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Gradient Regularity for Minimizers of Functionals Under $p-q$ Subquadratic Growth (*).

F. LEONETTI - E. MASCOLO - F. SIEPE

Sunto. – *Si prova la maggior sommabilità del gradiente dei minimi locali di funzionali integrali della forma*

$$\int_\Omega f(Du)\, dx\,,$$

dove f soddisfa l'ipotesi di crescita

$$|\xi|^p - c_1 \leqslant f(\xi) \leqslant c(1 + |\xi|^q)\,,$$

con $1 < p < q \leqslant 2$. *L'integrando* f *è* C^2 *e* DDf *ha crescita* $p-2$ *dal basso e* $q-2$ *dall'alto.*

1. – Introduction.

Let us consider the functional

$$\mathcal{F}(u, \Omega) = \int_\Omega f(Du(x))\, dx \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^n$, $n \geqslant 2$, Du is the gradient of a vector valued function $u : \Omega \to \mathbb{R}^N$, $N \geqslant 1$, and $f : \mathbb{R}^{nN} \to \mathbb{R}$.

In this paper we study the local regularity of minimizers of $\mathcal{F}$. In particular, we consider the case in which the integrand function f satisfies the $p-q$ growth condition

$$|\xi|^p - c_1 \leqslant f(\xi) \leqslant c(1 + |\xi|^q) \tag{1.2}$$

with $p < q$.

The regularity properties of minimizers, under assumption (1.2), has been intensely studied in the last years.

In the scalar case, i.e. when $N = 1$, Marcellini in [M2] and [M3], proved the $W^{1,\infty}$ regularity, provided p and q are not too far apart.

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In the setting of $p-q$ growth, minimizers may be unbounded in general, if no restriction on p and q is assumed (see [G2], [M1], [H]).

In the vectorial case there are recent results in [ELM1] and [ELM2], about higher integrability for the gradient of minimizers, in the case of $2 \leqslant p < q$.

Moreover, Marcellini in [M4] gives the local Lipschitz continuity of the local minimizers, when $f(\xi) = g(|\xi|)$ and g satisfies some general conditions which imply, if (1.2) holds, that $2 \leqslant p < q$.

Our aim is to study the case when (1.2) holds with $1 < p < q \leqslant 2$.

We will prove a higher integrability result for the gradient Du of local minimizers u of $\mathcal{F}$. More precisely there exists $\chi = \chi(n, p) > 1$ such that

$$Du \in L^{p\chi}_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN}). \tag{1.3}$$

This result will be obtained under the restrictions

$$\frac{2n}{n+2} < p < q \leqslant 2, \tag{1.4}$$

$f \in C^2$, DDf has $p-2$ growth from below and $q-2$ growth from above. The idea of the proof is the following. We consider a family of perturbed functionals of (1.1), defining, for $\sigma \in (0, 1)$

$$\mathcal{F}_\sigma(w, B_R) = \int_{B_R} f(Dw)\, dx + \sigma \int_{B_R} [(1 + |Dw|^2)^{q/2} - 1]\, dx,$$

where B_R is a ball such that $B_{4R} \subset\subset \Omega$.

Now $\mathcal{F}_\sigma$ has the same q growth from above and below. For local minimizers $v \in W^{1,q}(B_R, \mathbb{R}^N)$ of $\mathcal{F}_\sigma$, the following estimate holds

$$\|Dv\|_{L^{p\chi}(B_{\alpha R})} \leqslant c \left[1 + \int_{B_R} f(Dv)\, dx \right]^{2/p^2} \tag{1.5}$$

for $\alpha \in (0, 1)$ and a constant c that does not depend on σ.

As in [ELM2], if u is a local minimizer of $\mathcal{F}$, we mollify u and we get u_ε. Then we consider the *Dirichlet problems*

$$\min \{ \mathcal{F}_\sigma(w, B_R) : w \in u_\varepsilon + W_0^{1,q}(B_R, \mathbb{R}^N) \}.$$

If $v_{\varepsilon,\sigma}$ is the solution of such a problem, we write (1.5) for $v = v_{\varepsilon,\sigma}$. We will prove that letting first $\sigma \to 0$ and then $\varepsilon \to 0$, $Dv_{\varepsilon,\sigma}$ converges weakly to Du and we can pass to the limit in (1.5), thus obtaining (1.3).

In the case of $f(\xi) = g(|\xi|)$ where $g : [0, +\infty) \to [0, +\infty)$ is convex, $g(0) = 0$, $g \geqslant 0$ and $g \in \Delta_2$, we apply a recent result about local boundedness of minimizers of $\mathcal{F}$, contained in [DM]. This result allows us to get (1.3) without the restriction

$$\frac{2n}{n+2} < p,$$

contained in (1.4). Related results can be found in [FS], [Ch], [BL], [Li], and [CF].

2. – Statements and notations.

