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Iterative Differentiations in Rings of Unequal Characteristic.

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Sunto. – *Sia R un anello di caratteristica diseguale. Si stabiliscono formule generali per gli endomorfismi di una differenziazione F_a o F_c-iterativa di R, con c non zero-divisore di R. Tali formule sono note nel caso della caratteristica eguale.*

1. – Introduction.

It is known that if k is a ring and R a k-algebra a differentiation $\underline{D}$ of R is nothing else that a family of endomorphisms $D_i: R \to R$, $i \in N$, which satisfy the following conditions:

i) D_i is a k-linear map, for every i;

ii) $D_n(ab) = \sum_{i+j=n} D_i(a)\, D_j(b)$, for every n.

If $\underline{D}$ is a F_a (F_m, F_c) - iterative differentiation in the sense of [7] and char $k = 0$, it is possible to express every D_i in terms of D_1 which is a derivation ([7], [1], [2]).

In characteristic p>0 are freely present the endomorphisms D_{p^r}, $r \in N$, i.e. D_1, D_p, $\ldots$, and it is possible to express D_i in terms of a finite number of such endomorphisms. This number depends on the p-adic expression of i, for every i ([8]).

No result is known in unequal characteristic.

In this paper, we state these unknown formulas and we consequently complete the topic: characteristic zero, characteristic $p > 0$, unequal characteristic.

In section 2, there are some general remarks on the integrability of derivations in unequal characteristic.

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2. – Statements.

All rings are assumed to be commutative with a unit element. A local ring is assumed to be noetherian.

Let R be a ring. The set of all derivations of R into itself is denoted by $\mathrm{Der}(R)$. If k is a subring of R, the set of all derivations of R which vanish on k is denoted by $\mathrm{Der}_k(R)$.

We can easly prove that $\mathrm{Der}(R)$ (resp. $\mathrm{Der}_k(R)$) is the Lie algebra of all derivations $d : R \to R$ (resp. of all k-derivations $d : R \to R$) with $[d, d'] = dd' - d'd$.

DEFINITION 1. – *A differentiation $\underline{D}$ of R is a sequence $\underline{D} = \{D_0 = 1, D_1, \ldots, D_n, \ldots\}$ of additive endomorphisms $D_i\colon R \to R$ such that*

$$D_n(ab) = \sum_{i+j=n} D_i(a)\, D_j(b), \quad \textit{for every } n .$$

DEFINITION 2. – *Let k be a subring of R. A differentiation $\underline{D}$ of R is called a differentiation of R over k if $D_i(a) = 0$ for all $i > 0$ and for all $a \in k$.*

Let $\underline{D}$ be a differentiation of R, the subring $\{a \in R\colon D_i(a) = 0, \text{ for every } i\}$ in R is called the ring of invariants of R and is denoted by $R^{\underline{D}}$.

DEFINITION 3. – *A differentiation $\underline{D} = \{D_0 = 1, D_1, \ldots, D_n, \ldots\}$ of R is said to be F_a-iterative (or simply iterative) if*

$$D_i \circ D_j = \binom{i+j}{i} D_{i+j}, \quad \textit{for all } i, j .$$

Let c be not a zero-divisor in R.

DEFINITION 4. – *A differentiation $\underline{D} = \{D_0 = 1, D_1, \ldots, D_n, \ldots\}$ of R, $c \in R^{\underline{D}}$ is said to be F_c-iterative if*

$$D_i \circ D_j = \sum_r \binom{r}{i} \binom{i}{i+j-r} c^{i+j-r} D_r, \quad \textit{for all } i, j .$$

REMARK 1. – A differentiation $\underline{D}$ of a K-algebra R is said to be F_a-iterative (resp. F_c-iterative) because it is linked with the action of the additive formal group $F_a = X + Y$, over K (resp. $F_c = X + Y + cXY$) over the K-algebra R ([4], [5], [1], [2]). Note that if $c = 1$, F_c is the wellknown multiplicative formal group F_m.

Now we study the F_a-iterativity and the F_c-iterativity of a differentiation of a ring of unequal characteristic.

