Bollettino
Unione Matematica Italiana

Filip Defever, Ion Mihai, Leopold Verstraelen

B.-Y. Chen's inequalities for submanifolds of Sasakian space forms

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **4-B** (2001), n.2, p. 521–529.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2001_8_4B_2_521_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2001.

Bollettino U. M. I.
(8) 4-B (2001), 521-529

B.-Y. Chen's Inequalities for Submanifolds of Sasakian Space Forms.

Filip Defever (*) - Ion Mihai (**) - Leopold Verstraelen

Sunto. – *Recentemente, B.-Y. Chen ha introdotto una nuova serie di invarianti $\delta(n_1, \ldots, n_k)$ riemanniani per ogni varietà riemanniana. Ha anche ottenuto disuguaglianze strette per questi invarianti per sottovarietà di forme spaziali reali e complesse in funzione della loro curvatura media. Nel presente lavoro proviamo analoghe stime per gli invarianti $\delta(n_1, \ldots, n_k)$ per sottovarietà C-totalmente reali e CR di contatto di una forma spaziale di Sasaki $\widetilde{M}(c)$.*

1. – Introduction.

In [2] B.-Y. Chen defined a Riemannian invariant $\delta_M = \tau - \inf K$ for any Riemannian manifold M. Subsequently, sharp inequalities for this invariant were obtained for submanifolds in real and complex space forms in terms of their mean curvature; see also e.g. [3], [5], [6] for related results.

In [7] this question was studied for C-totally real submanifolds of a Sasakian space form. A general inequality was obtained between the main intrinsic invariants of a C-totally real submanifold M on one side, namely its sectional curvature function K and its scalar curvature function τ, and its main extrinsic invariant on the other side, namely its mean curvature function $|H|$. More precisely, in the Sasakian case, B.-Y. Chen's inequality, relating K, τ and H, reads:

$$\inf K \geqslant \tau - \frac{n^2(n-2)}{2(n-1)} |H|^2 - \frac{(n+1)(n-2)(c+3)}{8}. \tag{1}$$

In [4] B.-Y. Chen generalized his invariant δ_M, and defined a string of new Riemannian invariants $\delta(n_1, \ldots, n_k)$ including δ_M as a particular case. (For precise definitions, see Section 3). He also obtained sharp inequalities for these invariants for submanifolds in real and complex space forms.

The purpose of the present paper is to establish analogous inequalities for the new invariants $\delta(n_1, \ldots, n_k)$ for C-totally real and contact CR-submani-

(*) Postdoctoral Researcher F.W.O.-Vlaanderen, Belgium.
(**) Research Fellow of the Research Council of the K.U. Leuven

folds in a Sasakian space form, thus also generalizing (1). More precisely, we prove that the following estimates hold:

THEOREM 1. – For an n-dimensional $(n>2)$ C-totally real submanifold M^n of a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}^{2m+1}(c)$, we have that

$$\delta(n_1,\ldots,n_k)\leqslant\frac{n^2(n+k-1-\Sigma n_j)}{2(n+k-\Sigma n_j)}\,|H|^2+\frac{1}{8}\left[n(n-1)-\sum_{j=1}^{k}n_j(n_j-1)\right](c+3).$$

THEOREM 2. – For an n-dimensional $(n>2)$ contact CR-submanifold M^n of a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}^{2m+1}(c)$, we have that

$$\delta(n_1,\,\ldots,\,n_k)\leqslant\frac{n^2(n+k-1-\Sigma n_j)}{2(n+k-\Sigma n_j)}\,|H|^2+$$

$$\frac{1}{8}\left[n(n-1)-\sum_{j=1}^{k}n_j(n_j-1)\right](c+3)-\frac{1}{4}(n-1-3h)(c-1).$$

We also give characterizations for the situation in which the equality holds.

2. – Submanifolds of a Sasakian space form.

Let $(\widetilde{M},g)$ be a $(2m+1)$-dimensional Riemannian manifold endowed with an endomorphism ϕ of its tangent bundle $T\widetilde{M}$, a vector field ξ and a 1-form η such that

$$\begin{cases}\phi^2X=-X+\eta(X)\,\xi, \quad \phi\xi=0, \quad \eta\circ\phi=0, \quad \eta(\xi)=1,\\ g(\phi X,\phi Y)=g(X,Y)-\eta(X)\,\eta(Y), \quad \eta(X)=g(X,\xi),\end{cases}$$

for all vector fields $X, Y\in\Gamma(T\widetilde{M})$.