As we have seen in section 1, we deal with the local regularity properties for minimizers of functionals of type (1.1). Moreover we assume that $f \in C^2(\mathbb{R}^{nN})$, $f \geq 0$ satisfies the following growth conditions

$$|\xi|^p - c_1 \leq f(\xi) \leq L(1 + |\xi|^2)^{\frac{q}{2}} \tag{2.1}$$

$$|Df(\xi)| \leq L(1 + |\xi|^2)^{\frac{q-1}{2}} \tag{2.2}$$

$$|D^2 f(\xi)| \leq L(1 + |\xi|^2)^{\frac{q-2}{2}} \tag{2.3}$$

$$\langle D^2 f(\xi)\,\lambda, \lambda\rangle \geq \nu(1 + |\xi|^2)^{\frac{p-2}{2}}\,|\lambda|^2 \tag{2.4}$$

for every $\xi, \lambda \in \mathbb{R}^{nN}$ and some $L > 1$, $\nu > 0$, $c_1 \geq 0$. p, q are such that $1 < p < q \leq 2$. Note that growth condition (2.2) for Df can be derived by growth condition (2.1) for f and convexity (2.4).

We say that $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ is a local minimizer of $\mathcal{F}$ if $f(Du) \in L^1_{\mathrm{loc}}(\Omega)$, and

$$\int_{\mathrm{supp}(\varphi)} f(Du)\,dx \leq \int_{\mathrm{supp}(\varphi)} f(Du + D\varphi)\,dx,$$

for every $\varphi \in W^{1,1}(\Omega, \mathbb{R}^N)$ such that $\mathrm{supp}(\varphi) \subset\subset \Omega$.

By these assumptions, we observe immediately that $u \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. We will prove the following higher integrability result for u

THEOREM 2.1. – *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ be a local minimizer of functional* (1.1), *satisfying conditions* (2.1), (2.2), (2.3) *and* (2.4). *Then, if $p > 2n/(n+2)$, we have*

$$Du \in L^{\chi p}_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$$

for some $\chi = \chi(n, p) > 1$.

Moreover, if $x_0 \in \Omega$ and $R > 0$ are such that $B(x_0, 4R) \subset\subset \Omega$, and $\alpha \in (0, 1)$, *there exists a positive constant* $c \equiv c(n, N, p, q, L, \nu, \alpha, R)$ *such that*

$$\int_{B_{\alpha^3 R}} |Du|^{p\chi}\, dx \leq c \left(1 + \int_{B_R} f(Du)\, dx \right)^{\frac{2\chi}{p}} . \tag{2.5}$$

This Theorem can be improved when we consider a particular structure for the functional, that is when we suppose that $f(\xi) = g(|\xi|)$, where $g \in C^2([0, +\infty))$ is a convex, increasing N-function of class Δ_2, that is, $g:[0, +\infty) \to [0, +\infty)$ is such that $g(t) = 0$ if and only if $t = 0$ and for every $t > 0$ and every $\lambda > 1$

$$g(\lambda t) \leq \lambda^m g(t)$$

for some $m > 1$ (to be more precise, if this property holds, we say that $g \in \Delta_2^m$). Moreover g satisfies the following limit conditions

$$\lim_{t \to 0^+} \frac{g(t)}{t} = 0 \qquad \lim_{t \to +\infty} \frac{g(t)}{t} = +\infty .$$

Under these assumptions we prove the following

THEOREM. – 2.2. – *Let us suppose that* $u \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^N)$ *is a local minimizer of functional* (1.1), *satisfying conditions* (2.1)-(2.4). *Let us assume also that* $f(\xi) = g(|\xi|)$, *with* g *as above. Then*

$$Du \in L^{\chi p}_{\text{loc}}(\Omega, \mathbb{R}^{nN})$$

for some $\chi = \chi(n, p) > 1$, *and the following estimate holds*

$$\int_{B_{\alpha^3 R}} |Du|^{p\chi}\, dx \leq c \left(1 + \int_{B_R} g(|Du|)\, dx + \int_{B_R} g(|u|)\, dx \right)^{2\chi(3-p)} \tag{2.6}$$

for some positive $c = c(n, N, p, q, L, \nu, \alpha, R, m)$ *and every* $\alpha \in (0, 1)$.

Let us recall some known and technical results that will be useful later

LEMMA 2.1. – *For every* $\zeta, \xi \in \mathbb{R}^k$ *and* $\delta \in (-\frac{1}{2}, 0)$

$$1 \leq \frac{\int_0^1 (1 + |\zeta + t(\xi - \zeta)|^2)^\delta\, dt}{(1 + |\zeta|^2 + |\xi|^2)^\delta} \leq c(\delta) \tag{2.7}$$

$$0 < c_1(\delta) \leq \frac{|(1 + |\zeta|^2)^\delta \zeta - (1 + |\xi|^2)^\delta \xi|}{(1 + |\zeta|^2 + |\xi|^2)^\delta\, |\zeta - \xi|} \leq c_2(\delta, k) . \tag{2.8}$$

PROOF. – See [AF], [Gi, page 274].

Fix $h > 0$ and for $s = 1, \ldots, n$ a direction e_s in $\mathbb{R}^n$. For every vector valued function $G : \mathbb{R}^n \to \mathbb{R}^k$ we define

$$\tau_{s,h} G(x) = G(x + he_s) - G(x).$$

We state some properties of this difference function in connection with Sobolev spaces.