Let R be a ring and p a prime number. Assume i) p is neither zero or unit in R, and ii) all prime numbers other than p are units in R.

(The most important example is the case where R is a local ring of characteristic zero with residue field of characteristic $p>0$). Such a ring is called a ring of unequal characteristic.

Let R be a ring of unequal characteristic and p a prime number. Put $\overline{R}=R/pR$. Since every derivation (resp. every differentiation) D of R is trivial on the prime subring, it induces a derivation (resp. a differentiation) $\overline{D}$ of $\overline{R}$.

THEOREM 1. – *Let R be a ring of characteristic zero satisfying the conditions* i), ii) *stated above. Assume that p is not a zero-divisor in R.*

A differentiation $\underline{D}=\{D_0=1, D_1, \ldots, D_n, \ldots\}$ is F_a-iterative iff, putting $\delta_0=D_1$, $\delta_1=D_p$, $\ldots$, $\delta_i=D_{p^i}$, $\ldots$, we have

a) $\delta_i\delta_j=\delta_j\delta_i$, *for all i,j*

b) $\delta_i^p=(kp)\,\delta_{i+1}$, *for all i*
where $k=k_1\cdots k_s$ with k_j integers such that $(k_j, p)=1$ and

c) *for every $n>0$*

$$D_n=\delta_0^{n_0}\delta_1^{n_1}\ldots\delta_r^{n_r}/MN$$

where $n=n_0+n_1p+\ldots+n_rp^r(0\leqslant n_i<p)$ is the p-adic expansion of n and

$$M=\prod_{j=0}^{r}M_j$$

where $M_j=1$ when $n_j=0, 1$ and

$$M_j=\prod_{j=0}^{r}\left(\prod_{i=0}^{n_j-2}\binom{(n_j-i)\,p^j}{p^j}\right),\qquad when\ n_j\geqslant 2\,;$$

$$N=\prod_{t=0}^{r-1}\binom{n_rp^r+\sum_{j=0}^{r-1-t}n_{r-1-j}p^{r-1-j}}{n_tp^t}.$$

PROOF Let $\underline{D}=\{D_0=1, D_1, \ldots, D_n, \ldots\}$ be a F_a-iterative differentiation of R and let $\delta_i=D_{p^i}$.

a): It follows from the definition of F_a-iterativity.

b): By definition of F_a-iterativity we have that

$$\delta_i^p=\prod_{j=0}^{p-2}\binom{(p-j)\,p^i}{p^i}\,\delta_{i+1}.$$

Since

$$\prod_{j=0}^{p-2} \binom{(p-j)\,p^{i}}{p^{i}} = (p^{i+1})!/((p^{i})!)^{p}$$

by direct calculation it follows that

$$\delta_i^p = (kp)\,\delta_{i+1}, \qquad \text{for all } i$$

where $k = k_1 \cdots k_s$ with k_j integers such that $(k_j, p) = 1$.

In particular

$$\delta_i^p / k = p\delta_{i+1}.$$

c): Let $n = n_0 + n_1 p + \ldots + n_r p^r (0 \leqslant n_i < p)$ be the p-adic expansion of n. We have that:

$$\delta_j^{n_j} = \prod_{i=0}^{n_j-2} \binom{(n_j - i)\,p^{j}}{p^{j}} D_{n_j p^j},$$

when $n_j \geqslant 2$.

Put

$$M_j = \prod_{i=0}^{n_j-2} \binom{(n_j - i)\,p^{j}}{p^{j}},$$

it follows that

$$\delta_0^{n_0}\,\delta_1^{n_1} \ldots \delta_r^{n_r} = \left(\prod_{j=0}^{r} M_j\right) D_{n_0 p^0} D_{n_1 p^1} \ldots D_{n_r p^r}.$$

On the other hand, by direct calculations, we can prove that

$$D_{n_0 p^0} D_{n_1 p^1} \ldots D_{n_r p^r} = \prod_{t=0}^{r-1} \binom{n_r p^r + \sum_{j=0}^{r-1-t} n_{r-1-j}\, p^{r-1-j}}{n_t p^t} D_n.$$

Finally, put $M = \prod_{j=0}^{r} M_j$ and $N = \prod_{t=0}^{r-1} \binom{n_r p^r + \sum_{j=0}^{r-1-t} n_{r-1-j}\, p^{r-1-j}}{n_t p^t}$ we get the stated result.

