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Harmonic Functions on Classical Rank one Balls.

PHILIPPE JAMING (*)

Sunto. – *In questo articolo studieremo le relazioni fra le funzioni armoniche nella palla iperbolica (sia essa reale, complessa o quaternionica), le funzione armoniche euclidee in questa palla, e le funzione pluriarmoniche sotto certe condizioni di crescita. In particolare, estenderemo al caso quaternionico risultati anteriori dell'autore (nel caso reale), e di A. Bonami, J. Bruna e S. Grellier (nel caso complesso).*

1. – Introduction.

In this paper, we study the links between harmonic functions on the hyperbolic balls (real, complex or quaternionic), the euclidean harmonic functions on these balls and pluriharmonic functions. In particular we investigate whether growth conditions may separate these classes.

More precisely, let $\mathbb{F}=\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the quaternions) and let n be an integer, $n \geqslant 2$ ($n \geqslant 3$ if $\mathbb{F}=\mathbb{R}$). Let $\mathbb{B}_n$ be the euclidean ball in $\mathbb{F}^n$, let Δ be the euclidean laplacian operator on $\mathbb{B}_n$ and let $N = r\dfrac{\partial}{\partial r}$ be the normal derivation operator. For $k \in \mathbb{N}^*$ a function u of class $\mathcal{C}^{2k}$ is said to be k-hamonic if $\Delta^k u = 0$, in particular for $k=1$ this are the euclidean harmonic functions.

The ball $\mathbb{B}_n$ can also be endowed with the hyperbolic geometry. Let $D_{\mathbb{F}}$ be the associated Laplace-Beltrami operator. Let $\varrho = \dfrac{n-1}{2}$, n, $2n+1$ according to $\mathbb{F}=\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

It is well known that if u is euclidean harmonic or more generally k-harmonic for $k \in \mathbb{N}^*$ with a boundary distribution, then every normal derivative of u, $N^k u$, has also a boundary distribution. We will show that if u is a $D_{\mathbb{F}}$-harmonic function with a boundary distribution, then for every integer $k < \varrho$, $N^k u$ has also a boundary distribution.

Next, we define a pluriharmonic function as a function that is euclidean harmonic over every $\mathbb{F}$-line where $\mathbb{F}$ is seen as $\mathbb{R}^d$ with $d = \dim_{\mathbb{R}} \mathbb{F}$. This extends a classical definition from the case $\mathbb{F}=\mathbb{C}$ to the two other cases and seems to be the most pertinent definition for our study.

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It is shown in [7] for $\mathbb{F} = \mathbb{R}$ and n odd and in [2] for $\mathbb{F} = \mathbb{C}$, that if u is $D_{\mathbb{F}}$-harmonic with a boundary distribution, then $N^{\varrho} u$ has a boundary distribution if and only if u is also euclidean harmonic. Note that for $\mathbb{F} = \mathbb{R}$, ϱ is an integer if n is odd, whereas for n even, ϱ is a *half-integer*. In this last case, although one might give a meaning to N^{ϱ}, the above result is no longer true. Actually, if $\mathbb{F} = \mathbb{R}$ and n is even, we will show that if u is $D_{\mathbb{R}}$-harmonic then u is also $\frac{n}{2}$-harmonic (up to a change of variables), implying that u behaves more alike the euclidean-harmonic functions. In particular, as has already been shown in [7] by different methods, if u has a boundary distribution, then $N^k u$ has also a boundary distribution for every k. So, in even dimension, $D_{\mathbb{R}}$-harmonic functions behave like euclidean harmonic functions.

Further, in the case $\mathbb{F} = \mathbb{R}$, the only functions that are both $D_{\mathbb{R}}$-harmonic and euclidean harmonic (and more generally k-harmonic with $k \geqslant 1$) are the constants. In the case $\mathbb{F} = \mathbb{C}$, it is well known that the only functions that are both $D_{\mathbb{C}}$-harmonic and k-harmonic with $k \geqslant 1$ are the pluriharmonic functions (see [11]), in particular they are already euclidean harmonic.

We would also like to mention that in the complex case, this result appears as a particular case of a theorem by Ewa Damek & *al* (*see* [3]) stating that, in a Siegel tube domain, pluriharmonic functions satisfying some growth condition are characterized by only the invariant laplacian and some other elliptic operator. Moreover, here no assumptions on boundary values is needed and the second elliptic operator can be chosen as the euclidean laplacian.

In the case $\mathbb{F} = \mathbb{H}$ (as in the case $\mathbb{F} = \mathbb{R}$), a major difference occurs, namely that the pluri-harmonic functions are no longer $D_{\mathbb{F}}$-harmonic (except for the constant functions). Further there exist functions that are both $D_{\mathbb{H}}$-harmonic and 2-harmonic, and we will show that those $D_{\mathbb{H}}$-harmonic functions that are 2-harmonic but not 1-harmonic are linked to the pluriharmonic functions, and that this class is orthogonal on every sphere $r\mathbb{S}^{4n-1}$, $0 < r < 1$ to the $D_{\mathbb{H}}$-harmonic functions that are 1-harmonic. To conclude, if u is 2-harmonic with a boundary distribution, then $N^k u$ has also a boundary distribution. We will show that, among the $D_{\mathbb{H}}$-harmonic functions the converse is also true: let u be a $D_{\mathbb{H}}$-harmonic with a boundary distribution, then if $N^{\varrho} u$ has also a boundary distribution, then u is 2-harmonic.

The article is organised as follows: in the next section we give the setting of our problem, and we make clear the above mentionned links between the different notions of harmonicity in the real and the complex case. In section 3 we prove that for u $D_{\mathbb{F}}$-harmonic with a boundary distribution, $N^k u$ has a boundary distribution for $k < \varrho$. In the last section we deal with the quaternionic case.

2. – Setting and main results.

