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Sunto. – *Un sottogruppo H di un gruppo G si dice* nearly normal *se ha indice finito nella sua chiusura normale* H^G. *In questa nota si caratterizzano i gruppi in cui ogni sottogruppo che non sia nearly normal soddisfa una fissata condizione finitaria* χ *per diverse scelte naturali della proprietà* χ.

1. – Introduction.

A subgroup H of a group G is said to be *nearly normal* if it has finite index in its normal closure H^G. A well-known result of B. H. Neumann [8] states that all subgroups of a group G are nearly normal if and only if the commutator subgroup G' of G is finite. More recently, the structure of groups which are rich in some sense of nearly normal subgroups has been investigated; in particular groups have been characterized for which the set of subgroups that are not nearly normal satisfies either the minimal or the maximal condition (see [3] and [6]). On the other hand, Romalis and Sesekin started in 1966 the study of groups whose non-normal subgroups are abelian (see [11], [12] and [13]); this approach was also used by Bruno and Phillips in [2], where groups whose subgroups are normal or locally nilpotent were considered, and more recently by Franciosi, de Giovanni and Newell [5], who described groups in which every subgroup is either normal or polycyclic.

The aim of this article is the study of groups in which every subgroup that is not nearly normal satisfies a certain finiteness condition χ. Among other natural choices for the property χ we characterize groups whose non-periodic subgroups are nearly normal, and we obtain a complete description of radical groups in which every subgroup is either nearly normal or polycyclic (recall that a group is *radical* if it has an ascending series with locally nilpotent factors).

Most of our notation is standard and can for instance be found in [10].

2. – Groups with many periodic subgroups.

It has been mentioned in the introduction that a group G contains only nearly normal subgroups if and only if its commutator subgroup G' is finite.

This property can be used to obtain the following lemma, which will be applied several times in our arguments.

LEMMA 2.1. – *Let $\mathfrak{X}$ be a subgroup closed class of groups, and let G be a group in which every subgroup either is nearly normal or belongs to $\mathfrak{X}$. If H is a subgroup of G which is not an $\mathfrak{X}$-group, then the index $|G' : G' \cap H|$ is finite. In particular, if H is normal in G, then G/H is finite-by-abelian.*

PROOF. – Since the class $\mathfrak{X}$ is subgroup closed, every subgroup of G containing H does not belong to $\mathfrak{X}$. It follows that all subgroups of G/H^G are nearly normal, and so the commutator subgroup $G' H^G/H^G$ is finite. Moreover, H has finite index in H^G, and hence also the index $|G' : G' \cap H|$ is finite. ■

COROLLARY 2.2. – *Let G be a non-periodic group whose subgroups are either nearly normal or periodic. Then the commutator subgroup G' of G is cyclic-by-finite. In particular G is soluble-by-finite and locally-(polycyclic-by-finite).*

PROOF. – Let y be an element of infinite order of G. Then the index $|G' : G' \cap \langle y \rangle|$ is finite by lemma 2.1 and hence G' is cyclic-by-finite. ■

Our next result describes groups whose subgroups are either nearly normal or periodic.

THEOREM 2.3. – *Let G be a group. Then every subgroup of G is either nearly normal or periodic if and only if G satisfies one of the following conditions*:

(i) *G is periodic.*

(ii) *The commutator subgroup G' of G is finite.*

(iii) *G contains a finite normal subgroup E such that*

$$G/E = T/E \times D/E ,$$

where T/E is a periodic abelian group and D/E is infinite dihedral.

PROOF. – Suppose first that each subgroup of G is either nearly normal or periodic, and assume that neither G is periodic nor its commutator subgroup G' is finite. Then G' is non-periodic by Corollary 2.2 and so in particular G is not an FC-group. If a is any element of infinite order of G, the index $|\langle a \rangle G' : \langle a \rangle|$ is finite, and hence a has only finitely many conjugates (see [3], Lemma 2.1). Thus the FC-centre F of G contains all elements of infinite order of G, and so G cannot be generated by its elements of infinite order. Let T be