LEMMA 2.2. – *Let* $0 < \varrho < R$, $|h| < R - \varrho$, $p \geq 1$, *and* $G \in W^{1,p}(B_R, \mathbb{R}^k)$. *Then for every* $s = 1, \ldots, n$

$$\int_{B_\varrho} |\tau_{s,h} G(x)|^p \, dx \leq |h|^p \int_{B_R} |D_s G(x)|^p \, dx .$$

PROOF. – See [G1].

LEMMA 2.3. – *Let* $0 < \varrho < R$ *and* $G \in L^2(B_R, \mathbb{R}^k)$. *If for some* $a \in (0, 2)$, $M > 0$ *and* $\eta \in C_0^1(B_{\frac{R+\varrho}{2}})$ *such that* $0 \leq \eta \leq 1$ *and* $|D\eta| \leq 4/(R - \varrho)$ *in* $\mathbb{R}^n$, $\eta = 1$ *on* B_ϱ,

$$\sum_{s=1}^{n} \int_{B_R} \eta^2 |\tau_{s,h} G(x)|^2 \, dx \leq M^2 |h|^a$$

for every h *with* $|h| < R - \varrho$, *then* $G \in W^{b,2}(B_\varrho, \mathbb{R}^k) \cap L^{\frac{2n}{n-2b}}(B_\varrho, \mathbb{R}^k)$, *for every* $b \in (0, (a/2))$. *Moreover*

$$\|G\|_{L^{\frac{2n}{n-2b}}(B_\varrho)} \leq c(M + \|G\|_{L^2(B_R)}),$$

with $c \equiv c(n, k, b, a, R, \varrho)$.

PROOF. – See [A].

3. – Preliminary results.

In this section we consider a perturbation of the integrand of functional (1.1), given by

$$f_\sigma(\xi) = f(\xi) + \sigma[(1 + |\xi|^2)^{\frac{q}{2}} - 1]$$

where $\sigma \in (0, 1)$ under (2.1), ..., (2.4). The following Lemma contains some properties of this function f_σ. The proof is rather easy

LEMMA 3.1. – *$f_\sigma \in C^2(\mathbb{R}^{nN})$, $f_\sigma \geq 0$ and satisfies the following conditions*

$$\sigma\mu|\xi|^q + |\xi|^p - c_1 - \sigma \leq f_\sigma(\xi) \leq (L+1)(1+|\xi|^2)^{\frac{q}{2}} \tag{3.1}$$

$$|Df_\sigma(\xi)| \leq (L+q)(1+|\xi|^2)^{\frac{q-1}{2}} \tag{3.2}$$

$$|D^2 f_\sigma(\xi)| \leq (L+nNq^2)(1+|\xi|^2)^{\frac{q-2}{2}} \tag{3.3}$$

$$\langle D^2 f_\sigma(\xi)\,\lambda, \lambda\rangle \geq [\sigma q(q-1)(1+|\xi|^2)^{\frac{q-2}{2}} + \nu(1+|\xi|^2)^{\frac{p-2}{2}}]\,|\lambda|^2 \tag{3.4}$$

for every $\xi, \lambda \in \mathbb{R}^{nN}$, where L, ν and c_1 are those of (2.1), …, (2.4) *and $\mu = ((2^{q/2}-1)/2^{q/2}) \in (0,1)$.*

Moreover, if f satisfies the assumptions of Theorem 2.2, *then*

$$f_\sigma(\xi) = g_\sigma(|\xi|),$$

where

$$g_\sigma(t) = g(t) + \sigma[(1+t^2)^{q/2} - 1],$$

$g_\sigma : [0, +\infty) \to [0, +\infty)$ is convex, increasing, $g_\sigma(t) = 0$ if and only if $t = 0$, g_σ satisfies the Δ_2^s-condition, where $s = 2 \vee m = \max\{2, m\}$. We have also $g_\sigma \in C^2([0, +\infty))$ and

$$\lim_{t\to 0^+} \frac{g_\sigma(t)}{t} = 0\,, \qquad \lim_{t\to +\infty} \frac{g_\sigma(t)}{t} = +\infty\,.$$

Now we introduce for every $\sigma \in (0,1)$ and $R > 0$ such that $B_{4R} \subset\subset \Omega$, the functional

$$\mathcal{F}_\sigma(w) = \int_{B_R} f_\sigma(Dw)\,dx\,.$$

A local minimizer of functional $\mathcal{F}_\sigma$, will be a function $v \in W^{1,q}(B_R, \mathbb{R}^N)$ such that $\mathcal{F}_\sigma(v) \leq \mathcal{F}_\sigma(v+\varphi)$, for every $\varphi \in W_0^{1,q}(B_R, \mathbb{R}^N)$. Let us prove the following result.