If, in addition, $d\eta(X,Y)=g(\phi X,Y)$, then $\widetilde{M}$ is said to have a contact Riemannian structure (ϕ,ξ,η,g). If, moreover, the structure is normal, i.e. if $[\phi X,\phi Y]+\phi^2[X,Y]-\phi[X,\phi Y]-\phi[\phi X,Y]=-2d\eta(X,Y)\,\xi$, then the contact Riemannian structure is called a Sasakian structure and $\widetilde{M}$ is called a Sasakian manifold. For more details and background, we refer to the standard references [1], [10].

A plane section σ in $T_p\widetilde{M}$ of a Sasakian manifold $\widetilde{M}$ is called a ϕ-section if it is spanned by X and ϕX, where X is a unit tangent vector field orthogonal to ξ. The sectional curvature $\overline{K}(\sigma)$ w.r.t. a ϕ-section σ is called a ϕ-sectional curvature. If a Sasakian manifold $\widetilde{M}$ has constant ϕ-sectional curvature c, then it is called a Sasakian space form and is denoted by $\widetilde{M}(c)$.

The curvature tensor $\widetilde{R}$ of a Sasakian space form $\widetilde{M}(c)$ is given by [1]:

$$\widetilde{R}(X, Y) Z = \frac{c+3}{4}(g(Y, Z) X - g(X, Z) Y) + \tag{2}$$

$$\frac{c-1}{4}(\eta(X)\,\eta(Z)\,Y - \eta(Y)\,\eta(Z)\,X + g(X, Z)\,\eta(Y)\,\xi - g(Y, Z)\,\eta(X)\,\xi +$$

$$g(\phi Y, Z)\,\phi X - g(\phi X, Z)\,\phi Y - 2g(\phi X, Y)\,\phi Z),$$

for any tangent vector fields X, Y, Z to $\widetilde{M}(c)$.

An n-dimensional submanifold M of a Sasakian space form $\widetilde{M}(c)$ is called a *C-totally real submanifold* if ξ is a normal vector field on M. A direct consequence of this definition is that $\phi(TM) \subset T^{\perp}M$, i.e. that M is an anti-invariant submanifold of $\widetilde{M}(c)$, (hence their name of «contact»-totally real submanifolds); see e.g. [8].

On the other hand, a submanifold M tangent to ξ of a Sasakian space form $\widetilde{M}(c)$ is said to be a *contact CR-submanifold* if its tangent bundle TM splits into an invariant and an anti-invariant subbundle by ϕ, respectively, i.e.

$$TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \{\xi\},$$

$$\phi(\mathcal{D}) = \mathcal{D}, \qquad \phi(\mathcal{D}^{\perp}) \subset T^{\perp}M.$$

We want to mention that any contact CR-submanifold is foliated by C-totally real submanifolds, i.e. the distribution $\mathcal{D}^{\perp}$ is completely integrable.

Also, on a contact CR-submanifold M, the complex subbundle

$$\mathcal{B} = \{X - i\phi X;\ X \in \Gamma(\mathcal{D})\}$$

is involutive. Thus M is endowed with a *Cauchy-Riemann* structure in the sense of S. Greenfield [9].

3. – B.-Y. Chen's inequalities.

Let M be an n-dimensional Riemannian manifold. Denote by $K(\pi)$ the sectional curvature of the plane section $\pi \subset T_pM$, $p \in M$. For any orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space T_pM, the scalar curvature τ at p is defined by

$$\tau = \sum_{1 \le i < j \le n} K(e_i \wedge e_j).$$

For each point $p \in M$, we put

$$(\inf K)(p) = \inf\{K(\pi);\ \pi \subset T_pM,\ \dim \pi = 2\}.$$

Inf K is a well-defined function on M. Let δ_M denote the difference between the scalar curvature and inf K, i.e.:

$$\delta_M(p) = \tau(p) - \inf K(p);$$

δ_M is a well-defined Riemannian invariant. δ_M was introduced by B.-Y. Chen in [2], where he gave a sharp inequality of δ_M for submanifolds in real and complex space forms (see also [3]).

Afterwards, in [4], B.-Y. Chen generalized the previous concept, introducing a series of new Riemannian invariants, as follows. If L is a linear subspace of T_pM, $\dim L = r \geqslant 2$, and $\{u_1, \ldots, u_r\}$ an orthonormal basis of L, the scalar curvature $\tau(L)$ of L is defined by

$$\tau(L) = \sum_{1 \leqslant i < j \leqslant r} K(e_i \wedge e_j).$$

Clearly, $\tau(T_pM) = \tau(p)$.