the largest periodic normal subgroup of G, and let g be any element of T. Then the normal closure $\langle g\rangle^G$ is a periodic subgroup of the polycyclic-by-finite group $\langle g, G'\rangle$ and hence it is finite. It follows that T is contained in F. The factor group $\overline{G} = G/T$ is not torsion-free and does not contain periodic non-trivial normal subgroups, so that in particular it is not finite-by-abelian. Let $\overline{N}$ be any non-trivial normal subgroup of $\overline{G}$. Then $\overline{N}$ is not periodic, so that all subgroups of $\overline{G}/\overline{N}$ are nearly normal and $\overline{G}/\overline{N}$ has finite commutator subgroup. Therefore the group $\overline{G}$ is just-non-(finite -by-abelian). The commutator subgroup F' of F is obviously contained in T, so that $\overline{F} = F/T$ is abelian. Moreover, the Fitting subgroup $\overline{K} = K/T$ of $\overline{G}$ is torsion-free, so that K is generated by its elements of infinite order, and hence it is contained in F; thus $\overline{F}$ is the Fitting subgroup
of $\overline{G}$. As the group $\overline{G}$ is not torsion-free, we have $Z(\overline{G}) = \{1\}$ (see [14], Theorem 2.1), and so $\overline{F}$ is a faithful just-infinite $\overline{G}/\overline{F}$-module (see [14],Theorem 3.4). Let y be an element of infinite order of G, and put $\overline{y} = yT$. Then the indices $|\overline{F}:\langle\overline{y}\rangle^{\overline{G}}|$ and $|\langle\overline{y}\rangle^{\overline{G}}:\langle\overline{y}\rangle|$ are finite, so that the torsion- free abelian group $\overline{F}$ is cyclic-by-finite, and hence infinite cyclic and so $\overline{G}/C_{\overline{G}}(\overline{F})$ has order 2. So, since $C_{\overline{G}}(\overline{F}) = \overline{F}$, then $|\overline{G}:\overline{F}| = 2$. Let z be an element of G such that $\overline{F} = \langle zT\rangle$, and let $\overline{x}$ be any non-trivial element of finite order of $\overline{G}$. Then $\overline{x}$ has order 2 and $\overline{G} = \langle\overline{x}\rangle \ltimes \langle\overline{z}\rangle$ is an infinite dihedral group. The subgroup $L = \langle x, z\rangle$ is not periodic, so that the index $|L^G: L|$ is finite and L^G is polycyclic-by-finite. It follows that $E_0 = L^G \cap T$ is a finite normal subgroup of G. Moreover, the commutator subgroup T' of T is finite by Corollary 2.2, and so also the normal subgroup $E = E_0 T'$ of G is finite. Put $D = L^G E$. Clearly $G = TD$ and

$$T \cap D = T \cap L^G E = (T \cap L^G)\, E = E\,,$$

so that

$$G/E = T/E \times D/E\,,$$

where T/E is a periodic abelian group and $D/E \simeq G/T$ is infinite dihedral.

Conversely suppose that G satisfies the condition (iii) of the statement, and let H be any non-periodic subgroup of G. Then $HE \cap D$ is also non-periodic, and hence it has finite index in D. As T/E is contained in the centre of G/E, we have $(HE)^G = (HE)^D \leqslant HD$, so that

$$|H^G E : HE| \leqslant |HD : HE| = |D : HE \cap D|\,,$$

and so H has finite index in H^G. ■

If G is a group in which every subgroup is either nearly normal or a Černikov group, it is clear that G satisfies the minimal condition on subgroups that are not nearly normal. Then the above theorem and results obtained [3] can be used to obtain the following consequence.

COROLLARY 2.4. – *Let G be a group which is either non-periodic or locally finite. Then every subgroup of G is either nearly normal or a Černikov group if and only if G satisfies one of the following conditions*:

(i) *G is a Černikov group.*

(ii) *The commutator subgroup G' is finite.*

(iii) *G contains a finite normal subgroup E such that*

$$G/E = T/E \times D/E ,$$

where T/E is an abelian group satisfying the minimal condition and D/E is infinite dihedral.

PROOF. – Suppose first that every subgroup of G is either nearly normal or a Černikov group. If G is periodic, then it is locally finite and obviously satisfies the minimal condition on subgroups which are not nearly normal. Then either G is a Černikov group or the commutator subgroup G' is finite (see [3], Theorem 2.10). Assume now that G is not periodic and its commutator subgroup G' is infinite. Then it follows from Theorem 2.3 that G contains a finite normal subgroup E such that $G/E = T/E \times D/E$, where T/E is a periodic abelian group and D/E is infinite dihedral. Since G/T has infinite commutator subgroup, T/E must be a Černikov grup by Lemma 2.1. Conversely, let G satisfy condition (iii) of the statement, and let H be any subgroup of G which is not nearly normal. Then H is periodic by Theorem 2.3, and hence it is a Černikov group, as the group G is minimax. ∎

3. – Groups with many polycyclic subgroups.