2.1. *Gauss' Hypergeometric function.*

A number of hypergeometric functions will appear throughout. We use the classical notation ${}_2F_1(a, b, c; x)$ to note

$$ {}_2F_1(a, b, c; x) = \sum_{k=0}^{\infty} \frac{\Gamma(a+k)}{\Gamma(a)} \frac{\Gamma(b+k)}{\Gamma(b)} \frac{\Gamma(c)}{\Gamma(c+k)} \frac{x^k}{k!} $$

whith $c \neq 0, -1, -2, \ldots$ This can also be defined as being the solution of the differential equation

$$ (1-x)\, x \frac{d^2 u}{dx^2} + [c - (a+b+1)\, x] \frac{du}{dx} - abu = 0 $$

that is regular in 0. We refer to [4] for the theory of such functions.

2.2. *Classical rank one balls.*

Let us recall some facts about symetric spaces of rank 1 of the non-compact type and their realizations as the euclidean unit ball. This facts can be found for instance in [5] and their adaptation to the ball model are then straightforward computations.

Let $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and let $x \mapsto \overline{x}$ $(x \in \mathbb{F})$ be the standard involution on $\mathbb{F}$, put $|x| = x\overline{x}$ and $d = \dim_{\mathbb{R}} \mathbb{F}$.

Consider $\mathbb{F}^{n+1}$ as a right vector field over $\mathbb{F}$ and define the quadratic form $Q(x) = |x_1|^2 + \ldots + |x_n|^2 - |x_{n+1}|^2$ for $x = (x_1, \ldots, x_{n+1}) \in \mathbb{F}^{n+1}$. Then the connected component of the identity G of the group of all $\mathbb{F}$-linear transformations on $\mathbb{F}^{n+1}$ which preserve Q and which are of determinant one (except for the case $\mathbb{F} = \mathbb{H}$) is given as follows:

1. if $\mathbb{F} = \mathbb{R}$ then $G = SO_0(n, 1)$,
2. if $\mathbb{F} = \mathbb{C}$ then $G = SU(n, 1)$,
3. if $\mathbb{F} = \mathbb{H}$ then $G = Sp(n, 1)$.

Let $G = KAN$ be an Iwasawa decomposition for G. Then

$$ K = \left\{ k_{\widehat{k}, c} = \begin{pmatrix} \widehat{k} & 0 \\ 0 & c \end{pmatrix} : \widehat{k} \in SO(n, \mathbb{F}),\, c \in \mathbb{F},\, |c|^2 = 1 \right\}, $$

$$ A = \left\{ a_t = \begin{pmatrix} \operatorname{ch} t & 0 & \operatorname{sh} t \\ 0 & I_{n-1} & 0 \\ \operatorname{sh} t & 0 & \operatorname{ch} t \end{pmatrix} : t \in \mathbb{R} \right\} $$

and

$$N=\left\{ n_\xi=\begin{bmatrix} 1+y+\dfrac{\delta^2}{2} & -y-\dfrac{\delta^2}{2} & \xi_2 & \cdots & \xi_n \\ y+\dfrac{\delta^2}{2} & 1-y-\dfrac{\delta^2}{2} & \xi_2 & \cdots & \xi_n \\ \xi_2 & -\xi_2 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \\ \xi_n & -\xi_n & 0 & & 1 \end{bmatrix} : \begin{array}{c} \xi=(\xi_2,\dots,\xi_n)\in\mathbb{F}^{n-1} \\ y\in\mathbb{F},\ \overline{y}=-y \end{array} \right\}.$$

(where $\delta^2=|\xi|_2^2+\ldots+|\xi|_n^2$). Put $A_+=\{a_t\colon t>0\}$. The Cartan decomposition of G is given by $G=K\overline{A_+}K$.

Let M be the centralizer of A in K, *i.e.*

$$M=\left\{ m_{\widehat{m},\,c}=\begin{pmatrix} c & 0 & 0 \\ 0 & \widehat{k} & 0 \\ 0 & 0 & c \end{pmatrix} : \widehat{m}\in SO(n-1,\,\mathbb{F}),\ c\in\mathbb{F},\ |c|^2=1 \right\}.$$

If $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$ are in $\mathbb{F}^n$, set $\langle x,y\rangle=x_1\overline{y_1}+\ldots+x_n\overline{y_n}$ and $\|x\|^2=\langle x,x\rangle$. Then the unit ball $\mathbb{B}_n=\{x\in\mathbb{F}^n\colon \|x\|^2<1\}$ and its boundary $\mathbb{S}^{nd-1}$ (the unit sphere in $\mathbb{F}^n$) are identified with G/K and K/M. More precisely, an element of G/K is identified with the couple (a_t,ξ), $t\geqslant 0$, $\xi\in\mathbb{S}^{nd-1}\simeq K/M$ which is indentified with the point $(\operatorname{sh}t.\,\xi,\operatorname{ch}t)$ in the hyperboloid $Q(x_1,\ldots,x_n,x_{n+1})=-1$. This point is in turn identified with the point $(\operatorname{th}t)\,\xi\in\mathbb{B}_n$ (*see* figure 1).

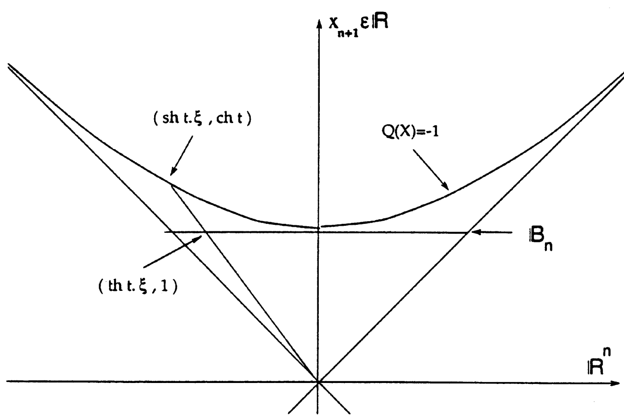

Fig. 1. – The identification of G/K with $\mathbb{B}_n$.