LEMMA 3.2. – *Let $v \in W^{1,q}(B_R, \mathbb{R}^N)$ be a local minimizer of functional $\mathcal{F}_\sigma$ with $(2n/(n+2)) < p < q \leq 2$. Then for every $\alpha \in (0,1)$, and for every b such that*

$$0 < b < \frac{n}{2} - \frac{n}{p} + 1\,,$$

we have that

$$Dv \in L^{\frac{np}{n-2b}}(B_{\alpha^3 R}, \mathbb{R}^{nN}). \tag{3.5}$$

Moreover there exists a constant $c \equiv c(n, N, p, q, L, \nu, c_1, R, \alpha, b)$ *such that*

$$\|Dv\|_{L^{\frac{np}{n-2b}}(B_{\alpha^3 R})} \leq c\left[1 + \int_{B_R} f(Dv)\, dx\right]^{\frac{2}{p^2}}. \tag{3.6}$$

PROOF. – Since v is a local minimizer for $\mathcal{F}_\sigma$, under growth conditions (3.1)-(3.4) we have that the Euler's equation

$$\int_{B_R} Df_\sigma(Dv)\, D\varphi\, dx = 0 \tag{3.7}$$

holds for every $\varphi \in W^{1,q}(B_R, \mathbb{R}^N)$ such that $\operatorname{supp}(\varphi) \subset\subset B_R$.

Let $\alpha \in (0, 1)$ and $\eta \in C_0^\infty(\mathbb{R}^n)$ be a cut-off function. More precisely we assume that $\operatorname{supp}(\eta) \subset B_{\frac{\alpha^3 R + \alpha^2 R}{2}}$, $\eta \equiv 1$ in $B_{\alpha^3 R}$, $0 \leq \eta \leq 1$, $|D\eta| \leq 4/(\alpha^2(1-\alpha)R)$.

Now let $|h| < R\alpha^2(1-\alpha)$ and for $s = 1, \ldots, n$ put $\varphi = \tau_{s,-h}(\eta^2 \tau_{s,h} v)$ as test function in (3.7). We get

$$\begin{aligned}(I) &= \int_{B_R} \eta^2 \tau_{s,h}(Df_\sigma(Dv))\, \tau_{s,h} Dv\, dx \\ &= -\int_{B_R} \tau_{s,h}(Df_\sigma(Dv))\, 2\eta\, D\eta \otimes \tau_{s,h} v\, dx = (II).\end{aligned} \tag{3.8}$$

Moreover, since

$$\tau_{s,h}(Df_\sigma(Dv)) = \int_0^1 D^2 f_\sigma(Dv + t\tau_{s,h}(Dv))\, dt \tau_{s,h} Dv\,,$$

we have that

$$(I) = \int_{B_R} \int_0^1 D^2 f_\sigma(Dv + t\tau_{s,h}(Dv))\, \eta\tau_{s,h} Dv \eta\tau_{s,h} Dv\, dt\, dx\,, \tag{3.9}$$

$$(II) = -\int_{B_R} \int_0^1 2D^2 f_\sigma(Dv + t\tau_{s,h}(Dv))\, \eta\tau_{s,h} Dv\, D\eta \otimes \tau_{s,h} v\, dt\, dx\,.$$

By the properties of f_σ we are in conditions to apply Cauchy-Schwartz inequality:

$$(3.10)\qquad (II) \leqslant \frac{1}{2}\int_{B_R}\int_0^1 D^2 f_\sigma(Dv + t\tau_{s,h}(Dv))\,\eta\tau_{s,h}Dv\eta\tau_{s,h}Dv\,dt\,dx$$

$$+2\int_{B_R}\int_0^1 D^2 f_\sigma(Dv + t\tau_{s,h}(Dv))\,D\eta\otimes\tau_{s,h}vD\eta\otimes\tau_{s,h}v\,dt\,dx$$

$$= \frac{1}{2}(I) + 2(III).$$

Since the integrals (I) and (III) are finite, by (3.8) and (3.10) we get

$$(I) \leqslant 4(III).$$

Moreover, by (3.4) and Lemma 2.1

$$(3.11)\qquad (I) \geqslant c\int_{B_R}\eta^2\,|\tau_{s,h}((1+|Dv|^2)^{\frac{p-2}{4}}Dv)|^2\,dx$$

for some positive constant $c \equiv c(\nu, p, n, N)$. Now by growth conditions (3.3) and the properties of η we have

$$(III) \leqslant c\int_{B_{\alpha^2 R}}\int_0^1 |D\eta|^2(1+|Dv + t\tau_{s,h}Dv|^2)^{\frac{q-2}{2}}\,|\tau_{s,h}v|^2\,dt\,dx\,,$$

where $c \equiv c(n, N, L, q)$. Since we suppose that $1 < p < q \leqslant 2$ we can drop $(1 + |Dv + t\tau_{s,h}Dv|^2)^{\frac{q-2}{2}}$ since it is less than 1. Then we have

$$(3.12)\qquad (III) \leqslant c(n, N, L, q, \alpha, R)\int_{B_{\alpha^2 R}}|\tau_{s,h}v|^2\,dx = (IV).$$

Let $a \in (0, p)$. Then

$$(3.13)\qquad (IV) = c\int_{B_{\alpha^2 R}}|\tau_{s,h}v|^a\,|\tau_{s,h}v|^{2-a}\,dx$$

$$\leqslant c\left(\int_{B_{\alpha^2 R}}|\tau_{s,h}v|^p\,dx\right)^{\frac{a}{p}}\left(\int_{B_{\alpha^2 R}}|\tau_{s,h}v|^{\frac{2-a}{p-a}p}\,dx\right)^{\frac{p-a}{p}}.$$

