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Diamonds in Thin Lie Algebras.

M. AVITABILE - G. JURMAN (*)

Sunto. – *In un'algebra di Lie graduata* thin, *la classe in cui compare il secondo diamante e la caratteristica del campo soggiacente determinano se l'algebra stessa abbia o meno dimensione finita ed in tal caso forniscono anche un limite superiore a tale dimensione.*

Introduction.

During the last few years there has been a growing interest in studying some *narrowness* conditions on p-groups and (graded) Lie algebras. The best known condition is having finite coclass, but other ones can be considered. A rather general condition is finiteness of width, that is, the existence of a constant that bounds the orders, or dimensions, of the lower central factors.

Although these conditions were initially born in a group-theoretical environment, they have been generalized to the class of graded Lie algebras over fields of arbitrary characteristic: in this case, order of the lower central factors reads as dimension of the homogeneous components. A strong narrowness condition is *thinness*.

A graded Lie algebra $L = \bigoplus_{i=1}^{\infty} L_i$ is *thin* if L_1 has dimension 2 and generates L as an algebra and if L satisfies the *covering property*: for each integer $i > 1$ and for each non-trivial element $z \in L_i$, one has $L_{i+1} = [L_1, z]$.

The covering property, combined with the dimension of L_1, implies that every homogeneous component L_i of L has dimension at most 2. A component of dimension two is called a *diamond*. When all the components L_i except the top one are one-dimensional, L is a graded Lie algebra of maximal class. These are studied in [CMN], [CN], [J2].

When this is not the case, there is at least one more two-dimensional homogeneous component of L. Let L_k be the next one (we say that the second diamond occurs in weight k): define the ideal $L^k = \bigoplus_{i \geqslant k} L_i$. As a first observation, it is proved in [CMNS] that k is odd when the quotient L/L^k is metabelian.

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In this paper we will study to what extent the number k restricts the dimension of the entire algebra.

When $k=3$ there are examples of infinite-dimensional thin Lie algebras over every field $\mathbb{F}$. Apart from the characteristic two case they arise as loop algebras of a simple algebra of type A_1. These are described in [CMNS] for $\mathrm{char}(\mathbb{F}) \geqslant 5$ and in [C1] for $\mathrm{char}(\mathbb{F}) = 3$. The same construction extends to the characteristic zero case. When the characteristic is two the situation is different, but still examples can be found: some of them are described in the paper [GMY].

Something similar happens for $k=5$ when the characteristic is not two. Again the example built in [CMNS] starting from a simple algebra of type A_2 where $\mathrm{char}(\mathbb{F})$ is at least 7 can be extended to the characteristic zero case. The papers [CM] and [C] deal with the cases $\mathrm{char}(\mathbb{F}) = 3$ and 5 respectively.

No more infinite-dimensional Lie algebras are known in characteristic zero and we will prove below that there are no more, while in the modular case further examples can be found. Denote by q a power of the characteristic p of the underlying field $\mathbb{F}$. When $k = 2q-1$ we have the (-1)-algebras built in [CM] starting from a graded Lie algebra of maximal class with parameter q; note that this construction method works even in characteristic two, provided the parameter of the starting algebra is bigger than two. For $k=q$, in the odd characteristic case, there are algebras related to the Nottingham group (see [C] and [C2]) and, finally, in characteristic two when $k = q-1$ and $q > 8$, we find the exceptional family of algebras described in [J].

We will show that for all other values of k the thin Lie algebra L is finite-dimensional.

Theorem 1, whose complete proof appears in the papers [CJ] and [J], justifies this claim when the quotient L/L^k is not metabelian.

We prove an analogous theorem (Theorem 2) which takes care of the case when the quotient L/L^k is metabelian; we give explicit bounds for the dimension of the algebras in terms of k and the characteristic of the underlying field. The examples above show that in this case we can assume $k \geqslant 5$ and even $k \geqslant 7$ for odd characteristic.