It is then easily seen that G acts transitively on $\mathbb{B}_n$ and on $\mathbb{S}^{nd-1}$ as follows:

$$g.(x_1, \ldots, x_n) = (y_1 y_{n+1}^{-1}, \ldots, y_n y_{n+1}^{-1})$$

where $(y_1, \ldots, y_n, y_{n+1}) = g(x_1, \ldots, x_n, 1)$. The balls $\mathbb{B}_n$ with that action of G are the classical rank 1 spaces of the non-compact type (or the real, complex and quaternionic hyperbolic balls depending on $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$).

Recall that $d = \dim_{\mathbb{R}} \mathbb{F}$. Let γ be the positive simple root of (G, A), and $m_1 = d(n-1)$, $m_2 = d-1$ be the multiplicities of γ and 2γ respectively. Let $\varrho = \frac{m_1}{2} + m_2$, so that $\varrho = \frac{n-1}{2}$, n, $2n+1$ according to $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

The Laplace-Beltrami operator on G/K is given by

$$\frac{d^2}{dt^2} + (m_1 \coth t + 2m_2 \coth 2t)\frac{d}{dt} + \frac{1}{\operatorname{sh}^2 t} L_{\omega_1} + \frac{1}{\operatorname{sh}^2 2t} L_{\omega_2}$$

where L_{ω_1} and L_{ω_2} are tangential operators (see e.g. [10] for precise expressions). Thus, on $\mathbb{B}_n$, the G-invariant laplacian is given by

$$D_{\mathbb{F}} = \frac{1-r^2}{4r^2}[(1-r^2)N^2 + (m_1 + m_2 - 1 + (m_2 - 1)r^2)N] + \frac{1-r^2}{r^2}\Delta_1 + \frac{(1-r^2)^2}{4r^2}\Delta_2$$

where $r = \|x\|$, $N = r\frac{\partial}{\partial r}$ and Δ_1, Δ_2 are two tangential operators having as eigenvectors the spherical harmonics.

EXAMPLE. – $\diamond$ If $\mathbb{F} = \mathbb{R}$ then $\Delta_1 = 0$ while $\Delta_2 = \Delta_\sigma$ the tangential part of the euclidean laplacian so that

$$D_{\mathbb{R}} = \frac{1-r^2}{4r^2}[(1-r^2)N^2 + (n-2-r^2)N] + \frac{(1-r^2)}{r^2}\Delta_\sigma.$$

$\diamond$ If $\mathbb{F} = \mathbb{C}$ design by $\mathcal{L}_{i,j} = \overline{z}_i \frac{\partial}{\partial z_j} - \overline{z}_j \frac{\partial}{\partial z_i}$. Then $\Delta_1 = \mathcal{L} = -\frac{1}{2}\sum_{i<j}(\overline{\mathcal{L}_{i,j}}\mathcal{L}L_{i,j} + \overline{\mathcal{L}_{j,i}}\mathcal{L}_{j,i})$ the *Kohn laplacian* and $\Delta_2 = 4T^2$ with $T = \operatorname{Im}\sum_{k=1}^{n} z_i \frac{\partial}{\partial z_i}$, so that

$$D_{\mathbb{C}} = \frac{1-r^2}{4r^2}[(1-r^2)N^2 + 2(n-1)N] + \frac{1-r^2}{4r^2}\mathcal{L} + \frac{(1-r^2)^2}{r^2}T^2$$

(the notation for N is not the same as in [2]).

The Poisson kernel associated to $D_{\mathbb{F}}$ is given by

$$\mathbb{P}_{\mathbb{F}}(x,\,\xi) = \left(\frac{1-\|x\|^2}{|1-\langle x,\,\xi\rangle\,|^2} \right)^{\varrho}$$

with $x \in \mathbb{B}_n$ and $\xi \in \mathbb{S}^{nd-1}$. The Poisson integral of a distribution f on $\mathbb{S}^{nd-1}$ is then defined in the usual way and written $\mathbb{P}_{\mathbb{F}}[f]$.

DEFINITION. – *Functions u on $\mathbb{B}_n$ such that $D_{\mathbb{F}} u = 0$ will be called $D_{\mathbb{F}}$-harmonic.*

If $\mathbb{F} = \mathbb{C}$, these are the $\mathcal{M}$-harmonic functions, whereas if $\mathbb{F} = \mathbb{R}$, the author called them $\mathcal{H}$-harmonic in [7] (however a different identification of $\mathbb{B}_n$ with G/K is used there).

2.3. *Boundary distribution.*

We focus in this article on functions that have a boundary distribution in the following sense:

DEFINITION. – *A function u on $\mathbb{B}_n$ has a* boundary distribution *if the limit*

$$\lim_{r\to 1} \int_{\mathbb{S}^{nd-1}} u(r\zeta)\,\Phi(\zeta)\,d\sigma(\zeta)$$

exists for every $\Phi \in \mathcal{C}^{\infty}(\mathbb{S}^{nd-1})$.

If u is $D_{\mathbb{F}}$-harmonic then u has a boundary distribution if and only if $u = \mathbb{P}_{\mathbb{F}}[f]$ for some distribution f on $\mathbb{S}^{nd-1}$. To see this, one may use Lewis' theorem [9] stating that the $D_{\mathbb{F}}$-harmonic functions that are Poisson integrals of distributions are exactly those $D_{\mathbb{F}}$-harmonic functions that have a polynomial growth and then prove as in [7] ($\mathbb{F} = \mathbb{R}$), [1] ($\mathbb{F} = \mathbb{C}$) that the $D_{\mathbb{F}}$-harmonic functions that have a polynomial growth are exactly those that have a boundary distribution. Alternatively, one may use the fact that a $D_{\mathbb{F}}$-harmonic function u is the Poisson integral of an hyperfunction μ and that u has a boundary distribution if and only if the hyperfunction μ is actually a distribution.