We denote by $S(n, k)$ the set of all k-tuples $(n_1, \ldots, n_k)$ of integers $\geqslant 2$ satisfying $n_1 + \ldots + n_k \leqslant n$ and $n_1 < n$. Let $(n_1, \ldots, n_k) \in S(n, k)$ and

$$S(n_1, \ldots, n_k)(p) = \inf\{\tau(L_1) + \ldots + \tau(L_k)\},$$

where $L_1, \ldots, L_k$ are mutually orthogonal subspaces of T_pM with $\dim L_j = n_j$, $j = 1, \ldots, k$. The Riemannian invariants $\delta(n_1, \ldots, n_k)$ are defined by [4]

$$\delta(n_1, \ldots, n_k)(p) = \tau(p) - S(n_1, \ldots, n_k)(p).$$

First, we prove inequalities satisfied by the $\delta(n_1, \ldots, n_k)$ for C-totally real submanifolds in a Sasakian space form:

THEOREM 1. – Let M be an n-dimensional $(n > 2)$ C-totally real submanifold of a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$. Then, for any $(n_1, \ldots, n_k) \in S(n, k)$, we have

$$(3) \qquad \delta(n_1, \ldots, n_k) \leqslant$$

$$\frac{n^2(n+k-1-\Sigma n_j)}{2(n+k-\Sigma n_j)}|H|^2 + \frac{1}{8}\left[n(n-1) - \sum_{j=1}^{k} n_j(n_j-1)\right](c+3).$$

Moreover, the equality holds at a point $p \in M$ if and only if there exist a tangent basis $\{e_1, \ldots, e_n\} \subset T_pM$ and a normal basis $\{e_{n+1}, \ldots, e_{2m}, e_{2m+1} = \xi\} \subset$

$T_p^{\perp} M$ such that the shape operators $A_r = A_{e_r}$ take the following forms

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix}, \tag{4}$$

with $a_1 + \ldots + a_{n_1} = \ldots = a_{n_1 + \ldots + n_{k-1}+1} + \ldots + a_{n_1 + \ldots + n_k} = a_{n_1 + \ldots + n_k + 1} = \ldots = a_n$, and

$$A_r = \begin{pmatrix} A_1^r & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & A_k^r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & \cdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}; \quad r \in \{n+2, \ldots, 2m\}, \tag{5}$$

where $\operatorname{tr} A_j^r = 0$, ${}^t A_j^r = A_j^r (j = \overline{1, k})$; $r \in \{n+2, \ldots, 2m\}$, and $A_\xi = 0$.

PROOF. – The Gauss equation and formula (2) imply that for a C-totally real submanifold of a Sasakian space form we have that

$$g(R(X, Y) Z, W) = \frac{1}{4}(c+3)(g(Y, Z) g(X, W) - g(X, Z) g(Y, W)) +$$
$$g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),$$

for all vector fields X, Y, Z, W tangent to M, where h denotes the second fundamental form and R the curvature tensor of M. It follows that

$$2\tau = n^2 |H|^2 - \|h\|^2 + \frac{n(n-1)(c+3)}{4} \tag{6}$$

Let $(n_1, \ldots, n_k) \in S(n, k)$ and denote

$$\gamma = n + k - \sum_{j=1}^{k} n_j, \qquad B = \frac{n^2(\gamma - 1)}{\gamma},$$

$$\varepsilon = 2\tau - n(n-1)\frac{c+3}{4} - B|H|^2. \tag{7}$$

Eliminating τ from the equations (6) and (7), one finds

$$n^2 |H|^2 = \gamma(\varepsilon + \|h\|^2). \tag{8}$$

Let $L_1, \ldots, L_k \subset T_p M$ be mutually orthogonal subspaces with $\dim L_j = n_j$, $j = \overline{1, k}$. We choose an orthonormal basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2m}, e_{2m+1} = \xi\}$ at $p \in M$ such that

$$L_j = \mathrm{sp}\{e_{n_1 + \ldots + n_{j-1} + 1}, \ldots, e_{n_1 + \ldots + n_j}\}, \qquad j = \overline{1, k},$$

$$e_{n+1} = \frac{H}{|H|}.$$

If we denote $a_i = h_{ii}^{n+1}$, the equation (7) becomes

$$\left(\sum_{i=1}^{n} a_i\right)^2 = \gamma\left[\varepsilon + \sum_{i=1}^{n} a_i^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2\right]. \tag{9}$$