Observe that for $n_j = 0, 1$, $M_j = 1$ and for $r = 0$, $N = 1$.

Remark 2. – Under the same hypotheses of Theorem 1, let $\overline{R}$ = R/pR and let us consider the induced differentiation $\underline{\overline{D}}$ of $\overline{R}$. It follows that *a*), *b*) become the followings in $\overline{R}$:

$a)'$ $\overline{\delta}_i \overline{\delta}_j = \overline{\delta}_j \overline{\delta}_i$, for all i, j

$b)'$ $\overline{\delta}_i^p = 0$, for all i.

Let us consider c). In $\overline{R}$ it can be written as

$$\overline{D}_n = \overline{\delta}_0^{n_0} \overline{\delta}_1^{n_1} \ldots \overline{\delta}_r^{n_r} / MN \, .$$

We observe that since $\operatorname{char}(\overline{R}) = p$, $N = 1$ and

$$M = \prod_{j=0}^{r} M_j = \prod_{j=0}^{r} \left(\prod_{i=0}^{n_j - 2} \binom{(n_j - i)\, p^j}{p^j} \right) =$$

$$\prod_{j=0}^{r} \left(\prod_{i=0}^{n_j - 2} (n_j - i) \right) = \prod_{j=0}^{r} n_j! = n_0!\, n_1! \ldots n_r! \, .$$

Finally we obtain the necessary and sufficient conditions since a differentiation of a ring of characteristic p is F_a-iterative ([8]).

Now we want to state necessary and sufficient conditions under which a differentiation of a ring R of unequal characteristic is F_c-iterative.

Let $\underline{D} = \{D_0 = 1, D_1, \ldots, D_n, \ldots\}$ be a F_c-iterative differentiation of a ring of characteristic zero satisfying the conditions i), ii) stated above and assume that p is not 0-divisor in R.

Let $n = n_0 + n_1 p + \ldots + n_r p^r$ $(0 \leqslant n_i < p)$ be the p-adic expansion of n. First of all by direct calculations we can prove that putting $\delta_0 = D_1$, $\delta_1 = D_p$, $\ldots$, $\delta_i = D_{p^i}$, $\ldots$, we have

$$\delta_j^{n_j} = \sum_{s_1^{(j)} = p^j}^{2p^j} \left[\binom{s_1^{(j)}}{p^j} \binom{p^j}{2p^j - s_1^{(j)}} c^{2p^j - s_1^{(j)}} T^{(j)} D_{s_{n_j - 2}^{(j)}} \right],$$

where $T^{(j)} = 1$, if $n_j = 2$ and

$$T^{(j)} = \prod_{i=1}^{n_j - 2} T_i$$

with

$$T_i = \sum_{s_i^{(j)} = s_{i-1}^{(j)}}^{s_{i-1}^{(j)} + p^j} \binom{s_i^{(j)}}{p^j} \binom{p^j}{p^j + s_{i-1}^{(j)} - s_i^{(j)}} c^{p^j + s_{i-1}^{(j)} - s_i^{(j)}} \quad \text{if } n_j > 2 \, .$$

It follows that

$$\delta_j^{n_j} = \sum_{i=1}^{n_j-1} A_i^{(j)} + B^{(j)}$$

where

$$A_i^{(j)} = \prod_{t=1}^{i} \binom{tp^j}{p^j} \sum_{s_i^{(j)}=ip^j}^{ip^j+p^j-1} \left[\binom{s_i^{(j)}}{p^j} \binom{p^j}{ip^j+p^j-s_i^{(j)}} c^{ip^j+p^j-s_i^{(j)}} \prod_{u=i+1}^{n_r-1} T_u D^{(j)}_{s_{n_r-1}} \right]$$

and

$$B^{(j)} = \prod_{i=0}^{n_j-2} \binom{(n_j-i)\,p^j}{p^j} D_{n_j p^j}.$$

Hence, putting $A^{(j)} = \sum_{i=0}^{n_j-1} A_i^{(j)}$,

$$\delta_0^{n_0}\delta_1^{n_1}\dots\delta_r^{n_r} = (A^{(0)}+B^{(0)})\circ(A^{(1)}+B^{(1)})\circ\dots\circ(A^{(r)}+B^{(r)}).$$