We here study the boundary behavior of normal derivatives $N^k u$ of $D_{\mathbb{F}}$-harmonic functions u that have a boundary distribution. In particular, we generalize lemma 2.1 in [2] in the complex case and Theorem 8 in [7] in the real case and give a unified proof independent of $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. We prove the following:

THEOREM 1. – *Let u be a $D_{\mathbb{F}}$-harmonic function with a boundary distribution. Let Υ be a tangential operator that commutes with N. Let k be an integer and $v = N^k \Upsilon u$. Then*

– *if $k < \varrho$, v has a boundary distribution,*

– *if $k = \varrho$, for every $\Phi \in \mathcal{C}^\infty(\mathbb{S}^{nd-1})$,*

$$\int_{\mathbb{S}^{nd-1}} v(r\zeta)\, \Phi(\zeta)\, d\sigma(\zeta) = O\left(\log \frac{1}{1-r}\right).$$

REMARK. – If Υ is tangential and if u has a boundary distribution, then Υu has also a boundary distribution. The operators Δ_1, Δ_2 and their products give examples of tangential operators that commute with N.

2.4. *Links between pluriharmonic, k-harmonic and euclidean harmonic functions.*

We will next clarify a few relations beetween different notions of harmonicity on $\mathbb{B}_n$.

To start with, we extend the definition of pluriharmonic in the complex case to the general case. The most relevant in our context is:

DEFINITION. – *Let u be a function of class $\mathcal{C}^2$ on $\mathbb{B}_n$.*

For $a, b \in \mathbb{F}^n$, define $u_{a,b}$ on $\mathbb{F}$ identified with $\mathbb{R}^d$ as $x \mapsto u(ax+b)$. Then u is said to be pluriharmonic if for every $a, b \in \mathbb{F}^n$, $u_{a,b}$ is harmonic on its domain.

Let $k \in \mathbb{N}^*$, then u is said to be k-harmonic if u is of class $\mathcal{C}^{2k}$ on $\mathbb{B}_n$ and if $\Delta^k u = 0$.

REMARK. – If u is pluriharmonic, then u is also harmonic. In particular if u is pluriharmonic with a boundary distribution, then all its derivatives also have a boundary distribution.

Let us first consider the cases of $\mathbb{R}$ and $\mathbb{C}$ for which references [7] and [2] are available.

Assume first that $\mathbb{F} = \mathbb{R}$. If u is pluriharmonic, then u is an affine function, in particular $\dfrac{\partial^2 u}{\partial r^2} = 0$ and $\Delta_\sigma u = 0$. Further, if u is also $D_{\mathbb{R}}$-harmonic, then $Nu = 0$ and the only affine functions such that $Nu = 0$ are the constant functions.

Assume now that u is both euclidean and $D_{\mathbb{R}}$-harmonic (in particular, u is continuous). But, the radial-tangential expression of the euclidean laplacian is:

$$\Delta = \frac{1}{r^2}[N^2 + (n-2)\, N + \Delta_\sigma]$$

thus, u satisfies

$$(1-r^2)\, N^2 u + (n-2)(1-r^2)\, Nu + (1-r^2)\, \Delta_\sigma u = 0\,.$$

Comparing with the radial-tangential expression of $D_{\mathbb{R}}$, one gets further that $Nu = 0$ *i.e.* u is homogeneous of degree 0. But the only continuous homogeneous functions are constant.

Finally, if $\mathbb{F} = \mathbb{R}$ then $\varrho = \frac{n-1}{2}$ thus the condition $k = \varrho$ in Theorem 1 has the above meaning only when n is *odd*. Moreover, Proposition 3 bellow shows that the behaviour of $D_{\mathbb{R}}$-harmonic functions is different in even and odd dimension. In [7] (1) the equivalence of 1, 4 and 5 in the following proposition has been proved:

PROPOSITION 2. – *Assume n is odd and let u be a $D_{\mathbb{R}}$-harmonic. The following are equivalent*:

1. *u is pluriharmonic (i.e. constant),*
2. *u is euclidean harmonic,*
3. *u is k-harmonic for some $k \geqslant 1$.*

Further, if u has a boundary distribution, this three conditions are equivalent to the following:

4. *for every $\Phi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$,*

$$\int_{\mathbb{S}^{n-1}} N^{\frac{n-1}{2}} u(r\zeta)\, \Phi(\zeta)\, d\sigma(\zeta) = o\left(\log \frac{1}{1-r}\right).$$

5. *$N^{\frac{n-1}{2}} u$ (i.e. $N^\varrho u$) has a boundary distribution.*

The situation in the case *n even* is different. Recall from Helgason [6] that every $D_{\mathbb{R}}$-harmonic function has a spherical harmonic expansion of the form

$$u(r\zeta) = \sum_{l \geqslant 0} f_l(r)\, r^l u_l(\zeta) \tag{1}$$

where u_l is a spherical harmonic of degree l and $f_l(r) = {}_2F_1\left(l, 1-\frac{n}{2}, l+\frac{n}{2}, r^2\right)$.

(1) Where ϱ has to be replaced by 2ϱ because of the different identification of $\mathbb{B}_n$ with G/K.

Then, if n is even, $1-\frac{n}{2}$, thus f_l is a polynomial of degree $\frac{n}{2}-1$. But then, a simple computation shows that $\Delta^k u = 0$ for $k \geqslant \frac{n}{2}$, that is:

PROPOSITION 3. – *For n even, every $D_{\mathbb{R}}$-harmonic function is $\frac{n}{2}$-harmonic.*

COROLLARY 4. – *If n is even and if $f \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ then $\mathbb{P}_{\mathbb{R}}[f] \in \mathcal{C}^\infty(\overline{B_n})$. Further, if u is $D_{\mathbb{R}}$-harmonic and has a boundary distribution, then, for every k, $N^k u$ has a boundary distribution.*

Assume now that $\mathbb{F} = \mathbb{C}$ ($\varrho = n$). In this case, pluriharmonic functions are both euclidean harmonic and $D_{\mathbb{C}}$-harmonic. The converse is also true (see [11], Theorem 4.4.9). Moreover, we will show that if u is k-harmonic and $D_{\mathbb{C}}$-harmonic, then u is pluriharmonic, a fact for which we have not found any reference. Our proof is again based on the fact from [6] that every $D_{\mathbb{C}}$-harmonic function has a spherical harmonic expansion of the form:

$$u(z) = \sum_{p,\, q \in \mathbb{N}} {}_2F_1(p,\, q,\, p+q+n,\, |z|^2)\, u_{p,\,q}(z) \tag{2}$$

where $u_{p,\,q}$ is a spherical harmonic of degree p in z and q in $\bar{z}$. Moreover, this series converges uniformly over compact sets of $\mathbb{B}_n$.