Since $v \in W^{1,p}(B_R, \mathbb{R}^N)$, $B_{\alpha^2 R} \subset B_{\alpha R} \subset B_R$ and $|h| < \alpha^2 R - \alpha^3 R < \alpha R - \alpha^2 R$, by Lemma 2.2 we have

$$(IV) \leqslant c|h|^{a}\left(\int_{B_{aR}} |Dv|^{p}\,dx\right)^{\frac{a}{p}}\left(\int_{B_{a^2R}} |\tau_{s,h}v|^{\frac{2-a}{p-a}p}\,dx\right)^{\frac{p-a}{p}}. \tag{3.14}$$

Now we use the assumption $p > 2n/(n+2)$: let us choose a in such a way that

$$\frac{2-a}{p-a}p = p^* = \frac{np}{n-p} \quad \text{that is} \quad a = n+2-2\frac{n}{p}.$$

We remark that a satisfies the required properties since we suppose that

$$p > \frac{2n}{n+2}.$$

With these assumptions and applying Sobolev inequality in (3.14) we obtain

$$(IV) \leqslant c|h|^{a}\left(\int_{B_{aR}} |Dv|^{p}\,dx\right)^{\frac{a}{p}}\left(\int_{B_{aR}} |Dv|^{p}\,dx\right)^{\frac{2-a}{p}} \tag{3.15}$$

and finally, by (2.1) and (3.11)

$$\int_{B_R} \eta^2\,|\tau_{s,h}((1+|Dv|^2)^{\frac{p-2}{4}}Dv)|^2\,dx \leqslant 4\,\tilde{c}\,|h|^{a}\left(1+\int_{B_R} f(Dv)\,dx\right)^{\frac{2}{p}} \tag{3.16}$$

for some positive constant $\tilde{c} \equiv \tilde{c}(n, N, p, q, L, \nu, c_1, a, R)$. By this estimate and Lemma 2.3 it follows that

$$(1+|Dv|^2)^{\frac{p-2}{4}}Dv \in W^{b,2}(B_{a^3R}, \mathbb{R}^{nN}) \cap L^{\frac{2n}{n-2b}}(B_{a^3R}, \mathbb{R}^{nN}),$$

for every $b \in (0, (a/2))$. In particular, if we set

$$M = 2\sqrt{\tilde{c}n}\left(1+\int_{B_R} f(Dv)\,dx\right)^{\frac{1}{p}} \tag{3.17}$$

we have

$$\sum_{s=1}^{n}\int_{B_R} \eta^2\,|\tau_{s,h}((1+|Dv|^2)^{\frac{p-2}{4}}Dv)|^2\,dx \leqslant M^2\,|h|^{a} \tag{3.18}$$

from which it follows that

$$\|(1+|Dv|^2)^{\frac{p-2}{4}}Dv\|_{L^{\frac{2n}{n-2b}}(B_{\alpha^3 R})} \leq \hat{c}(M+\|(1+|Dv|^2)^{\frac{p-2}{4}}Dv\|_{L^2(B_R)})$$

for some $\hat{c}\equiv\hat{c}(n, N, b, p, R, \alpha)$.

It is easy to show that for every $z\in\mathbb{R}^k$, $\vartheta>0$ and $p\in(1, 2)$ we have

$$|z|^{p\vartheta}\leq 1+2^{\frac{(2-p)\vartheta}{2}}[(1+|z|^2)^{\frac{p-2}{2}}|z|^2]^{\vartheta}.$$

By this fact, since $(n/(n-2b))>1$ and $p<2$, it follows that

$$\int_{B_{\alpha^3 R}}|Dv|^{\frac{np}{n-2b}}dx\leq c(n,p,b,R,\alpha)\left(1+\int_{B_{\alpha^3 R}}((1+|Dv|^2)^{\frac{p-2}{2}}|Dv|^2)^{\frac{n}{n-2b}}dx\right)$$

$$\leq c(n,N,p,b,R,\alpha)\left[1+(M+\|(1+|Dv|^2)^{\frac{p-2}{4}}Dv\|_{L^2(B_R)})^{\frac{2n}{n-2b}}\right]$$

$$\leq c\left[1+\left(1+\int_{B_R}f(Dv)\,dx\right)^{\frac{1}{p}}+\left(1+\int_{B_R}f(Dv)\,dx\right)^{\frac{1}{2}}\right]^{\frac{2n}{n-2b}}$$

$$\leq c(n, N, p, q, L, \nu, c_1, R, \alpha, b)\left(1+\int_{B_R}f(Dv)\,dx\right)^{\frac{2n}{p(n-2b)}}$$

that is just estimate (3.6). Then the proof is concluded. ■

4. – Proof of Theorem 2.1.

Our next goal is to prove that Lemma 3.2 holds also for the minimizer u of our original functional (1.1). We use an approximation argument.

Let $0<\varepsilon<\min\{1, R\}$ and consider a sequence of smooth functions u_ε, obtained by u by mean of standard mollifiers. We have that $u_\varepsilon\in W^{1,q}(B_R, \mathbb{R}^N)$ and $u_\varepsilon\to u$ in $W^{1,p}$.

By the growth conditions about $\mathcal{F}_\sigma$, we are able to define the solution $v_{\varepsilon,\sigma}\in u_\varepsilon+W_0^{1,q}(B_R, \mathbb{R}^N)$ of the Dirichlet problem

$$\min\left\{\int_{B_R}f_\sigma(Dw)\,dx:\ w\in u_\varepsilon+W_0^{1,q}(B_R, \mathbb{R}^N)\right\} \tag{4.1}$$

according to direct methods of the calculus of variations.