Note that the paper [CMNS] dealt with the same problem for thin Lie algebras associated with thin pro-p-groups. In that case, the second diamond always occurs in class less than the characteristic of the field and it was proved that, if $5 < k < p$ then the associated Lie algebra has class at most $k+2$. In particular, the structure of pro-p-groups associated with the thin Lie algebras built in this paper has been investigated in [M]. A discussion of pro-p-groups whose associated Lie algebra has second diamond in class 3 or 5 can be found in [KL-GP].

We conclude, in Section 3, with a final remark on deriving the characteristic of the field from the value of k.

Although none of the results relies on machine computations, many numerical examples have been worked out by using the p-Quotient Program developed at the Australian National University (ANU pQ, see [HNO]) in order to get the information required to construct the theory presented here. This software shows the structure of some finite-dimensional quotients of the algebras involved and suggests Jacobi expansions needed to prove the statements.

1. – The theorems.

For notation and background material, we refer mainly to the papers [CMN] and [CMNS]: in particular, all iterated commutators are left-normed and exponential notation is used as shorthand

$$[yx^n] = [y\underbrace{x\dots x}_{n}].$$

The generalized Jacobi identity

$$[u[yx^n]] = \sum_{i=0}^{n}(-1)^i\binom{n}{i}[ux^i yx^{n-i}]$$

is often used without specific mention.

We will also use Lucas' Theorem (see [L], [KW]) several times. Let a and b be two non-negative integers, and p a prime. Write a and b in p-adic form,

$$a = a_0 + a_1 p + \dots + a_{n-1}p^{n-1} + a_n p^n$$

$$b = b_0 + b_1 p + \dots + b_{n-1}p^{n-1} + b_n p^n,$$

so that $0 \leqslant a_i,\ b_i \leqslant p-1$, for all i, then

$$\binom{a}{b} \equiv \prod_{i=0}^{n}\binom{a_i}{b_i} \pmod{p}.$$

Let $L = \bigoplus_{i=1}^{\infty} L_i$ be a thin graded Lie algebra over a field $\mathbb{F}$ with second diamond in class k. Choose a minimal generating set for L_1 by taking a non-trivial element y in the two-step centralizer $C_{L_1}(L_2)$ and $x \notin \mathbb{F}y$, so that $L_1 = \langle x, y\rangle$. Then L/L^k is a graded Lie algebra of maximal class, so we can apply the theory developed in [CMN]. Suppose that the quotient L/L^k is not metabelian. Then the characteristic of $\mathbb{F}$ is a positive number p and the set

$$A = \{\alpha \in \mathbb{N} : 2 < \alpha < k-1,\ C_{L_1}(L_\alpha) \neq C_{L_1}(L_2)\}$$

is not empty. Let m and M be respectively the minimum and the maximum of

A, and call $b = m - 2$ and $t = k - 1 - M$: in view of the theory of graded Lie algebras of maximal class, b must be of the form $2r - 2$ for some power $r = p^h$ of the characteristic. We refer to r as the *parameter* of the algebra, while t must be of the form $2r - p^s - 1$ where s ranges in the set $\{-\infty, 0, 1, \dots, h\}$.

Then, the claim reads as follows.

THEOREM 1 ([CJ], [J]). – *Let L be a graded thin Lie algebra with second diamond in class k and with L/L^k not metabelian.*

Then L has positive characteristic p. Let $r = p^h$ be its parameter and let t be defined as above.

Then the following holds:

- $t < 2r - 1$;
- *when $t = 2r - 2$ and p is odd,*
  L has class at most $k + r - 1$;
- *when $t = 2r - 2$ and p is 2, let $n = (k+1)/8r$;*
  - *when n is not integer,*
    L has class at most $k + 4r - 2$;
  - *when n is integer but not a power of 2,*
    L has class at most $k + 4nr - 1$;
- *when $t = 2r - p^s - 1$ with $s \in \{1, \dots, h\}$,*
  L has class at most $k + r - 1$, and even
  L has class at most k for $p^s \neq 2$.

The proof of the above result is in the paper [CJ] for the odd characteristic case and in [J] for the case of characteristic two.

So from now on we can take L/L^k to be metabelian. This includes the characteristic zero case.