Now, write $f_{p,\,q}(r) = {}_2F_1(p,\, q,\, p+q+n,\, r^2)$. If we further ask for u to be euclidean harmonic or more generally k-harmonic, then applying Δ^k to (2) implies that

$$\sum_{p,\, q \in \mathbb{N}} T^k_{p,\,q} f_{p,\,q}(r)\, u_{p,\,q}(z) = 0$$

where $T_{p,\,q} = \frac{1}{r^2}(N^2 + 2n(p+q)\,N)$. Thus, for every p, q such that $u_{p,\,q} \neq 0$, $T^k_{p,\,q} f_{p,\,q}(r) = 0$ for $0 \leqslant r < 1$. But, the only functions φ that are regular in 0 such that $T^k_{p,\,q}\varphi = 0$ are polynomials of degree at most k. Thus $f_{p,\,q}$ has to be a polynomial. Note that a hypergeometric function ${}_2F_1(a,\, b,\, c,\, x)$ (with $c > 0$) is a polynomial if and only if $a \leqslant 0$ or $b \leqslant 0$. Thus $u_{p,\,q} = 0$ unless $p = 0$ or $q = 0$ *i.e.* the sum in (2) is reduced to summing over $\{(p,\, 0) : p \in \mathbb{N}\}$ and $\{(0,\, q) : q \in \mathbb{N}\}$, that is, u is pluriharmonic.

Further, in [2], pluriharmonic functions have been characterized among $D_{\mathbb{C}}$-harmonic functions with a boundary distribution. This gives equivalence of 1, 4 and 5 of the following:

PROPOSITION 5. – *Let u be an $D_{\mathbb{C}}$-harmonic function. The following are equivalent*:

1. *u is pluriharmonic,*
2. *u is euclidean harmonic,*
3. *u is k-harmonic for some $k \in \mathbb{N}^*$.*

Further, if u has a boundary distribution, this three conditions are equivalent to the following:

4. $N^n u$ (*i.e.* $N^{\varrho} u$) *has a boundary distribution*,
5. *for every* $\Phi \in \mathcal{C}^\infty(\mathbb{S}^{2n-1})$,

$$\int_{\mathbb{S}^{2n-1}} N^n u(r\zeta)\, \Phi(\zeta)\, d\sigma(\zeta) = o\left(\log \frac{1}{1-r}\right).$$

We will prove a similar result in the quaternionic case ($\varrho = 2n+1$). However, the result will be more elaborate, as the class of «pluriharmonic» functions and the class of functions that are both euclidean and $D_{\mathbb{H}}$-harmonic do no longer coincide. We postpone the description of results to section 4.

3. – Proof of theorem 1.

Let us prove Theorem 1 by induction on k. For $k=0$ this is just the hypothesis on u.

If u is $D_{\mathbb{F}}$-harmonic, then

$$(1-r^2)\, N^2 u + (m_1 + m_2 - 1 + (m_2 - 1)\, r^2)\, Nu + \varDelta_1 u + \frac{1-r^2}{4} \varDelta_2 u = 0\,.$$

If we apply N^{k-1} and isolate terms in N^{k+1} and N^k, we obtain

$$\begin{aligned}(1-r^2)\, N^{k+1} u - 2(k-1)\, r^2 N^k u + (m_1 + m_2 - 1 + (m_2-1)\, r^2)\, N^k u = \\ r^2 \sum_{j=2}^{k-1} \binom{k-1}{j} 2^j N^{k+1-j} u - (m_2 - 1)\, r^2 \sum_{j=1}^{k-1} \binom{k-1}{j} 2^j N^{k-j} u - \\ -N^{k-1} \varDelta_1 u - \frac{1-r^2}{4} N^{k-1} \varDelta_2 u + r^2 \sum_{j=1}^{k-1} \binom{k-1}{j} 2^{j-2} N^{k-1-j} \varDelta_2 u\,.\end{aligned}$$

Let $\mathbb{Y}$ be a tangential operator that commutes with N then

$$\begin{aligned}(3) \quad (1-r^2)\, N^{k+1} \mathbb{Y} u + (m_1 + m_2 - 1 + (m_2 - 2k + 1)\, r^2)\, N^k \mathbb{Y} u = \\ r^2 \sum_{j=2}^{k-1} \binom{k-1}{j} 2^j N^{k+1-j} \mathbb{Y} u - (m_2 - 1)\, r^2 \sum_{j=1}^{k-1} \binom{k-1}{j} 2^j N^{k-j} \mathbb{Y} u - \\ N^{k-1} \mathbb{Y} \varDelta_1 u - \frac{1-r^2}{4} N^{k-1} \mathbb{Y} \varDelta_2 u + r^2 \sum_{j=1}^{k-1} \binom{k-1}{j} 2^{j-2} N^{k-1-j} \mathbb{Y} \varDelta_2 u\,.\end{aligned}$$

By the induction hypothesis, all the terms in the right member of (3) have a boundary distribution. If we fix $\Phi \in \mathcal{C}^\infty(\mathbb{S}^{n-1})$ and write

$$\psi_k(r) = \int_{\mathbb{S}^{nd-1}} N^k \Upsilon u(r\zeta)\, \Phi(\zeta)\, d\sigma(\zeta)$$

we get that

$$g_k(r) \equiv (1-r^2)\, N\psi_k + (m_1 + m_2 - 1 + (m_2 + 1 - 2k)\, r^2)\, \psi_k \tag{4}$$

has a limit L when $r \to 1$.