We introduce the following notations

$$\begin{cases} b_1 = a_1, \\ b_2 = a_2 + \ldots + a_{n_1}, \\ b_3 = a_{n_1+1} + \ldots + a_{n_1+n_2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ b_{k+1} = a_{n_1+\ldots+n_{k-1}+1} + \ldots + a_{n_1+\ldots+n_k}, \\ b_{k+2} = a_{n_1+\ldots+n_k+1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ b_{\gamma+1} = a_n. \end{cases}$$

Then, the equation (9) may be expressed by

$$\left(\sum_{i=1}^{\gamma} b_i\right)^2 = \gamma\left[\varepsilon + \sum_{i=1}^{\gamma} b_i^2 + \sum_{i \neq j} (h_{ij}^{n+1})^2 + \sum_{r=n+2}^{2m} \sum_{i,j=1}^{n} (h_{ij}^r)^2 - \right.$$

$$\left. \sum_{2 \leqslant \alpha_1 \neq \beta_1 \leqslant n_1} a_{\alpha_1} a_{\beta_1} - \sum_{\alpha_2 \neq \beta_2} a_{\alpha_2} a_{\beta_2} - \ldots - \sum_{\alpha_k \neq \beta_k} a_{\alpha_k} a_{\beta_k}\right],$$

where $n_1 + 1 \leqslant \alpha_2$, $\beta_2 \leqslant n_1 + n_2, \ldots, n_1 + \ldots + n_{k-1} + 1 \leqslant \alpha_k$, $\beta_k \leqslant n_1 + \ldots + n_k$.

Now, we can invoke the following Lemma of [2]:

Let $b_1, \ldots, b_n, \varepsilon$ be $n+1$ real numbers such that

$$\left(\sum_{i=1}^{n} b_i\right)^2 = (n-1)\left(\varepsilon + \sum_{i=1}^{n} b_i^2\right).$$

Then $2b_1 b_2 \geqslant \varepsilon$, with equality holding if and only if $b_1 + b_2 = b_3 = \ldots = b_n$.

Applying this Lemma, and following the same way as in [4], we then obtain

$$\tau(L_1) + \ldots + \tau(L_k) \geqslant \frac{\varepsilon}{2} + \frac{1}{8}\sum_{j=1}^{k} n_j(n_j - 1)(c+3), \tag{10}$$

which immediately leads to the inequality to prove.

If the equality holds at a point $p \in M$ and we choose $e_1, \ldots, e_n$ such that $h_{ij}^{n+1} = 0$, the shape operators indeed take the forms (4), (5). This finishes the proof of Theorem 1. ∎

Next, we turn our attention to contact CR-submanifolds of a Sasakian space form $\widetilde{M}(c)$ and prove the following inequalities for $\delta(n_1, \ldots, n_k)$ in this case.

THEOREM 2. – Let M be an n-dimensional $(n > 2)$ contact CR-submanifold of a $(2m+1)$-dimensional Sasakian space form $\widetilde{M}(c)$. Then, for any $(n_1, \ldots, n_k) \in S(n, k)$, we have

$$\delta(n_1, \ldots, n_k) \leqslant \frac{n^2(n+k-1-\Sigma n_j)}{2(n+k-\Sigma n_j)}|H|^2 + \frac{1}{8}\left[n(n-1) - \sum_{j=1}^{k} n_j(n_j-1)\right](c+3) - \frac{1}{4}(n-1-3h)(c-1). \tag{11}$$

Moreover, the equality holds at a point $p \in M$ if and only if there exist a tangent basis $\{e_1, \ldots, e_n\} \subset T_pM$ and a normal basis $\{e_{n+1}, \ldots, e_{2m}, e_{2m+1} = \xi\} \subset T_p^{\perp}M$ such that the shape operators $A_r = A_{e_r}$ take the following forms

$$A_{n+1} = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix},$$

with $a_1 + \ldots + a_{n_1} = \ldots = a_{n_1 + \ldots + n_{k-1} + 1} + \ldots + a_{n_1 + \ldots + n_k} = a_{n_1 + \ldots + n_k + 1} = \ldots = a_n$, and

$$A_r = \begin{pmatrix} A_1^r & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & \cdots & A_k^r & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & \cdots \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}; \quad r \in \{n+2, \cdots, 2m\},$$

where tr $A_j^r = 0$, ${}^t A_j^r = A_j^r (j = \overline{1, k})$; $r \in \{n+2, \ldots, 2m\}$, and $A_\xi = 0$.

PROOF. – The arguments for the present case follow quite closely the line of the proof of Theorem 1. Therefore, we confine ourselves to indicate the points where the most significant formulas take a different form.