Denote by $P_1, \dots, P_v$ $(v = 4(r-1))$ the terms that we obtain from the previous identity applying the distributive property and let $P_v = \prod_{j=0}^{r} B^{(j)}$. We have

$$P_v = \prod_{j=0}^{r} \left(\prod_{i=0}^{n_j-2} \binom{(n_j-i)\,p^j}{p^j} \right) D_{n_0p^0} D_{n_1p^1}\dots D_{n_rp^r}.$$

Put

$$N = \prod_{t=0}^{r-1} \binom{n_r p^r + \sum_{j=0}^{r-1-t} n_{r-1-j} p^{r-1-j}}{n_t p^t}.$$

Let us consider $D_{n_0p^0}D_{n_1p^1}\dots D_{n_rp^r}$. We write

$$D_{n_0p^0}D_{n_1p^1}\dots D_{n_rp^r} = \sum_{i=0}^{r-1} Q_i + ND_{n_0p^0+n_1p^1+\dots+n_rp^r} = \sum_{i=0}^{r-1} Q_i + ND_n,$$

where

$$Q_0 = \sum_{d_0=n_rp^r}^{n_rp^r+n_{r-1}p^{r-1}-1} \left[\binom{d_1}{n_{r-1}p^{r-1}} \binom{n_{r-1}p^{r-1}}{n_rp^r+n_{r-1}p^{r-1}-d_1} c^{n_rp^r+n_{r-1}p^{r-1}-d_1} \prod_{j=2}^{r} Z_j D_{d_r} \right]$$

with

$$Z_j = \sum_{d_j = d_{j-1}}^{d_{j-1}+n_{r-j}p^{r-j}} \binom{d_j}{n_{r-j}p^{r-j}} \binom{n_{r-j}p^{r-j}}{n_{r-j}p^{r-j}+d_{j-1}-d_j} c^{n_{r-j}p^{r-j}+d_{j-1}-d_j},$$

and for $i > 0$, put $w = \sum_{t=0}^{i-1} n_{r-t}p^{r-t}$ and $q = \sum_{t=0}^{i} n_{r-t}p^{r-t} - 1$

$$Q_i = \prod_{t=1}^{i} \left[\binom{n_r p^r + \sum_{s=1}^{t} n_{r-s}p^{r-s}}{n_{r-t}p^{r-t}} \left(\sum_{d_i = w}^{q} \binom{d_i}{n_{r-i}p^{r-i}} \binom{n_{r-i}p^{r-i}}{w-d_i} c^{w-d_i} \right) \prod_{j=i+2}^{r} Z_j D_{d_r} \right].$$

Finally, put $Q = \sum_{i=0}^{r-1} Q_i$ and $M = \prod_{j=0}^{r} \left(\prod_{i=0}^{n_j - 2} \binom{(n_j - i)\, p^j}{p^j} \right)$, we have

$$D_n = [\delta_0^{n_0} \delta_1^{n_1} \ldots \delta_r^{n_r} - (P_1 + \ldots + P_{v-1} + QM)]/MN\,.$$

Hence we state the following result:

THEOREM 2. – *Let R be a ring of characteristic zero satisfying the conditions* i), ii) *stated above. Assume that p is not a zero-divisor in R.*