But, solving the differential equation (4) $\left(N = r\dfrac{d}{dr}\right)$ leads to

$$\psi_k(r) = \frac{(1+r)^{\varrho - k}}{r^{m_1 + m_2 - 1}} (1-r)^{\varrho - k} \int_0^r \frac{g_k(s)\, s^{m_1 + m_2 - 2}}{(1+s)^{\varrho + 1 - k}} (1-s)^{-(\varrho - k) - 1} ds\,.$$

Thus, if $k < \varrho$, $\psi_k(r)$ has limit $\dfrac{L}{2\varrho}$ wheras if $k = \varrho$, $\psi_k(r)$ has logarithmic growth. ■

4. – Boundary behavior of $2n+1^{th}$ derivative in the quaternionic case.

In this section we will restrict our attention to the case $\mathbb{F} = \mathbb{H}$, and we will compare pluriharmonic functions, euclidean harmonic functions and $D_{\mathbb{H}}$-harmonic functions. Our study will rely on the spherical harmonic expansion of $D_{\mathbb{H}}$-harmonic functions, therefore we will recall the theory of spherical harmonics adapted to the analysis on $\mathbb{S}^{4n-1}$, the unit sphere of $\mathbb{H}^n$, as can be found in [8].

4.1. *Spherical harmonics in the case* $\mathbb{F} = \mathbb{H}$.

Let $\Lambda = \{(p, q) \in \mathbb{N}^2 \colon p \in \mathbb{N},\ q - p \in 2\mathbb{N}\}$.

Denote by $w_1, \ldots, w_n$ the standard coordinates on $\mathbb{H}^n$, $w_s = x_s + ix_{n+s} + jx_{2n+s} + kx_{3n+s}$ where $x_s \in \mathbb{R}$ $(1 \leqslant s \leqslant 4n)$. The polar coordinates are given as follows:

$$\begin{cases} w_1 = r\cos\xi(\cos\Phi + y\sin\Phi) \\ w_s = r\sigma_s \sin\xi \end{cases}$$

where $r = \|(w_1, \ldots\|, w_n)$, $0 \leqslant \xi \leqslant \frac{\pi}{2}$, $0 \leqslant \Phi \leqslant 2\pi$, $y \in \mathbb{H}$ with $|y|^2 = 1$ and $\Re(y) = 0$, $\sigma_s \in \mathbb{H}$ with $\sum_{s=2}^{n} |\sigma_s|^2 = 1$. It is easy to see that an M-invariant function on $\mathbb{H}^n$ depends only on r, $r_1 = w_1 + \overline{w_1}$ and $r_2 = |w_1|^2$.

Let $\widehat{K}$ denote the equivalence classes of irreducible unitary representations of K and $\widehat{K}_M = \{(\tau, V_\tau) \in \widehat{K} \colon \dim V_\tau^M \neq 0\}$ where V_τ^M denotes the subspace of V_τ consisting of M-fixed vectors. Since G is of rank one, $\dim V_\tau^M = 1$ if $(\tau, V_\tau) \in \widehat{K}_M$. The Peter-Weyl theorem implies that

$$L^2(\mathbb{S}^{nd-1}) = \sum_{\tau \in \widehat{K}_M} V_\tau$$

as a representation space of K. The actual parametrization of $\tau \in \widehat{K}_M$ and the spherical harmonics that span V_τ^M are given by (see [8]) the following formula:

$$\Phi_{p,q} = r^q \frac{\sin((p+1)\,\Phi)}{\sin^{-1}\Phi} \cos^q \xi\, {}_2F_1\left(\frac{p-q}{2}, -\frac{p+q+2}{2}, 2(n-1); -\tan^2 \xi\right),$$

for $p, q \in \Lambda$.

The corresponding matrix coefficient $\langle \tau_k \Phi_{p,q}, \Phi_{p,q} \rangle$ is an M-invariant spherical function on K. The span of these coefficients are nothing but the spherical harmonics when restricted to $\mathbb{S}^{nd-1}$. We will write $H(p, q)$ ($(p, q) \in \Lambda$) for the set of spherical harmonics obtained in this way.

We will use the fact that $\{H(p, q) : (p, q) \in \Lambda\}$ provides a complete orthonormal set of joint eigenfunctions of Δ_1 and Δ_2. More precisely, for $\varphi_{p,q} \in H(p, q)$,

$$\frac{1}{4}\Delta_2 \varphi_{p,q} = -p(p+2)\,\varphi_{p,q}$$

and

$$\left(\Delta_1 + \frac{1}{4}\Delta_2\right) \varphi_{p,q} = -q(q+4n-2)\,\varphi_{p,q}.$$

For convenience, for $\zeta \in \mathbb{B}_n \setminus \{0\}$ we write

$$\dot{\zeta} = \frac{\zeta}{\|\zeta\|} = \left(\frac{\zeta_1}{\|\zeta\|}, \ldots, \frac{\zeta_n}{\|\zeta\|}\right).$$

4.2. *Spherical harmonics expansion of $D_{\mathbb{H}}$-harmonic hunctions.*

Let u be $D_{\mathbb{H}}$-harmonic. By the Peter-Weyl theorem, u has an expansion into spherical harmonics

$$u(\zeta) = \sum_{p, q \in \Lambda} \psi_{p,q}(r)\, \varphi_{p,q}(\dot{\zeta})$$

where $r = \|\zeta\|$ and

$$\psi_{p,q}(r) = \int_K u(k.\,\zeta)\, \overline{\Phi_{p,q}(k.\,\dot{\zeta})}\, dk\,.$$

Then, using the radial-tangential expression of $D_{\mathbb{H}}$ and the fact that Δ_1, Δ_2 are self-adjoint, we get

$$(1-r^2)\, r^2 \psi''_{p,q}(r) + (m_1 + m_2 + (m_2 - 2)\, r^2)\, r\psi'_{p,q}(r) - \\ [q(q+4n-2) - r^2 p(p+2)]\, \psi_{p,q}(r) = 0\,.$$

Let us look for solutions of the form $r^q F_{p,q}(r^2)$. The function $F_{p,q}$ satisfies

$$(1-t)\, tF''_{p,q}(t) + [q+2n-qt]F'_{p,q}(t) - \frac{1}{4}[q(q-2) - p(p+2)]\, F_{p,q}(t) = 0\,.$$

As $\psi_{p,q}$ is regular in 0, this leads to

$$F_{p,q}(t) = {}_2F_1\left(\frac{q-p-2}{2}, \frac{p+q}{2}, q+2n; t\right).$$

This may be summarized in the following lemma (Helgason - [6]):

LEMMA 6. – *Every $D_{\mathbb{H}}$-harmonic function u admits a decomposition into spherical harmonics of the form*

$$u(r\dot{\zeta}) = \sum_{(p,q)\in\Lambda} {}_2F_1\left(\frac{q-p-2}{2}, \frac{p+q}{2}, q+2n; r^2\right) r^q \varphi_{p,q}(\dot{\zeta}) \tag{5}$$

where $\varphi_{p,q} \in H(p,q)$.