Let us fix $\alpha\in(0, 1)$. We are going to apply estimate (3.6) for $v_{\varepsilon,\sigma}$. There exists a constant $c\equiv c(n, N, p, q, R, \alpha, \nu, c_1, L, b)$ not depending neither on

ε nor σ, such that

$$(4.2)\qquad \left(\int_{B_{a^3R}} |Dv_{\varepsilon,\sigma}|^{\frac{np}{n-2b}}\,dx\right)^{\frac{p(n-2b)}{2n}} \leqslant c\left(1+\int_{B_R} f(Dv_{\varepsilon,\sigma})\,dx\right)$$

$$\leqslant c\left(1+\int_{B_R} f_\sigma(Du_\varepsilon)\,dx\right)$$

$$\leqslant c\left(1+\int_{B_{R+\varepsilon}} f(Du)\,dx+\sigma\int_{B_R}(1+|Du_\varepsilon|^2)^{\frac{q}{2}}\,dx\right)$$

by the minimality of $v_{\varepsilon,\sigma}$ and Jensen inequality.

Moreover we have also

$$(4.3)\qquad \int_{B_R} |Dv_{\varepsilon,\sigma}|^p\,dx \leqslant \int_{B_R} f(Dv_{\varepsilon,\sigma})\,dx + c_1\,|B_R|$$

and

$$\int_{B_R} f(Dv_{\varepsilon,\sigma})\,dx \leqslant \int_{B_R} f_\sigma(Dv_{\varepsilon,\sigma})\,dx \leqslant \int_{B_R} f_\sigma(Du_\varepsilon)\,dx$$

$$(4.4)\qquad \leqslant \int_{B_R} f(Du_\varepsilon)\,dx + \sigma\int_{B_R}(1+|Du_\varepsilon|^2)^{\frac{q}{2}}\,dx$$

$$\leqslant \int_{B_{R+\varepsilon}} f(Du)\,dx + \sigma\int_{B_R}(1+|Du_\varepsilon|^2)^{\frac{q}{2}}\,dx\,.$$

Since $\sigma<1$, by (4.3) and (4.4) we deduce that $Dv_{\varepsilon,\sigma}$ is uniformly bounded in $L^p(B_R,\mathbb{R}^{nN})$ with respect to σ. Then up to a subsequence

$$Dv_{\varepsilon,\sigma} \rightharpoonup Dw_\varepsilon \quad \text{weakly in } L^p(B_R) \quad \text{as } \sigma\to 0\,,$$

for some $w_\varepsilon \in u_\varepsilon + W_0^{1,p}(B_R,\mathbb{R}^N)$. By lower semicontinuity we can let $\sigma\to 0$ in (4.2) and (4.4) obtaining

$$(4.5)\qquad \left(\int_{B_{a^3R}} |Dw_\varepsilon|^{\frac{np}{n-2b}}\,dx\right)^{\frac{p(n-2b)}{2n}} \leqslant c\left(1+\int_{B_{R+\varepsilon}} f(Du)\,dx\right),$$

and

$$(4.6)\qquad \int_{B_R} f(Dw_\varepsilon)\,dx \leqslant \int_{B_{R+\varepsilon}} f(Du)\,dx$$

so that

$$\int_{B_R} |Dw_\varepsilon|^p \, dx \leq \int_{B_{R+\varepsilon}} f(Du) \, dx + c_1 |B_R| \, . \tag{4.7}$$

Now, since $w_\varepsilon \in u_\varepsilon + W_0^{1,p}(B_R, \mathbb{R}^N)$ and Du_ε converges to Du strongly in L^p, by (4.7) we deduce that up to a subsequence

$$Dw_\varepsilon \rightharpoonup Dw \quad \text{weakly in } L^p(B_R) \quad \text{as } \varepsilon \to 0 \, ,$$

for some $w \in u + W_0^{1,p}(B_R, \mathbb{R}^N)$. Finally, letting $\varepsilon \to 0$ in (4.5) and (4.6), by semicontinuity we have

$$\int_{B_{a^3 R}} |Dw|^{\frac{np}{n-2b}} \, dx \leq c \left(1 + \int_{B_R} f(Du) \, dx \right)^{\frac{2n}{p(n-2b)}}$$

and

$$\int_{B_R} f(Dw) \, dx \leq \lim \inf_{\varepsilon \to 0} \int_{B_R} f(Dw_\varepsilon) \, dx \leq \int_{B_R} f(Du) \, dx \, . \tag{4.8}$$

Inequality (4.8) and the strict convexity of f implies that $Dw = Du$ a.e. in B_R. Moreover, since $w = u$ on ∂B_R, Poincaré inequality gives $u = w$. This concludes the proof of Theorem 2.1. ∎

5. – Proof of Theorem 2.2.

Before we prove Theorem 2.2, we give a precise statement of the boundedness result contained in [DM].