THEOREM 2. – *Let L be a graded thin Lie algebra with second diamond in (odd) class k and L/L^k metabelian. Let $k \geq 7$ in odd characteristic and $k \geq 5$ if the characteristic is 0 or 2.*

Then the following holds:

- *for characteristic 0,*
  L has class at most $k + 2$;
- *for positive characteristic p,*
  - *when $k \equiv -1$ modulo p (for odd p) or modulo 4 (when $p = 2$), write $k = 2nq - 1$ where q is a p-power and $(n, p) = 1$ then*
    - *if $n = 1$*
      there exist infinite-dimensional Lie algebras L (see [CM]);

* *if* $n > 1$
    - *for odd characteristic,* *L has class at most* $k + q - 1$ ([CM]);
    - *for characteristic* 2, *L has class at most* $2^{\lfloor \log_2(k) \rfloor + 1} - 3$;

- *when* $k \equiv 0$ (mod $p > 2$), *write* $k = nq$ *where* q *is a* p*-power and* $(n, p) = 1$*; then*
    * *if* $n = 1$ *there exist infinite-dimensional Lie algebras L* (*see* [C]);
    * *if* $n > 1$ *L has class at most* $k + q - 1$;
- *when* $k \not\equiv -1, 0$, *L has class at most* $k + 2$.

If $k \equiv 1$ *modulo* p *the bound can be improved. In this case L has class at most* k.

2. – The proof.

The following result was originally proved in [CMNS] for $\mathbb{F} = \mathbb{F}_p$, the field with p elements, with $p > 5$, but nothing changes in the proof if $\mathbb{F}$ is any field of characteristic $p > 5$.

LEMMA 1. – *Under the hypotheses of Theorem* 2, *if the characteristic of the field* $\mathbb{F}$ *is greater than* k (*including zero*), *then* L *has class at most* $k + 2$.

A first immediate consequence is that we can always suppose, in the modular case, that $k \geqslant \max\{7, p\}$. The same proof works in characteristic zero; this proves completely the claim for the characteristic zero case.

From now on suppose $\mathrm{char}(\mathbb{F}) = p > 0$.

Denote by v an element of weight $k - 1$. The additional hypothesis L/L^k metabelian means that

$$\mathbb{F}y = C_{L_1}(L_2) = C_{L_1}(L_3) = \ldots = C_{L_1}(L_{k-2}),$$

which implies that we can choose $v = [yx^{k-2}]$. Now define $\lambda = (k - 1)/2$ and

$v^{-t} = [yx^{k-2-t}]$ so that

$$[v^{-t}x^{t}] = v\,.$$

Since $L_k = \langle [vx], [vy] \rangle$, in class $k+1$ we have a spanning set with four elements

$$[vxx], [vxy], [vyx], [vyy]\,.$$

Two standard calculations show relations among them, namely

$$0 = [v^{-1}[yxy]] = -[vyy]\,,$$

and

$$0 = [[yx^{\lambda}][yx^{\lambda}]] = \sum_{i=0}^{\lambda}(-1)^{i}\binom{\lambda}{i}[yx^{\lambda+i}yx^{\lambda-i}]$$

$$= (-1)^{\lambda-1}\lambda[vyx] + (-1)^{\lambda}[vxy]\,,$$

that, together with the covering property, imply

$$[vxy] = \lambda[vyx] \qquad [vxx] = \mu[vyx]\,, \tag{1}$$

for some $\mu \in \mathbb{F}$.

Suppose $k \equiv -1$ modulo p if p is odd, or modulo 4 if $p = 2$, *i.e.* $k = 2qn - 1$ for some power q of $p = \operatorname{char}(\mathbb{F})$ and some n coprime with p: it is equivalent to saying that $\lambda \equiv -1 \pmod p$.