This section is devoted to the description of radical groups whose non-polycyclic subgroups are nearly normal. However, our first result deals with groups having finite Prüfer rank or finite abelian section rank. Recall that a group G is said to have *finite abelian section rank* if it has no sections which are infinite abelian groups of prime exponent.

THEOREM 3.1. – *Let G be a radical group. Then*:

(a) *Every subgroup of G either is nearly normal or has finite abelian section rank if and only if either G is finite-by-abelian or it has finite abelian section rank.*

(b) *Every subgroup of G either is nearly normal or has finite Prüfer rank if and only if either G is finite-by-abelian or it has finite Prüfer rank.*

PROOF. – (a) Suppose that every subgroup of G either is nearly normal or has finite abelian section rank, but G does not have finite abelian section rank. Then G also has infinite subgroup rank (see [1]), and hence it contains an abelian subgroup A which is a direct product of infinitely many isomorphic cyclic non-trival groups. Let $A = A_1 \times A_2$, where both A_1 and A_2 have infinite abelian section rank. Then for $i = 1, 2$ the subgroup $G' \cap A_i$ has finite index in G' by Lemma 2.1. Therefore G' is finite, and G is finite-by-abelian.

(b) Suppose that every subgroup of G either is nearly normal or has finite Prüfer rank, but G does not have finite Prüfer rank. Then G contains an abelian subgroup A with infinite Prüfer rank (see [1]). By (a), we may assume without loss of generality that G has finite abelian section rank, so that A can be chosen to be periodic, and $A = A_1 \times A_2$, where both A_1 and A_2 have infinite Prüfer rank. It can be proved now as in (a) that G is finite-by-abelian. ■

LEMMA 3.2. – *Let G be a radical group containing a finite normal subgroup N such that every subgroup of G/N is either nearly normal or polycyclic. Then every subgroup of G is either nearly normal or polycyclic.*

PROOF. – Let H be a non-polycyclic subgroup of G. Then HN/N is not polycyclic, and so HN/N is a nearly normal subgroup of G/N. It follows that HN has finite index in $H^G N$, so that also the index $|H^G : H|$ is finite, and H is nearly normal in G. ■

A soluble group G is said to be an $\mathfrak{S}_1$-group if it has finite abelian section rank and the set $\pi(G)$ of all prime numbers that are orders of elements of G is finite.

LEMMA 3.3. – *Let G be a radical group with infinite commutator subgroup. If every subgroup of G either is nearly normal or polycyclic, then G is an $\mathfrak{S}_1$-group. Moreover, if G' is not polycyclic, then G is a minimax group.*

PROOF. – The group G has finite Prüfer rank by Theorem 3.1. If T is the largest periodic normal subgroup of G, it follows that G/T has a series of finite length whose factors are abelian subgroups with finite torsion subgroups (see [10] Part 2, Lemma 9.34). Assume that the locally finite group T is not a Černikov group, so that it does not satisfy the minimal condition on abelian subgroups by a theorem of Šunkov [15]. Then T contains an infinite abelian subgroup A which is the direct product of infinitely many subgroups of prime order. Put $A = A_1 \times A_2$ where both A_1 and A_2 are infinite. The indices $|G' : G' \cap A_1|$ and $|G' : G' \cap A_2|$ are finite by Lemma 2.1, so that the commutator subgroup G' is finite. This contradiction proves that T is a Černikov group, and hence G

is an $\mathfrak{S}_1$-group. Suppose now that G' is not a minimax group, so that it contains an abelian subgroup B which is not minimax (see [10] Part 2, Theorem 10.35). The subgroup B_1 of B, consisting of all elements of finite order of B, is a Černikov group, and hence $B = B_1 \times B_2$ where B_2 is torsion-free. Let E be a finitely generated subgroup of B_2 such that B_2/E is periodic. Clearly B_2/E is not a Černikov group, and so it contains a direct product $B_3/E \times B_4/E$, with both factors infinite. It follows that the indices $|G' : G' \cap B_3|$ and $|G' : G' \cap B_4|$ are finite, so that $G' \cap E$ has finite index in G', and G' is polycyclic. ■

LEMMA 3.4. – *Let G be a torsion-free locally nilpotent group which is not finitely generated. If every subgroup of G is either nearly normal or finitely generated, then G is nilpotent of class at most* 2.