THEOREM 5.1. – *Let $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be a minimizer of the functional*

$$\mathcal{F}(u) = \int_\Omega g(|Du|) \, dx \, ,$$

where g is a N-function, $g \in \Delta_2^m$. Then u is locally bounded in Ω and the following estimate holds

$$\sup_{B_{\alpha R}} |u| \leq c(m, \alpha, R) \left(1 + \int_\Omega g(|u|) \, dx \right) \tag{5.1}$$

for every $R > 0$ such that $B_R \subset \Omega$ and every $\alpha \in (0, 1)$.

It is remarkable that since $g \in \Delta_2$, from $g(|Du|) \in L^1_{\text{loc}}(\Omega)$ it follows that also $g(|u|) \in L^1_{\text{loc}}(\Omega)$.

Let us go on with the proof of Theorem 2.2. We proceed as in the proof of

Lemma 3.2. So, let v be a minimizer of

$$\mathcal{F}_\sigma(w) = \int_{B_R} g(|Dw|)\,dx + \sigma\int_{B_R}[(1+|Dw|^2)^{\frac{q}{2}} - 1]\,dx\,.$$

By (3.12) and Lemma 2.2 we have

$$\begin{aligned}(III) &\leqslant c(n, N, L, q, \alpha, R)\int_{B_{\alpha^2 R}} |\tau_{s,h} v|^2\,dx \\ &\leqslant c(n, N, L, p, q, \alpha, R)\left(\sup_{B_{\alpha R}}|v|\right)^{2-p}\int_{B_{\alpha^2 R}}|\tau_{s,h} v|^p\,dx \\ &\leqslant c(n, N, L, p, q, \alpha, R)\left(\sup_{B_{\alpha R}}|v|\right)^{2-p}|h|^p\int_{B_{\alpha R}}|Dv|^p\,dx\,.\end{aligned}$$

This estimate is similar to (3.15) of the previous proof. Then by Lemma 2.3 we have, as in conclusion of Lemma 3.2

$$\left(\int_{B_{\alpha^3 R}}|Dv|^{\frac{np}{n-2b}}dx\right)^{\frac{n-2b}{2n}} \leqslant c\left(1+\left(\sup_{B_{\alpha R}}|v|\right)^{\frac{2-p}{2}}\right)\left(1+\int_{B_R} g(|Dv|)\,dx\right)^{\frac{1}{2}} \tag{5.2}$$

for every $b \in (0, (p/2))$.

Let now u be a local minimizer of $\mathcal{F}$. We mollify u as in section 4, in order to have $u_\varepsilon \in W^{1,q}(B_R, \mathbb{R}^N)$ and $u_\varepsilon \to u$ in $W^{1,p}$. Moreover we consider the Dirichlet problem

$$\min\left\{\int_{B_R} g_\sigma(|Dw|)\,dx:\ w \in u_\varepsilon + W_0^{1,q}(B_R)\right\}. \tag{5.3}$$

Let $v_{\varepsilon,\sigma} \in u_\varepsilon + W_0^{1,q}(B_R, \mathbb{R}^N)$ be the solution of (5.3). Then (5.2) implies

$$\left(\int_{B_{\alpha^3 R}}|Dv_{\varepsilon,\sigma}|^{\frac{np}{n-2b}}\,dx\right)^{\frac{n-2b}{2n}} \leqslant \tag{5.4}$$

$$c\left(1+\left(\sup_{B_{\alpha R}}|v_{\varepsilon,\sigma}|\right)^{\frac{2-p}{2}}\right)\left(1+\int_{B_R} g(|Dv_{\varepsilon,\sigma}|)\,dx\right)^{\frac{1}{2}}.$$

Now we use Theorem 5.1 obtaining

$$\sup_{B_{\alpha R}}|v_{\varepsilon,\sigma}| \leqslant \tilde{c}\left(1+\int_{B_R} g_\sigma(|v_{\varepsilon,\sigma}|)\,dx\right) \tag{5.5}$$

where $\tilde{c}$ is a positive constant, independent of ε and σ. We use Δ_2 condition and convexity of g_σ:

$$\int_{B_R} g_\sigma(|v_{\varepsilon,\sigma}|)\,dx \leqslant c\left(\int_{B_R} g_\sigma\left(\frac{|v_{\varepsilon,\sigma}-(v_{\varepsilon,\sigma})_R|}{2R}\right)dx + \int_{B_R} g_\sigma(|(v_{\varepsilon,\sigma})_R|)\,dx\right),$$

where $(v_{\varepsilon,\sigma})_R = |B_R|^{-1}\int_{B_R} v_{\varepsilon,\sigma}\,dx$.