The following lemma deals with the odd characteristic case:

LEMMA 2 [Proposition 1 of [CM]]. – *With the above hypotheses, if $n > 1$, then L has class at most $k + q - 1$.*

So assume $p = 2$. Let η be the integer $\lfloor \log_2(k) \rfloor$, so that we can write k as $2^{\eta} + 4\xi + 3$, where $0 \leqslant 4\xi + 3 < 2^{\eta}$. Now, in class $k+1$, the relations (1) are the following:

$$[vyy] = 0\,,$$

$$[vxy] = [vyx]\,,$$

$$[vxx] = \mu[vyx]\,.$$

First of all, we show by induction that the elements of weight $k + s$ are centralized by y when $1 \leqslant s \leqslant 2^{\eta} - 2$. For $s = 1, 2$ the result follows from standard arguments, expanding $[v[yxy]]$ and $[v^{-3}[yx^{5}y]]$ respectively. Then suppose $s > 2$ and expand the following Jacobi identity for elements in class $k + 1 + s$

for $3 \leqslant s \leqslant 2^{\eta} - 2$:

$$\begin{aligned}
0 &= [v^{-(2^{\eta}-2-s)}[yx^{2^{\eta}-2}y]] \\
&= [v^{-(2^{\eta}-2-s)}[yx^{2^{\eta}-2}]\, y] \\
&= \sum_{i=0}^{2^{\eta}-2} \binom{2^{\eta}-2}{i} [v^{-(2^{\eta}-2-s)} x^{i} y x^{2^{\eta}-2-i} y] \\
&= \binom{2^{\eta}-2}{2^{\eta}-2-s} [vyx^{s}y] \\
&\quad + \binom{2^{\eta}-2}{2^{\eta}-1-s} [vxyx^{s-1}y] \\
&= \left(\binom{2^{\eta}-2}{2^{\eta}-2-s} + \binom{2^{\eta}-2}{2^{\eta}-1-s} \right) \cdot [vyx^{s}y] \\
&= \binom{2^{\eta}-1}{2^{\eta}-1-s} [vyx^{s}y] \\
&= [vyx^{s}y].
\end{aligned}$$

Now add the hypothesis that $k+1$ is not a power of two: this implies $\xi < 2^{\eta-2} - 1$. Let $\alpha = 2^{\eta} - 2$: then $k + 2 < 2\alpha + 2$, in view of the above restriction on the range of ξ, so it makes sense to expand the following Jacobi identity in class $2\alpha + 2$:

$$\begin{aligned}
0 &= [[yx^{\alpha}][yx^{\alpha}]] \\
&= \binom{\alpha}{k-2-\alpha} \cdot [vyx^{2\alpha+2-k}] \\
&\quad + \binom{\alpha}{k-1-\alpha} \cdot [vxyx^{2\alpha+1-k}] \\
&= \left(\binom{\alpha}{k-2-\alpha} + \binom{\alpha}{k-1-\alpha} \right) \cdot [vyx^{2\alpha+2-k}] \\
&= \binom{\alpha+1}{k-1-\alpha} [vyx^{2\alpha+2-k}] \\
&= \binom{2^{\eta}-1}{4(\xi+1)} [vyx^{2\alpha+1-k}x].
\end{aligned}$$

But $2\alpha + 1 - k < 2^{\eta} - 2$ and thus the element $[vyx^{2\alpha+1-k}]$ is central: then the class of L is at most $2^{\lfloor \log_2(k) \rfloor + 1} - 3$ as claimed.

Now we can assume $\lambda \not\equiv -1$: thus in (1) we can redefine x as

$$x - \frac{\mu}{\lambda + 1} y ,$$

reducing to the case

$$[vxx] = 0 .$$

Again, the covering property implies $L_{k+1} = \langle [vyx] \rangle$ and in class $k+1$ we have the following situation

$$[vxx] = [vyy] = 0 , \tag{2a}$$

$$[vxy] = \lambda [vyx] . \tag{2b}$$

If $k \equiv 1$ modulo p for p odd and modulo 4 for $p = 2$, then $\lambda \equiv 0 \pmod p$, and by the covering property $L_{k+1} = \{0\}$.

This exhausts the characteristic two case, so assume $p > 2$.