4.3. *Euclidean-harmonic, k-harmonic and pluriharmonic $D_{\mathbb{H}}$-harmonic functions.*

If u is euclidean harmonic on $\mathbb{B}_n$ and $D_{\mathbb{H}}$-harmonic then the same proof as for the complex case in section (2.4) implies that the only spherical harmonics that can occur in (5) are those for which ${}_2F_1\left(\frac{q-p-2}{2}, \frac{p+q}{2}, q+2n; r^2\right)$ is constant. But an hypergeometric function ${}_2F_1(a, b, c, x)$ is constant if and only if $a=0$ or $b=0$, so that the only spherical harmonics that occur in (5) are those for $q = p+2$ or $q = p = 0$.

Let us now turn to pluriharmonic functions. Recall that a function u on $\mathbb{B}_n$ is *pluriharmonic* if for every $a, b \in \mathbb{H}^n$, the function $u_{a,b}\colon \mathbb{R}^4 = \mathbb{H} \mapsto \mathbb{F}$ defined by $u_{a,b}(z) = u(az+b)$ is harmonic on its domain.

With this definition, the only pluriharmonic spherical harmonics are the functions in $H(p,p)$, $p\in\mathbb{N}$. But ${}_2F_1(-1, p, p+2n; r^2) = \left(1 - \frac{p}{p+2n} r^2\right)$, so that the $D_{\mathbb{H}}$ extension from $\mathbb{S}^{nd-1}$ to $\mathbb{B}_n$ of a function in $H(p,p)$ is no longer pluriharmonic, unless $p=0$. So as in the real case, the only pluriharmoic functions that are D_{H}-harmonic are the constants. This leads us to the following notion:

DEFINITION. – *We will say that a function u is the D_{H}-partner of a pluriharmonic function if u has a spherical harmonic expansion*

$$u(r\zeta)=\sum_{p=1}^{+\infty}\left(1-\frac{p}{p+2n}r^2\right)u_{p,p}(r\zeta)\,. \tag{6}$$

In this case, a direct computation shows that $\Delta^2 u=0$, that is, the D_{H}-partners of pluriharmonic functions are D_{H}-harmonic functions that are 2-harmonic but not 1-harmonic. Moreover, the same proof as for the caracterization of D_{C}-harmonic functions that are k-harmonic shows that every D_{H}-harmonic function that is k-harmonic is already 2-harmonic, and thus a sum of a 1-harmonic function and of a D_{H}-partner of a pluriharmonic function.

Finally,

$$\begin{aligned}\left(1-\frac{p}{p+2n}r^2\right)r^p&=(1-r^2)\,r^p+\frac{2n}{p+2n}r^{2+p}\\&=(1-r^2)\,r^p+\frac{2n}{r^{2(n-1)}}\int_0^r s^{p+2n-1}ds\,.\end{aligned}$$

From this fact, the definition of a D_{H}-partner of a pluriharmonic function, given a priori in terms of a spherical harmonics expansion, can be reformulated via an integral operator:

LEMMA 7. – *A function u is a D_{H}-partner of a pluriharmonic function if and only if there exists a pluriharmonic function v such that*

$$u(r\zeta)=(1-r^2)\,v(r\zeta)+\frac{2n}{r^{2(n-1)}}\int_0^r s^{2n-1}v(s\zeta)\,ds\,. \tag{7}$$

Moreover, u has a boundary distribution if and only if v has a boundary distribution.

PROOF. – If v has a boundary distribution, formula (7) immediatly implies that u has also a boundary distribution.

For the converse, differentiating (7) leads to the differential equation

$$r\frac{\partial v}{\partial r}+(1+(2n-3)\,r^2)\,v=2(n-1)\,u+Nu\,.$$

Solving this equation in v leads to

(8) $v(r\zeta) =$

$$\frac{\exp\left(-\frac{2n-3}{2}r^2\right)}{r}\int_0^r (2(n-1)\,u(s\zeta)+Nu(s\zeta))\exp\left(\frac{2n-3}{2}s^2\right)ds\,.$$

But if u has a boundary distribution, then by Theorem 1, Nu has also a boundary distribution. Thus (8) implies that v has a boundary distribution. ■

REMARK. – Note also that, according to the fact that spherical harmonics for different parameters are orthogonal, the class of $D_{\mathbb{H}}$-partners of pluriharmonic functions and the class of $D_{\mathbb{H}}$-harmonic and euclidean harmonic functions are orthogonal on every sphere $r\mathbb{S}^{4n-1}$, $0<r<1$ (thus on B_n).

4.4. *Boundary behavior of the* $2n+1^{th}$ *derivative.*

We will now establish the following theorem:

THEOREM 8. – *Let u be a $D_{\mathbb{H}}$-harmonic function. Then the following are equivalent*:

1. *u is k-harmonic for some $k \geqslant 2$,*
2. *u is 2-harmonic,*
3. *u is the sum of an euclidean harmonic function and of the $D_{\mathbb{H}}$-partner of a pluriharmonic function.*

Further if u has a boundary distribution, then the three above assertions are also equivalent to the following:

4. *$N^{2n+1}u$ has a boundary distribution,*
5. *for every $\Phi \in \mathcal{C}^\infty(\mathbb{S}^{4d-1})$,*

$$\int_{\mathbb{S}^{4d-1}} N^{2n+1}u(r\zeta)\,\Phi(\zeta)\,d\sigma(\zeta) = o\left(\log\frac{1}{1-r}\right).$$

Moreover, in this case, both the euclidean part and the pluriharmonic partner part of u have a boundary distribution.