Then we apply Poincaré inequality (see [BL]):

$$\int_{B_R} g_\sigma\left(\frac{|v_{\varepsilon,\sigma}-(v_{\varepsilon,\sigma})_R|}{2R}\right)dx \leqslant c\int_{B_R} g_\sigma(|Dv_{\varepsilon,\sigma}|)\,dx\,.$$

Moreover

$$\begin{aligned}|(v_{\varepsilon,\sigma})_R| &\leqslant \frac{1}{|B_R|}\left(\int_{B_R}|v_{\varepsilon,\sigma}-u_\varepsilon|\,dx + \int_{B_R}|u_\varepsilon|\,dx\right)\\ &\leqslant \frac{c}{|B_R|}\left(\int_{B_R}|Dv_{\varepsilon,\sigma}|\,dx + \int_{B_R}|Du_\varepsilon|\,dx + \int_{B_R}|u_\varepsilon|\,dx\right),\end{aligned}$$

thus, using Jensen inequality and integrating over B_R,

$$\int_{B_R} g_\sigma(|(v_{\varepsilon,\sigma})_{B_R}|)\,dx \leqslant c\left(\int_{B_R} g_\sigma(|Dv_{\varepsilon,\sigma}|)\,dx + \int_{B_R} g_\sigma(|Du_\varepsilon|)\,dx + \int_{B_R} g_\sigma(|u_\varepsilon|)\,dx\right).$$

Eventually we put together the previous inequalities and we use the minimality of $v_{\varepsilon,\sigma}$ with respect to u_ε:

$$\begin{aligned}(5.6)\quad \int_{B_R} g_\sigma(|v_{\varepsilon,\sigma}|)\,dx &\leqslant c\left(\int_{B_R} g_\sigma(|Dv_{\varepsilon,\sigma}|)\,dx + \int_{B_R} g_\sigma(|Du_\varepsilon|)\,dx + \int_{B_R} g_\sigma(|u_\varepsilon|)\,dx\right)\\ &\leqslant c\left(2\int_{B_R} g_\sigma(|Du_\varepsilon|)\,dx + \int_{B_R} g_\sigma(|u_\varepsilon|)\,dx\right)\\ &\leqslant c\left(\int_{B_R} g(|Du_\varepsilon|)\,dx + \sigma\int_{B_R}(1+|Du_\varepsilon|^2)^{\frac{q}{2}}\,dx\right.\\ &\quad\left.+\int_{B_R} g(|u_\varepsilon|)\,dx + \sigma\int_{B_R}(1+|u_\varepsilon|^2)^{\frac{q}{2}}\,dx\right).\end{aligned}$$

(5.4), (5.5), (5.6) and Jensen merge into

$$(5.7)\quad \left(\int_{B_{\alpha^3 R}} |Dv_{\varepsilon,\sigma}|^{\frac{np}{n-2b}}\,dx\right)^{\frac{n-2b}{2n}} \leqslant$$

$$c\left(1+\int_{B_{R+\varepsilon}} g(|Du|)\,dx + \int_{B_{R+\varepsilon}} g(|u|)\,dx + \sigma\int_{B_R}(1+|Du_\varepsilon|^2)^{\frac{q}{2}}\,dx + \sigma\int_{B_R}(1+|u_\varepsilon|^2)^{\frac{q}{2}}\,dx\right)^{3-p}.$$

Moreover, as in (4.3) and (4.4),

$$\int_{B_R} |Dv_{\varepsilon,\sigma}|^p\,dx \leq \int_{B_R} g(|Dv_{\varepsilon,\sigma}|)\,dx + c_1\,|B_R| \tag{5.8}$$

$$\int_{B_R} g(|Dv_{\varepsilon,\sigma}|)\,dx \leq \int_{B_{R+\varepsilon}} g(|Du|)\,dx + \sigma \int_{B_R} (1+|Du_\varepsilon|^2)^{\frac{q}{2}}\,dx\,. \tag{5.9}$$

Since $\sigma < 1$, these estimates are uniform with respect to σ. Thus there exists $w_\varepsilon \in u_\varepsilon + W_0^{1,p}(B_R, \mathbb{R}^N)$ such that, up to a subsequence,

$$Dv_{\varepsilon,\sigma} \rightharpoonup Dw_\varepsilon \quad \text{weakly in } L^p(B_R),\ \text{as } \sigma \to 0\,,$$

then, by semicontinuity and (5.7), (5.8), (5.9) we get

$$\left(\int_{B_{a^3 R}} |Dw_\varepsilon|^{\frac{np}{n-2b}}\,dx \right)^{\frac{n-2b}{2n}} \leq c \left(1 + \int_{B_{R+\varepsilon}} g(|Du|)\,dx + \int_{B_{R+\varepsilon}} g(|u|)\,dx \right)^{3-p}$$

and

$$\int_{B_R} |Dw_\varepsilon|^p\,dx \leq \int_{B_{R+\varepsilon}} g(|Du|)\,dx + c_1\,|B_R|$$

Therefore, since $Du_\varepsilon \to Du$ strongly in L^p, there exists $w \in u + W_0^{1,p}(B_R, \mathbb{R}^N)$ such that

$$Dw_\varepsilon \rightharpoonup Dw \quad \text{as } \varepsilon \to 0\,,$$

weakly in $L^p(B_R)$. Again we use semicontinuity:

$$\left(\int_{B_{a^3 R}} |Dw|^{\frac{np}{n-2b}}\,dx \right)^{\frac{n-2b}{2n}} \leq c \left(1 + \int_{B_R} g(|Du|)\,dx + \int_{B_R} g(|u|)\,dx \right)^{3-p},$$

and

$$\int_{B_R} g(|Dw|)\,dx \leq \liminf_{\varepsilon \to 0} \int_{B_R} g(|Dw_\varepsilon|)\,dx \leq \int_{B_R} g(|Du|)\,dx\,.$$

As in Theorem 2.1 we conclude that $u = w$. ■

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