The two expansions in class $k+2$

$$0 = [v[yxy]] = 2[vyxy] - [vxyy] ,$$

and

$$0 = [[yx^{k-4}][yxxxy]] = 3[vyxy] - [vxyy] ,$$

imply $[vxyy] = [vyxy] = 0$ and $L_{k+2} = \langle [vyxx] \rangle$. Note that the second identity above holds since we are assuming $k \geqslant 7$. We refer to [CMNS] for the case $k = 5$. Finally, move to class $k+3$:

$$\begin{aligned} 0 &= [vx[yxy]] \\ &= 2[vxyxy] - [vxxyy] - [vxyyx] \\ &= 2\lambda [vyxxy] , \end{aligned}$$

so that $L_{k+3} = \langle [vyxxx] \rangle$.

If $k \equiv 0$ modulo p, then we write $k = nq$ with $(n, p) = 1$. As observed in the Introduction, n must be odd and there exist infinite-dimensional examples when $n = 1$, so write n as $2h + 1$, with $h \geqslant 1$.

We show that $L_{k+l} = \langle [vyx^l] \rangle$ for $1 \leqslant l \leqslant q$: to do this, we prove by induction that

$$[vyx^l y] = 0 ,$$

where $0 \leqslant l \leqslant q - 1$. The cases $l \leqslant 2$ come from previous calculations, so suppose the equation is satisfied for $l=b-1$ and prove it for $l=b$. The expansion

$$\begin{aligned}0&=[v[yx^b y]]\\&=[vyx^b y]-b[vxyx^{b-1}y]-(-1)^b[vyx^b y]\\&=(1-b\lambda-(-1)^b)[vyx^b y],\end{aligned}$$

holds for any $b<q$, since $p>2$ and $n>1$; the coefficient

$$1-b\lambda-(-1)^b\equiv 1+\frac{b}{2}+(-1)^{b+1}\qquad(\mathrm{mod}\ p)$$

vanishes either when b is an even multiple of p, or when b is odd and congruent to -4 modulo p.

Take a power p^t of the characteristic and an integer φ in the range $2<p^t+\varphi<k-b$ and expand the following Jacoby identity:

$$\begin{aligned}0&=[yx^{k-\varphi-p^t}[yx^{b+p^t+\varphi-2}y]]\\&=(-1)^{p^t+\varphi-2}\binom{b+p^t+\varphi-2}{p^t+\varphi-2}[vyx^b y]\\&\quad+(-1)^{p^t+\varphi-1}\binom{b+p^t+\varphi-2}{p^t+\varphi-1}[vxyx^{b-1}y]\\&=(-1)^{p^t+\varphi-2}\left(\binom{b+p^t+\varphi-2}{p^t+\varphi-2}-\lambda\binom{b+p^t+\varphi-2}{p^t+\varphi-1}\right)[vyx^b y]\\&=(-1)^{p^t+\varphi-2}\eta[vyx^b y].\end{aligned}\tag{3}$$

In the former case, write $b=\beta p^g$ where $\beta\not\equiv 0\ (\mathrm{mod}\ p)$ and let $t=g$ and $\varphi=0$ in equation (3); then $p^t+\varphi$ satisfies the required bounds and the coefficient η is

$$\eta=\binom{\beta p^t+p^t-2}{p^t-2}-\lambda\binom{\beta p^t+p^t-2}{p^t-1}\equiv 1\,.$$

The latter case requires a distinction.

When $p>3$, write $b+4$ as βp^g where $\beta\not\equiv 0\ (\mathrm{mod}\ p)$, and take $t=g$ and $\varphi=2$ in equation (3). The number $p^t+\varphi$ still satisfies the constraints and η becomes

$$\eta=\binom{\beta p^t+p^t-4}{p^t}-\lambda\binom{\beta p^t+p^t-4}{p^t+1}=\beta(1+4\lambda)=-\beta\not\equiv 0\quad(\mathrm{mod}\ p).$$

When $p=3$ the values of b to consider are those of the kind $3^g(2z+1)+2$

for some non-negative integer z. Moreover, when $p=3$ the coefficient λ is 1. Choose $t=g$ and $\varphi=-1$ and evaluate η:

$$\eta=\binom{3^t(2z+1)+3^t-1}{3^t-3}-\binom{3^t(2z+1)+3^t-1}{3^t-2}\equiv 1-2=-1\,.$$