PROOF. – The equivalence of 1, 2 and 3 has already been established. Now let u be a $D_{\mathbb{H}}$-harmonic function with a boundary distribution and assume 3. Write $u=u_1+u_2$ where u_1 is $D_{\mathbb{H}}$ and euclidean harmonic and u_2 is a $D_{\mathbb{H}}$-partner of a pluriharmonic function. Then by orthogonality of u_1 and of u_2 on every sphere, it is obvious that u_1 and u_2 both have boundary distributions. In particular, $N^{2n+1}u_1$ has a boundary distribution.

Further, Lemma 7 implies first that u_2 is the $D_{\mathbb{H}}$-partner of a pluriharmonic function with a boundary distribution and then that $N^{2n+1}u_2$ also has a boundary distribution. So 3 implies 4. The implication 4 $\Rightarrow$ 5 is obvious. Let us prove 5 $\Rightarrow$ 3. Let u be $D_{\mathbb{H}}$-harmonic with a boundary distribution.

Lemma 6 tells us that u admits an expansion in spherical harmonics

$$u(r\dot{\zeta}) = \sum_{(p,\,q)\in\Lambda} f_{p,\,q}(r^2)\, r^q \varphi_{p,\,q}(\dot{\zeta}) \tag{9}$$

where $\varphi_{p,\,q} \in H(p,\,q)$ and $f_{p,\,q}$ is the hypergeometric function

$$f_{p,\,q}(x) = {}_2F_1\left(\frac{q-p-2}{2},\ \frac{p+q}{2},\ q+2n;\ x\right).$$

Moreover the sum 9, as well as its derivatives converge uniformly on compact subsets of $\mathbb{B}_n$, in particular

$$\|\varphi_{p,\,q}\|_{L^2(\mathbb{S}^{n-1})} N^k(f_{p,\,q}(r^2)\, r^q) = \int_{\mathbb{S}^{n-1}} N^k u(r\zeta)\, \varphi_{p,\,q}(\zeta)\, d\sigma(\zeta). \tag{10}$$

We will need the three following facts (*see* [4]):

i) ${}_2F_1(a,\,b,\,c;\,x)$ has a limit when $x \to 1$ if and only if at least one of the following holds:

α) $a \leqslant 0$, β) $b \leqslant 0$, or γ) $\mathrm{Re}\,(c-a-b) > 0$ and $c \neq 0,\,-1,\,-2,\,\ldots$;

ii) ${}_2F_1(a,\,b,\,c;\,x) \geqslant C\left(\log\dfrac{1}{1-x}\right)$ in the cases not covered by i.

iii) $\dfrac{d^k}{dx^k}{}_2F_1(a,\,b,\,c;\,x) = \dfrac{\Gamma(a+k)\,\Gamma(b+k)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c+k)}{}_2F_1(a+k,\,b+k,\,c+k;\,x)$

But hypothesis 5 says that the right hand member of 10 has a limit when $r \to 1$. Thus, property *iii*) implies that, if $\varphi_{p,\,q} \neq 0$, then

$${}_2F_1\left(\frac{q-p-2}{2} + 2n+1,\ \frac{p+q}{2} + 2n+1,\ q+4n+1,\ x\right)$$

has a limit when $x \mapsto 1$. Thus properties i) and ii) imply that $\varphi_{p,\,q} = 0$ unless ($(p,\,q) \in \Lambda$):

◇ $\dfrac{q-p-2}{2} \leqslant 0$ (property i_α), that is $q = p+2$ — the euclidean harmonic part — or $p = q$ — the pluriharmonic partner —

◇ $\dfrac{p+q}{2} = 0$ (property i_β), that is if $(p,q) = (0,0)$ the constant part of u.

◇ or $p - q \leqslant 0$ (property i_γ), that is again $p = q$.

Summarizing, u has a spherical harmonics expansion

$$u(r\dot{\zeta}) = \sum_{p=0}^{\infty} r^{p+2} \varphi_{p,\,p+2}(\dot{\zeta}) + \varphi_{0,\,0}(\dot{\zeta}) + \sum_{p=1}^{\infty} \left(1 - \frac{p}{p+2n} r^2\right) r^p \varphi_{p,\,p}(\dot{\zeta})$$

where $\varphi_{p,\,p} \in H(p,\,p)$, $\varphi_{p,\,p+2} \in H(p,\,p+2)$, thus u is of the desired form.

The fact that both parts have a boundary distribution results directly from the orthogonality mentioned above and Lemma 7. ■

5. – Further remarks on pluriharmonic functions.

1. *The notion of pluriharmonicity is* not *invariant under* $Sp(n,\,1)$.

Indeed, at 0, $D_{\mathbb{H}}$ and Δ coïncide. Moreover, a pluriharmonic function is euclidean harmonic at 0, thus $D_{\mathbb{H}}$-harmonic at 0. Thus, if the notion of pluriharmonicity was invariant under the action of $Sp(n,\,1)$, pluriharmonic functions would be $D_{\mathbb{H}}$-harmonic which, as we have seen, is not the case.

2. A theorem of Forelli in the case $\mathbb{F} = \mathbb{C}$ asserts that a function u is pluriharmonic if and only if, for every $\zeta \in \mathbb{S}^{2n-1}$, the function $u_\zeta \colon z \mapsto u(z\zeta)$ is harmonic (see [11], theorem 4.4.9). *In case* $\mathbb{F} = \mathbb{H}$ *such a theorem can not hold.*

Indeed, as the slices $z\zeta$, $z \in \mathbb{C}$, $\zeta \in \mathbb{S}^{4n-1}$ are invariant under the action of $Sp(n,\,1)$, this would imply the invariance of the notion of pluriharmonicity, a contradiction with the previous fact.

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