 2F'\, u_x\}\, e^{2\alpha\beta t} = 2u_x\{g[u_t - f] + \alpha u + F'\}\, e^{2\alpha\beta t}, \tag{2.2}$$

$$\Psi_{xx} = \{2G_{xx}^2 + 2G_x G_{xxx} + 2\alpha u_x^2 + 2\alpha u u_{xx} + 2F''\, u_x^2 + 2F'\, u_{xx}\}\, e^{2\alpha\beta t}. \tag{2.3}$$

With $G_x = gu_x$, $G_{xxx} = u_{tx} - f'\,u_x$, $gu_{xx} = u_t - f - g'\,u_x^2$, we obtain

$$(2.4)\qquad g\Psi_{xx} = \{2gG_{xx}^2 + 2g^2 u_x[u_{tx} - f'\,u_x] + 2g(\alpha + F'')\,u_x^2 + 2(F' + \alpha u)[u_t - f - g'\,u_x^2]\}\,e^{2\alpha\beta t}.$$

Next, we compute

$$(2.5)\quad \Psi_t = \{2G_x G_{xt} + 2\alpha u u_t + 2F'\,u_t + 2\alpha\beta G_x^2 + 2\alpha^2\beta u^2 + 4\alpha\beta F\}\,e^{2\alpha\beta t}.$$

With $G_{xt} = g'\,u_x u_t + gu_{xt}$, we obtain

$$(2.6)\qquad \Psi_t = \{2gu_x[g'\,u_x u_t + gu_{xt}] + 2\alpha u u_t + 2F'\,u_t + 2\alpha\beta g^2 u_x^2 + 2\alpha^2\beta u^2 + 4\alpha\beta F\}\,e^{2\alpha\beta t}.$$

Combining (2.4) and (2.6),we obtain after some reduction

$$(2.7)\qquad g\Psi_{xx} - \Psi_t = \{2gG_{xx}^2 - 2f'\,g^2 u_x^2 + 2g(\alpha + F'')\,u_x^2 - 2fF' - 2\alpha uf - 2g'\,u_x^2[gu_t + \alpha u + F'] - 2\alpha\beta g^2 u_x^2 - 2\alpha^2\beta u^2 - 4\alpha\beta F\}\,e^{2\alpha\beta t}.$$

Moreover we have from (2.2)

$$(2.8)\qquad G_{xx} = -\frac{1}{g}(\alpha u + F') + \frac{1}{2gu_x}\Psi_x e^{-2\alpha\beta t},$$

i.e.

$$(2.9)\qquad gG_{xx}^2 = \frac{1}{g}(\alpha u + F')^2 + \dots\,.$$

In (2.9) and later dots stand for a term of the form $k(x, t)\,u_x^{-2}\,\Psi_x$ where $k(x, t)$ is regular in $(-L, L) \times (0, \infty)$. We may extract another term of the same form from the right hand side of (2.7). We have in fact from (2.2)

$$(2.10)\qquad gu_t + \alpha u + F' = fg + \dots\,.$$

Inserting (2.9) and (2.10) into (2.8) we obtain after some reduction

$$(2.11)\quad L\Psi := g\Psi_{xx} - \Psi_t + \dots = \left\{2\alpha\left(\frac{1}{g} - \beta\right)[g^2 u_x^2 + \alpha u^2] - 2gu_x^2(fg - F')' + 2F'\left[\frac{2\alpha u}{g} - f + \frac{F'}{g}\right] - 2\alpha uf - 4\alpha\beta F\right\}e^{2\alpha\beta t} = \left\{2\alpha\left(\frac{1}{g} - \beta\right)[g^2 u_x^2 + \alpha u^2] + 2\alpha[uf - 2\beta F]\right\}e^{2\alpha\beta t}.$$

From (1.7) we have $\beta \leqslant \dfrac{1}{g}$. This inequality implies that

$$uf - 2\beta F \geqslant \frac{1}{g}(ufg - 2F) \geqslant 0\,, \tag{2.12}$$

where the last inequality follows from (1.4) integrated by parts. This leads to the desired parabolic inequality

$$L\Psi \geqslant 0\,. \tag{2.13}$$

It then follows from Nirenberg's maximum principle [6, 11] that Ψ takes its maximum either at $x = \pm L$ for some $t > 0$, or initially at some point $x \in (-L, L)$, or at a critical point $(\bar{x}, \bar{t})$ of $u(x, t)$. However the first possibility cannot occur in view of Friedmann's maximum principle [4, 11]. In fact we have $u_t(\pm L, t) = 0$ from (1.2), which implies $G_{xx}(u(\pm L, t)) = 0$, and (2.2) gives $\Psi_x(\pm L, t) = 0$, so that Ψ cannot have its maximum at $x = \pm L$. This establishes Theorem 1.