A differentiation $\underline{D} = \{D_0 = 1, D_1, \ldots, D_n, \ldots\}$ is F_c-iterative iff, putting $\delta_0 = D_1$, $\delta_1 = D_p$, $\ldots$, $\delta_i = D_{p^i}$, $\ldots$, we have

a) $\delta_i \delta_j = \delta_j \delta_i$, *for all* i, j

b) $\delta_i^p = \sum_{j=1}^{p-1} A_j + c^{p-1} \delta_i$, *for all* i, *where*

$$A_j = \sum_{r_j = p^i+1}^{2p^i} \left[\binom{r_j}{p^i} \binom{p^i}{2p^i - r_j} c^{2p^i - r_j} \prod_{s=j+1}^{p-1} \sum_{r_s = r_{s-1}}^{p^i + r_{s-1}} \binom{r_s}{p^i} \binom{p^i}{p^i + r_{s-1} - r_s} c^{p^i + r_{s-1} - r_s} D_{r_s} \right]$$

c) *for every* $n > 0$

$$D_n = [\delta_0^{n_0} \delta_1^{n_1} \ldots \delta_r^{n_r} - (P_1 + \ldots + P_{v-1} + QM)]/MN\,,$$

where Q, M, N, P_i, for $i = 1, \ldots, \nu - 1$ are the integers defined above.

PROOF. – a): It follows from the definition of F_c-iterativity.

b) It sufficies to observe that

$$\delta_i^p = \sum_{r_1 = p^i}^{2p^i} \left[\binom{r_1}{p^i} \binom{p^i}{2p^i - r_1} c^{2p^i - r_1} \right.$$

$$\left. \prod_{s=2}^{p-1} \left(\sum_{r_s = r_{s-1}}^{p^i + r_{s-1}} \binom{r_s}{p^i} \binom{p^i}{p^i + r_{s-1} - r_s} c^{p^i + r_{s-1} - r_s} D_{r_s} \right) \right].$$

3. – Integrable derivations in unequal characteristic.

In this section we want to state some constructive theorems related to the problem of extending a system of derivations to differentiations in the unequal characteristic case. The formulation of these theorems are the same of those in [7]. Nevertheless we want to construct exactly the differentiations. Our considerations concern liftings to differentiations which are not necessarily F_a or F_c-iterative (in this case a differentiation is an action of formal group). This situation was partially studied in [3].

DEFINITION 5. – *We say that a derivation $D \in \mathrm{Der}(R)$ is integrable if there exists a differentiation $\underline{D} = \{D_0 = 1, D_1, \ldots, D_n, \ldots\}$ of R with $D_1 = D$. We also say that $\underline{D}$ lifts D.*

If k is a subring of R we say that a derivation $D \in \mathrm{Der}(R)$ is integrable over k if there exists a differentiation $\underline{D} = \{D_0 = 1, D_1, \ldots, D_n, \ldots\}$ of R over k with $D_1 = D$.

We will denote the set of all integrable derivations of R over k with $\mathrm{Ider}_k(R)$.

REMARK 4. – It is obvious that if R is a ring of unequal characteristic and p a prime number, if D is an integrable derivation of R the induced derivation $\overline{D}$ of $\overline{R} = R/pR$ is integrable too.

DEFINITION 6. – *Let (A, m) be a local ring of characteristic zero with residue field of characteristic $p > 0$. A subring C of A is called a coefficient ring if it satisfies the following conditions*:

(i) *C is a complete DVR with maximal ideal generated by p;*

(ii) *$C/pC \cong A/m$ by the canonical map.*

REMARK 5. – Let (A, m) be a local k-algebra and let $(A, m)^{\frown} = \widehat{A}$ the m-adic completion of A. Then

$$\mathrm{Ider}_k(\widehat{A}) = \mathrm{Ider}_k(A) \otimes \widehat{A},$$

i.e. every differentiation of A over k can be extended to a differentiation of $\widehat{A}$ over k.

Infact, if $\underline{D} = \{1, D_1, \ldots, D_n, \ldots\}$ is a differentiation of A over k, we have $D_n(m^{\nu}) \subseteq m^{\nu - n}$, for $\nu > n$, and so each D_n is uniformly continous in the m-adic topology and can be unequely extended to the completion $\widehat{A}$.

PROPOSITION 1. – *Let (A, m) be a local ring and let I be a subring which is a DVR with prime element p such that $m \cap I = pI$. Assume moreover that A/m is separably finite over I/pI. Then $\widehat{A}$ contains a coefficient ring C which is finite over the pI-adic completion of I, $\widehat{I}$.*

PROOF. – See [9], Proposition 2.