PROOF. – Suppose that G is not abelian. Then the commutator subgroup G' of G is infinite, and G has finite Prüfer rank by Theorem 3.1. It follows that G is nilpotent, so that its centre $Z(G)$ is not finitely generated (see [16], p. 421), and hence $G'Z(G)/Z(G)$ is finite by Lemma 2.1. On the other hand, the group $G/Z(G)$ is torsion-free, and so G has nilpotency class at most 2. ■

LEMMA 3.5. – *Let G be a radical group containing an infinite periodic normal subgroup N. If the commutator subgroup G' of G is residually finite and every subgroup of G is either nearly normal or polycyclic, then G' is finite.*

PROOF. – Assume that G' is infinite, so that G is an $\mathfrak{S}_1$-group by Lemma 3.3. Since N is not polycyclic, it follows from Lemma 2.1 that $G'/G' \cap N$ is finite, so that G' is periodic, and hence it is a Černikov group. This contradiction proves the lemma. ■

LEMMA 3.6. – *Let G be a radical group whose commutator subgroup is not polycyclic. If every subgroup of G is either nearly normal or polycyclic, and X/Y is is an infinite periodic section of G with Y polycyclic, then X/Y is a finite extension of a Prüfer group.*

PROOF. – The group G is minimax by Lemma 3.3, and so X/Y is a Černikov group. Assume that the finite residual J/Y of X/Y is not a Prüfer group, so that $J/Y = J_1/Y \times J_2/Y$, where both factors are infinite. Then the indices $|G' : G' \cap J_1|$ and $|G' : G' \cap J_2|$ are finite by Lemma 2.1 so that $G' \cap Y$ has finite index in G', and G' is polycyclic. This contradiction shows that J/Y is a Prüfer group. ■

Let A be a torsion-free abelian group. A finitely generated subgroup X of A is said to be an IG-separator for A if $X/X \cap B$ is finite for every non-polycyclic subgroup B of A. The torsion-free abelian group A is called IG -separated if all its cyclic subgroups are IG-separators. Obviously every torsion-free abelian group of rank 1 is IG-separated. It is also clear that all torsion-free abelian groups containing non-trivial IG-separators must have finite rank, so that in particular IG-separated groups have finite rank. The structure of torsion-free abelian groups containing non-trivial IG-separators has been studied in [4], where it was proved in particular that, if a torsion-free abelian group A contains a non-trivial IG-separator, then there exists a finitely generated subgroup E of A such that A/E is torsion-free of rank 1.

Recall that a subgroup H of a group G is said to be *almost normal* if it has finitely many conjugates in G, i.e. if the normalizer $N_G(H)$ has finite index in G.

We are now in a position to prove the main result of this section.

THEOREM 3.7. – *Let G be a radical group. Then every subgroup of G either is nearly normal or polycyclic if and only if G satisfies one of the following conditions*:

(i) *G is polycyclic.*

(ii) *The commutator subgroup G' of G is finite.*

(iii) *G is a minimax group whose finite residual J is a Prüfer group, G/J is finite-by-abelian and all abelian subgroups of G are Min-by-Max.*

(iv) *G is nilpotent of class* 2, *$G/Z(G)$ is finitely generated and G' is an IG-separator for $Z(G)$.*

(v) *G is nilpotent of class* 2, *G' is not polycyclic and contains an IG-separator X such that G/X is of type* (iii).

(vi) *G is not nilpotent and has a series $V < K < G$, where V is finitely generated, K is torsion-free abelian, K/V is a finite extension of a Prüfer group, G/K is a finitely generated central-by-finite group and for every non-polycyclic subgroup L of K the index $|K : L|$ is finite.*

(vii) *G contains a finite normal subgroup N such that G/N is of type* (iv), (v) *or* (vi).

PROOF. – Assume that every subgroup of G either is nearly normal or polycyclic, but G is a non-polycyclic group with infinite commutator subgroup. Suppose first that G' is not residually finite, so that in particular G is a soluble minimax group by Lemma 3.3, and its finite residual J is the direct product of finitely many Prüfer groups (see [10] Part 2, Theorem 10.33). Then J is a