3. – Decay bounds. The particular case: $f(s) := \mu s / g(s)$.

In this section we investigate the following initial boundary value problem

$$[G(\tilde{u})]_{xx} + \frac{\mu \tilde{u}}{g(\tilde{u})} = \tilde{u}_t\,, \qquad x \in (-L, L)\,, \qquad t \in (0, T)\,, \tag{3.1}$$

$$\tilde{u}(\pm L, t) = 0\,, \qquad t \in (0, T)\,, \tag{3.2}$$

$$\tilde{u}(x, 0) = h(x) \geqslant 0\,, \qquad x \in (-L, L)\,, \tag{3.3}$$

under the assumptions of Section 1, where μ is some constant to be specified. We have the following result:

THEOREM 2. – *Suppose that the constant μ in* (3.1) *satisfies the inequality*

$$0 \leqslant \mu < \alpha_0 := \frac{\pi^2 g_m^2}{4L^2}\,. \tag{3.4}$$

We then conclude that the solution $\tilde{u}(x, t)$ of (3.1)-(3.3) *exists for all time* (*i.e.* $T = \infty$ *in* (3.1), (3.2)). *Moreover we have the following decay estimate*:

$$g^2 \tilde{u}_x^2 + \alpha_0 \tilde{u}^2(x, t) \leqslant H^2 e^{-2(\alpha_0 - \mu)\beta t}, \tag{3.5}$$

where β satisfies (1.7) *and with*

$$H^2 := \max_{x \in (-L, L)} \{ g^2(h)\, h'^2 + \alpha_0 h^2 \}\,. \tag{3.6}$$

For the proof of Theorem 2, we introduce the auxiliary function $\widetilde{\Psi}_\alpha(x, t)$ defined on $\tilde{u}(x, t)$ by (1.6), (1.8) as

$$\tilde{\Psi}_\alpha(x, t) := \{g^2(\tilde{u})\,\tilde{u}_x^2 + (\alpha+\mu)\,\tilde{u}^2\}\, e^{2\alpha\beta t}, \qquad x \in (-L, L),\ 0 < t < T. \tag{3.7}$$

Clearly $\tilde{\Psi}_\alpha(x, t)$ satisfies either (i) or (ii) in (2.1). Let us assume that $\tilde{\Psi}_\alpha$ satisfies (i) in (2.1), i.e. assume

$$\tilde{\Psi}_\alpha(x, t) \leqslant \tilde{\Psi}_\alpha(\bar{x}, \bar{t}), \tag{3.8}$$

with $\tilde{u}_x(\bar{x}, \bar{t}) = 0$. Inequality (3.8) evaluated at $t = \bar{t}$ gives

$$g_m^2 \tilde{u}_x^2(x, \bar{t}) \leqslant g^2(\tilde{u})\,\tilde{u}_x^2(x, \bar{t}) \leqslant (\alpha+\mu)[\tilde{u}_M^2 - \tilde{u}^2(x, \bar{t})] \tag{3.9}$$

with

$$\tilde{u}_M := \max_{x \in (-L, L)} \tilde{u}(x, \bar{t}). \tag{3.10}$$

Inequality (3.9) may be rewritten as

$$\frac{|d\tilde{u}(x, \bar{t})|}{\sqrt{\tilde{u}_M^2 - \tilde{u}^2(x, \bar{t})}} \leqslant \frac{\sqrt{\alpha+\mu}}{g_m}\, dx. \tag{3.11}$$

Integrating (3.11) from the critical point $\bar{x}$ to the nearest endpoint of the interval $[-L, L]$, we obtain the inequality

$$\alpha + \mu \geqslant \frac{\pi^2 g_m^2}{4L^2} =: \alpha_0. \tag{3.12}$$

If (3.12) does not hold, i.e. if we have

$$0 \leqslant \alpha < \alpha_0 - \mu, \tag{3.13}$$

we conclude that $\tilde{\Psi}_\alpha(x, t)$ cannot satisfy the first possibility (i), and must therefore satisfy the second possibility (ii) in (2.1). This shows that blow-up cannot occur i.e. $T = \infty$ Moreover (ii) in (2.1) with $\alpha \to \alpha_0 - \mu$ reduces to (3.5). This achieves the proof of Theorem 2.