PROPOSITION 2. – *Let (A,m) be a local ring of characteristic zero with residue field of characteristic p. Let I be a subring which is a DVR with prime element p such that $m \cap I = pI$. Assume moreover that A/m is separably finite over I/pI. Then*

$$\mathrm{Der}_{\widehat{I}}(C) = \mathrm{Ider}_{\widehat{I}}(C),$$

where C is a coefficient ring of A which is finite over $\widehat{I}$.

PROOF. – C is a regular complete local ring of maximal ideal pC and p is not an element of p^2C. Moreover $A/m \cong C/pC$ is separable over $I/pI = \widehat{I}/p\widehat{I}$. Hence, by ([8], Lemma 1), C is formally smooth over the subring $\widehat{I}$ and the assertion follows from ([7], Theorem 8).

COROLLARY 1. – Under the same hypotheses of Proposition 2. Let $p \notin m^2$. Then

$$\mathrm{Ider}_I(A) = \mathrm{Ider}_C(\widehat{A}) = \mathrm{Der}_C(\widehat{A}).$$

PROOF. – First of all we observe that due to ([6], Lemma 1), A is formally smooth over C and so $\mathrm{Ider}_C(\widehat{A}) = \mathrm{Der}_C(\widehat{A})$ ([10], Theorem 8). Since $\mathrm{Ider}_C(\widehat{A})$ is a submodule of $\mathrm{Ider}_I(A)$, we have only to prove that $\mathrm{Ider}_I(A) \subseteq \mathrm{Ider}_C(\widehat{A})$.

Let $D \in \mathrm{Ider}_I(A)$ and let $D' = D/I$ be the restriction of D to I. If $\widetilde{D}$ is its extension to $\widehat{I}$, $\widetilde{D} \in \mathrm{Ider}_{Z_{pZ}}(\widehat{I}) = \mathrm{Ider}(\widehat{I})$. But $\mathrm{Ider}(\widehat{I}) = 0$ and so since C is an integral extension of $\widehat{I}$, $\mathrm{Ider}(C) = 0$ too. Infact if $k \subset A \subset B$ are integral domains and B is an integral extension of A with $\mathrm{Der}_k(A) = 0$ it is easy to prove that $\mathrm{Der}_k(B) = 0$. Finally $D \in \mathrm{Ider}_C(\widehat{A})$.

In what follows, given a k-algebra A, a subset Γ of A, and a function $f : \Gamma \to A[\![X_1, \ldots, X_m]\!]$, we will denote by $f_\alpha : \Gamma \to A$, $\alpha \in N^m$, the functions determined by the equality $\sum_\alpha f_\alpha(y) X^\alpha = f(y)$, $y \in \Gamma$, where $X^\alpha = X_1^{\alpha_1} \ldots X_m^{\alpha_m}$ for $\alpha = (\alpha_1, \ldots, \alpha_m) \in N^m$. Note that if $D : A \to A[\![[X_1, \ldots, X_m]]\!]$ is a morphism of k-algebras with $D_0 = \mathrm{id}_A$, then $D_\alpha : A \to A$ is a k-derivation for any $\alpha \in N^m$ with $|\alpha| = \alpha_1 + \ldots + \alpha_m = 1$.

THEOREM 3. – *Let (A, m) be a complete regular local ring of characteristic zero and of krull dimension n with residue field of characteristic p. Let I be a subring which is a DVR with prime element p such that $m \cap I = pI$ and $p \notin m^2$. Assume moreover that A/m is separably finite over I/pI and that $(p, x_2, \ldots, x_n)$ is a regular system of parameters of A.*

Then for any I-derivations $d_1, \ldots, d_m : A \to A$ there is a morphism of I-algebras $D : A \to A[\![X]\!]$ such that $D_0 = \mathrm{id}_A$ and $D_{(j)} = d_j$, for $j = 1, \ldots, m$, where $(j) = (0, \ldots, 0, 1, 0, \ldots, 0) \in N^m$ with 1 on the j-th position.