The Gauss equation and formula (2) imply that for a contact CR-submanifold of a Sasakian space form we have that

$$g(R(X, Y)Z, W) = \frac{1}{4}(c+3)(g(Y, Z)\, g(X, W) - g(X, Z)\, g(Y, W)) +$$

$$\frac{1}{4}(c-1)(\eta(X)\,\eta(Z)\, g(Y, W) - \eta(Y)\,\eta(Z)\, g(X, W) +$$

$$\eta(Y)\,\eta(W)\, g(X, Z) - \eta(X)\,\eta(W)\, g(Y, Z) +$$

$$g(\phi X, W)\, g(\phi Y, Z) - g(\phi X, Z)\, g(\phi Y, W) - 2g(\phi X, Y)\, g(\phi Z, W)) +$$

$$g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),$$

from which there follows that

$$(12) \qquad 2\tau = n^2 |H|^2 - \|h\|^2 + \frac{1}{4} n(n-1)(c+3) - \frac{1}{2}(n-1-3h)(c-1),$$

with $2h = \dim \mathcal{D}$. Let $(n_1, \ldots, n_k) \in S(n, k)$ and denote

$$\gamma = n + k - \sum_{j=1}^{k} n_j, \qquad B = \frac{n^2(\gamma - 1)}{\gamma},$$

$$(13) \qquad \varepsilon = 2\tau - \frac{1}{4} n(n-1)(c+3) - B|H|^2 - \frac{1}{2}(n-1-3h)(c-1).$$

Eliminating τ from the equations (12) and (13), one finds

$$(14) \qquad n^2 |H|^2 = \gamma(\varepsilon + \|h\|^2).$$

From here on, the calculations run parallel to those in the proof of Theorem 1; note however that ε is defined differently. With the same notations, one also arrives at the same formula (10), but with ε now given by (13) instead. This immediately leads to the inequalities (11). Also the conditions for the equality-case can be read off in the same way, thus finishing the proof of Theorem 2. ∎

REFERENCES

[1] D. E. BLAIR, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math., vol. **509**, Springer, Berlin, 1976.

[2] B.-Y. CHEN, *Some pinching and classification theorems for minimal submanifolds*, Archiv Math., **60** (1993), 568-578.

[3] B.-Y. CHEN, *A Riemannian invariant for submanifolds in space forms and its applications*, Geometry and Topology of Submanifolds, **6** (1994), 58-81, World Scientific, Singapore.

[4] B.-Y. CHEN, *Some new obstructions to minimal and Lagrangian isometric immersions*, submitted.

[5] B.-Y. CHEN - F. DILLEN - L. VERSTRAELEN - L. VRANCKEN, *Totally real submanifolds of* $\boldsymbol{CP}^n$ *satisfying a basic equality*, Archiv. Math., **63** (1994), 553-564.

[6] B.-Y. CHEN, F. DILLEN, L. VERSTRAELEN AND L. VRANCKEN, *An exotic totally real minimal immersion of* S^3 *in* $\boldsymbol{CP}^3$ *and its characterization*, Proc. Roy. Soc. Edinburgh Sect. A, **126** (1996), 153-165.

[7] F. DEFEVER - I. MIHAI - L. VERSTRAELEN, *B.-Y. Chen's inequality for C-totally real submanifolds of Sasakian space forms*, Boll. Un. Mat. Ital., **11-B** (1997), 365-374.

[8] K. YANO - M. KON, *Anti-invariant Submanifolds*, Lecture Notes in Pure and Applied Mathematics, Marcel Dekker, New York, 1976.

[9] K. YANO - M. KON, *CR-Submanifolds of Kaehlerian and Sasakian Manifolds*, Birkhäuser, Boston, 1983.

[10] K. YANO - M. KON, *Structures on Manifolds*, Series in Pure Mathematics, vol. **3**, World Scientific, Singapore, 1984.

Filip Defever: Zuivere en Toegepaste Differentiaalmeetkunde
Departement Wiskunde, K.U. Leuven, Celestijnenlaan 200 B,
B - 3001 Leuven, BELGIUM. E-mail: Filip.Defever@wis.kuleuven.ac.be

Ion Mihai: Faculty of Mathematics, University of Bucharest
Str. Academiei 14, 70109 Bucharest, Romania. E-mail: imihai@math.math.unibuc.ro

Leopold Verstraelen: Departement Wiskunde, K.U. Leuven, Celestijnenlaan 200 B
B - 3001 Leuven, Belgium, and Group of Exact Sciences, K. U. Brussel, Vrijheidslaan 17
B - 1081 Brussel, Belgium. E-mail: Leopold.Verstraelen@wis.kuleuven.ac.be

Pervenuta in Redazione
il 29 giugno 1999