Prüfer group by Lemma 3.6, and G/J is a finite-by-abelian group by Lemma 2.1. Let A be any abelian subgroup of G. Then $A = A_1 \times A_2$, where A_1 is a Černikov group and A_2 is torsion-free. As G' is periodic, we have $G' \cap A_2 = \{1\}$, and hence A_2 is polycyclic by Lemma 2.1. Therefore every abelian subgroup of G is Min-by-Max and G satisfies the condition (iii). Suppose now that G' is residually finite, so that the largest periodic normal subgroup T of G is finite by Lemma 3.5. Replacing G by the factor group G/T, it can be assumed that G does not contain periodic non-trivial normal subroups, and in this case we have to prove that G satisfies one of the conditions (iv), (v), (vi). If G' is polycyclic, then also the factor group $G/Z_2(G)$ is polycyclic (see [10] Part 1, p. 119), so that $Z_2(G)$ is not polycyclic and $G/Z_2(G)$ is finite-by-abelian by Lemma 2.1. It follows that G is finite-by-nilpotent, and hence it is a torsion-free nilpotent group. Moreover G has nilpotency class 2 by Lemma 3.4. Let H be any non-polycyclic subgroup of G. Then the group $G/G' \cap H$ is finite-by-abelian by Lemma 2.1. Since G has finite Prüfer rank by Theorem 3.1, it follows that $G/G' \cap H$ is central-by-finite, and so H is almost normal in G. Therefore every subgroup of G is either polycyclic or almost normal, so that $G/Z(G)$ is finitely generated and G' is an IG-separator for $Z(G)$ (see [4], Theorem 3.6). Suppose finally that G' is residually finite but not polycyclic, so that G is a soluble minimax group again by Lemma 3.3. If G is nilpotent, then it has class 2 by Lemma 3.4. Let X be a finitely generated subgroup of G' such that G'/X is periodic. Thus G'/X is not residually finite and so G/X has the structure described in (iii). Moreover, it follows from Lemma 2.1 that X is an IG-separator for G', so that G satisfies the condition (v). Assume that G is not nilpotent. As G' is not polycyclic, it contains an abelian subgroup that also is not polycyclic (see [10] Part 1, Theorem 3.31), and hence G' is abelian-by-finite by Lemma 2.1. Obviously the Fitting subgroup K of G' is torsion-free, so that it is abelian. Let V be a finitely generated subgroup of K such that K/V is periodic. Then K/V is a finite extension of a Prüfer group by Lemma 3.6. Moreover, if L is any non-polycyclic subgroup of K, the index $|G' : L|$ is finite by Lemma 2.1, so that in particular L has finite index in K. Since the minimax group G has no periodic non-trivial normal subgroups, its Fitting subgroup F is a torsion-free nilpotent group, and G/F is polycyclic. Assume that F' is not polycyclic. Then G/F' is finite-by-abelian by Lemma 2.1, so that G is finite-by-nilpotent (see [7], Lemma 2.1), and hence even nilpotent. This contradiction shows that F' is polycyclic, so that $Y = VF'$ is a polycyclic subgroup of K. The abelian group F/K contains a finitely generated subgroup M/K such that F/M is periodic. Put $M = KE$, where E is a finitely generated subgroup of F. Clearly $M/YE = KE/YE \simeq K/K \cap YE$ is an infinite periodic group, and so also F/YE is periodic. As YE is polycyclic, it follows from Lemma 3.6 that F/YE is a finite extension of a Prüfer group. Therefore M has finite index in F, and hence G/K is finitely generated. It follows that G/K is a finitely generated finite-by-abelian group, so that it is central-by-finite, and G satisfies the condition (vi) of the statement.

Conversely let H be any non-polycyclic subgroup of G and let A be an abelian non-polycyclic subgroup of H. If G satisfies condition (iii), we have that A contains a Prüfer subgroup, and hence $J \leqslant A \leqslant H$. As G/J is finite-by-abelian , it follows that H is nearly normal in G. If G satisfies condition (v) and J/X is the finite residual of G/X, the subgroup $G' \cap J$ is not polycyclic, so that $G' \cap J/X$ is infinite and J is contained in G'. Moreover AJ/J is polycyclic, so that $A \cap J$ is not polycyclic, and hence $X/A \cap X$ is finite. Then $J/A \cap X$ is an extension of a finite subgroup by a Prüfer group, and so the index $|J : A \cap J|$ is finite. Therefore $G/A \cap J$ is finite-by-abelian, and H is a nearly normal subgroup of G. Suppose now that G satisfies condition (iv). Then the subgroup $H \cap Z(G)$ is not polycyclic, and hence $G/G' \cap H$ is finite-by-abelian, and H is nearly normal. in G. If G satisfies condition (vi), the subgroup $H \cap K$ is not polycyclic and so has finite index in K. Thus also the index $|H^G \cap K : H \cap K|$ is finite. Since G/K is finite-by-abelian, we have that HK has finite index in $H^G K$, so that the index $|H^G : H|$ is finite and H is a nearly normal subgroup of G. Suppose finally that G satisfies condition (vii). Then it follows from the above arguments and from Lemma 3.2 that every subgroup of G is either nearly normal or polycyclic. The theorem is proved. ■

Observe finally that soluble groups in which all abelian subgroups are Min-by-Max have been considered in [9]. Moreover examples of groups satisfying (v) or (vi) of the above statement can be found in [4].

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