Now $3^t+\varphi$ satisfies the required constraints unless $g=1$. In this case choose $\varphi=1$ and $t=g$, so that the coefficient η results

$$\eta=\binom{3(2z+2)+1}{2}-\binom{3(2z+2)+1}{3}\equiv-(2z+2)\,.$$

This method fails when $2z+2\equiv 0 \pmod 3$. When this occurs, write $2z+2=3^a\gamma$ with $0<a$ and $\gamma\not\equiv 0 \pmod 3$, therefore $b=3^{a+1}\gamma-1$. Taking $\varphi=3^{a+1}-2$ and $t=1$ in equation (3), we obtain

$$\eta=\binom{3^{a+1}\gamma+3^{a+1}-2}{3^{a+1}-1}-\binom{3^{a+1}\gamma+3^{a+1}-2}{3^{a+1}}\equiv-\gamma\,.$$

Thus, in every case, $[vyx^l y]$ is zero for every $0\leqslant l\leqslant q-1$. Now let

$$\gamma=\frac{k+q}{2}-1\qquad \alpha=\frac{k-q}{2}-1$$

and expand the following identity in class $k+q$:

$$\begin{aligned}
0&=[[yx^\gamma][yx^\gamma]]\\
&=(-1)^\alpha\binom{\gamma}{\alpha}[vyx^q]+(-1)^{\alpha+1}\binom{\gamma}{\alpha+1}[vxyx^{q-1}]\\
&=(-1)^{h-1}\left(\binom{\gamma}{\alpha}-\lambda\binom{\gamma}{\alpha+1}\right)[vyx^q]\\
&=(-1)^{h-1}\left(\binom{hq+q-1}{(h-1)\,q+q-1}-\lambda\binom{hq+q-1}{hq}\right)[vyx^q]\\
&=(-1)^{h-1}(h-\lambda)[vyx^q]\,.
\end{aligned}$$

The coefficient $h-\lambda\equiv h+1/2$ is not zero since p is coprime to $n=2h+1$, therefore the component of weight $k+q$ of L vanishes. The above case was the last one for characteristic three.

So assume $p>3$ for the final cases.

If $k \equiv a$ modulo p, where $a \in \{2, \ldots, p-2\}$ and $p > 3$, then the expansion

$$\begin{aligned} 0 &= [[yx][yx^k]] \\ &= (-1)^{k-3}\binom{k}{k-3}[vyxxx] + (-1)^{k-2}\binom{k}{k-2}[vxyxx], \end{aligned}$$

gives the relation

$$[vxyxx] = \frac{k-2}{3}[vyxxx],$$

which, together with the one obtained by commuting (2*b*) twice with x

$$[vxyxx] = \lambda[vyxxx],$$

yields

$$0 = \left(\lambda - \frac{k-2}{3}\right)[vyxxx] = (k+1)[vyxxx].$$

Since $k+1 \not\equiv 0$, we obtain

$$[vyxxx] = 0$$

and thus $L_{k+3} = \{0\}$.

3. – A final comment.

Suppose we are given a modular graded thin Lie algebra of infinite dimension over a field of unknown odd characteristic, whose second diamond has weight $k > 5$ and whose structure is known up to elements in weight $k+1$. Is it possible to find the characteristic of the field?

The above result yields that k is of the form q or $2q-1$ for some prime power q. The only ambiguous case occurs when both k and $(k+1)/2$ are prime powers. In this situation, the answer can be given by looking at the elements just after the second diamond: if $[vxy] = -[vyx]$, then the characteristic is the only prime factor of the prime power $(k+1)/2$ and the algebra is a (-1)-algebra, otherwise p is the only prime factor of the number k and the algebra is of Nottingham type.

In fact, when $k = 2q-1$, then $\lambda = q-1$ and it cannot happen that $\lambda + 1$ is both a power of a prime p and a multiple of a different prime r.

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