4. – Decay bounds: The general case.

In this section we want to determine restrictions on the initial data of problem (1.1)-(1.3) sufficient to force the solution $u(x, t)$ to decay exponentially in time. Consequently we obtain explicit decay bounds for u and $|u_x|$ valid under somewhat stronger restrictions. As indicated earlier, the solution of (1.1)-(1.3) may blow-up at some time $\hat{t}$ that may be finite or infinite [3, 5]. However if blow-up does occur at $\hat{t}$, the solution $u(x, t)$

of problem (1.1)-(1.3) will exist in $(0, \hat{t})$. Our first analysis will be confined on any time interval $(0, T)$ with T prior an (hypothetic) blow-up time $\hat{t}$.

In a first step we establish the following comparison result:

LEMMA 1. – *Under the assuptions of the first section, the solution $u(x, t)$ of problem* (1.1)-(1.3) *may be estimated as follows*:

$$0 \leqslant u(x, t) \leqslant U e^{-(\alpha_0 - \mu)\beta t}, \quad x \in (-L, L), t \in (0, T), \tag{4.1}$$

with β satisfying (1.7) *and with*

$$\alpha_0 := \frac{\pi^2 g_m^2}{4L^2}, \tag{4.2}$$

$$\mu := \frac{f(u_M)\, g(u_M)}{u_M}, \tag{4.3}$$

$$u_M := \max_{x \in (-L, L) \times (0, T)} u(x, t), \tag{4.4}$$

$$U := \max_{x \in (-L, L)} \sqrt{\frac{g^2(h)\, h'^2}{\alpha_0} + h^2}. \tag{4.5}$$

For the proof of Lemma 1, we note that $\dfrac{f(s)\, g(s)}{s}$ is nondecreasing in $s > 0$. This follows from (1.4):

$$\left(\frac{fg}{s}\right)' = \frac{1}{s^2}[s(fg)' - fg] \geqslant 0, \quad \forall s > 0. \tag{4.6}$$

From (1.1), (4.6) and (4.3) we compute

$$(G(u))_{xx} + \frac{\mu u}{g(u)} - u_t = \frac{u}{g(u)}\left[\mu - \frac{f(u)\, g(u)}{u}\right] \geqslant 0. \tag{4.7}$$

Using a standar comparison result [11] we may compare $u(x, t)$ with the solution $\tilde{u}(x, t)$ of problem (3.1)-(3.3). This leads to the desired result:

$$0 \leqslant u(x, t) \leqslant \tilde{u}(x, t) \leqslant U e^{-(\alpha_0 - \mu)\beta t}, \tag{4.8}$$

where U, α_0, μ, β are given in Lemma 1.

In a second step we establish the next result:

LEMMA 2. – *Assuming the hypotheses of the first section and assuming that the initial data $h(x)$ is small enough in the following sense*

$$\frac{f(U)\, g(U)}{U} < \alpha_0 \left(:= \frac{\pi^2 g_m^2}{4L^2} \right), \tag{4.9}$$

where U is defined in (4.5), *we then conclude that the solution $u(x, t)$ of* (1.1)-(1.3) *exists for all time* (*i.e.* $T = \infty$). *Moreover we have*

$$\max_{x \in (-L, L)} \frac{f(u(x, t))\, g(u(x, t))}{u(x, t)} < \alpha_0, \qquad \forall t > 0\,. \tag{4.10}$$

For the prooof of Lemma 2 we first observe that (4.5), (4.9) and the monotonicity of $\dfrac{f(s)\, g(s)}{s}$ imply the inequality

$$\frac{f(h)\, g(h)}{h} \leqslant \frac{f(U)\, g(U)}{U} < \alpha_0. \tag{4.11}$$

Suppose now that (4.10) does not hold for all time. In view of (4.11) there must be a first time T at which we have

$$\mu := \frac{f(u_M)\, g(u_M)}{u_M} \leqslant \frac{f(u(x, T))\, g(u(x, T))}{u(x, T)} = \alpha_0, \tag{4.12}$$

with $u_M := \max_{(x, t) \in (-L, L) \times (0, T)} u(x, t)$. It then follows from Lemma 1 that

$$u(x, t) \leqslant U e^{-(\alpha_0 - \mu)\beta t} \leqslant U\,, \qquad x \in (-L, L),\ 0 \leqslant t \leqslant T\,. \tag{4.13}$$