PROOF. – Let us consider a function

$$s : \{x_2, \ldots, x_n\} \to A[\![X]\!] = A[\![X_1, \ldots, X_m]\!]$$

with $s_0(x_i) = x_i$ for $i = 2, \ldots, n$ and $m \geq 1$.

Then there exists a unique morphism of k-algebras $D : A \to A[\![X]\!]$ such that $D_0 = \mathrm{id}_A$ and $D(x_i) = s(x_i)$, for $i = 2, \ldots, n$.

Infact, by I. S. Cohen's Theorem and Proposition 1, A has a coefficient ring C which contains I and is finite over I. Moreover if $(p, x_2, \ldots, x_n)$ is a system of parameters of A we have

$$A = C[\![x_2, \ldots, x_n]\!] .$$

We define I-linear maps $D_\alpha : A \to A$, $\alpha \in N^m$, such that $D(a) = \sum_\alpha D_\alpha(a) X^\alpha$, $a \in A$, will be the desired morphism of I-algebras.

For every $c \in C[x_2, \ldots, x_n]$

$$c = \sum c_{e_2 \ldots e_n} x_2^{e_2} \ldots x_n^{e_n}, \qquad c_{e_2 \ldots e_n} \in C ,$$

we define D_α over $x_2^{e_2} \ldots x_n^{e_n} = x^\mu$ as the coefficient at X^α in $s(x_2)^{e_2} \ldots s(x_n)^{e_n} \in A[\![X]\!]$.

Finally for $z \in C$ and x^μ as above we set

$$D_\alpha(zx^\mu) = zD_\alpha(x^\mu) .$$

This formula determines a I-linear map $D_\alpha : C[x_2, \ldots, x_n] \to C[x_2, \ldots, x_n]$. Since $A = C[\![x_2, \ldots, x_n]\!] = (C[x_2, \ldots, x_n])^\frown$, for the $(p, x_2, \ldots, x_n)$-adic topology and any differentiation can be extended to the completion (Theorem 4), we have

$$D_\alpha : A \to A\,, \qquad \alpha \in N^m,$$

such that $D_0 = \mathrm{id}_A$ and $D_\alpha(x_i) = s_\alpha(x_i)$, for every i.

This means that $D : A \to A[\![X]\!]$, with $D(a) = \sum_\alpha D_\alpha(a)\, X^\alpha$, $a \in A$, is a I-linear map with $D_0 = \mathrm{id}_A$ and $D(x_i) = s(x_i)$, for every i.

By direct calculations we can prove that D_α is a morphism of rings.Finally the uniqueness of D is a simple consequence of the definition $D_\alpha(x_i) = s_\alpha(x_i)$.

Now let us define the function $s : \{x_2, \dots, x_n\} \to A[\![X]\!]$ by $s(x_i) = x_i + \sum_{i=0}^{m} d_i(x_i)\, X_i$, for every i. Then there exists a morphism of I-algebras $D : A \to A[\![X]\!]$ such that $D_0 = \mathrm{id}_A$ and $D(x_i) = s(x_i)$. Hence $D_{(j)}(x_i) = d_j(x_i)$ for every i, which implies that $D_{(j)} = d_j$ for $j = 1, \dots, m$.

Corollary 2. – *Under the assumptions of Theorem* 3 *we have*:

$$\mathrm{Der}_I(A) = \mathrm{Ider}_I(A)\,.$$

Proof. – It sufficies to consider $m = 1$ and to apply Corollary 1.

Remark 6. – We recall that if k is a ring and A a k-algebra. A n-dimensional differentiation $\underline{D}$ of A is a set of linear maps $\{D_\alpha : A \to A,\ \alpha \in N^n\}$ such that $D_0 = \mathrm{id}_A$ and

$$D_\gamma(ab) = \sum_{\alpha+\beta=\gamma} D_\alpha(a)\, D_\beta(b)\,, \qquad \text{for every } \gamma\,.$$

Hence we can easly verify that the derivations $d_1, \dots, d_m$ can be extended to a n-dimensional differentiation.

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