From (4.13) and (4.9) we obtain

$$\max_{x \in (-L, L)} \frac{f(u(x, T))\, g(u(x, T))}{u(x, T)} \leqslant \frac{f(U)\, g(U)}{U} < \alpha_0, \tag{4.14}$$

from which we conclude indeed that (4.10) cannot be violated for any finite value of T. This achieves the proof of Lemma 2.

In a last step we establish a decay bound for u and $|u_x|$ formulated in the next theorem.

THEOREM 3. – *Assuming the hypotheses of the first section and assuming that the initial data are small enough in the sense that there exists a constant $\alpha_1 > 0$ such that*

$$\frac{f(U)\, g(U)}{U} < \alpha_0 - \alpha_1 = \frac{\pi^2 g_m^2}{4L^2} - \alpha_1, \tag{4.15}$$

with U defined in (4.5), *we have the following decay estimate*:

$$g^2 u_x^2 + \alpha_1 u^2 + 2F(u) \leqslant \mathcal{H}^2 e^{-2\alpha_1 \beta t}, \qquad x \in (-L, L),\ t > 0\,, \tag{4.16}$$

with β satisfying (1.7) *and with*

$$\mathcal{H}^2 := \max_{x \in (-L, L)} \{ g^2(h)\, h'^2 + \alpha_1 h^2 + 2F(h) \}\,. \tag{4.17}$$

For the proof of Theorem 3 we first observe that (4.15) implies (4.9) so that the solution $u(x, t)$ of (1.1)-(1.3) does not blow-up in any finite time. Clearly $\Psi_{\alpha_1}(x, t)$ defined in (1.6) must satisfy either one of the two possibilities in (2.1). Let us assume that $\Psi_{\alpha_1}(x, t)$ satisfies the first possibility (i) in (2.1), i.e. assume

$$\Psi_{\alpha_1}(x, t) \leqslant \Psi_{\alpha_1}(\overline{x}, \overline{t}), \tag{4.18}$$

with $u_x(\overline{x}, \overline{t}) = 0$.

Inequality (4.18) evaluated at $t = \overline{t}$ reduces to

$$g_m^2 u_x^2(x, \overline{t}) \leqslant g^2 u_x^2(x, \overline{t}) \leqslant \alpha_1 [u_M^2 - u^2(x, \overline{t})] + 2[F(u_M) - F(u)], \tag{4.19}$$

with $u_M := u(\overline{x}, \overline{t})$. Together with

$$F(u_M) - F(u(x, \overline{t})) = \int_u^{u_M} \frac{f(s)\, g(s)}{s} s\, ds \leqslant \tag{4.20}$$

$$\leqslant \frac{f(u_M)\, g(_M)}{u_M} \int_u^{u_M} s\, ds = \frac{\mu}{2} [u_M^2 - u^2(x, \overline{t})],$$

we obtain

$$g_m^2 u_x^2(x, \overline{t}) \leqslant (\alpha_1 + \mu)[u_M^2 - u^2(x, \overline{t})], \tag{4.21}$$

with $\mu = \dfrac{f(u_M)\, g(u_M)}{u_M}$. Inequality (4.21) is analogous to (3.9). As in Section 3, we conclude from (4.21) that

$$\alpha_1 + \mu \geqslant \frac{\pi^2 g_m^2}{4L^2} =: \alpha_0. \tag{4.22}$$

It remains to show that (4.22) cannot hold under assumption (4.15). Indeed from (4.10) we obtain the strict inequality

$$\mu := \frac{f(u_M)\, g(u_M)}{u_M} \leqslant \alpha_0, \tag{4.23}$$

so that $u \leqslant U$ by (4.1). Since $\dfrac{f(s)\, g(s)}{s}$ is nondecreasing, it follows from (4.15) that

$$\mu \leqslant \frac{f(U)\, g(U)}{U} < \alpha_0 - \alpha_1, \tag{4.24}$$

in contradiction of (4.22). This achieves the proof of Theorem 3.

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