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Heat Diffusion on Homogeneous Trees (Note on a Paper by Medolla and Setti) (*).

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Sunto. – *Medolla e Setti* [6] *studiano l'andamento della diffusione del calore generata dal Laplaciano discreto su un albero omogeneo e dimostrano che il calore è asintoticamente concentrato in «anelli» che viaggiano verso l'infinito a velocità lineare e la cui larghezza divisa per* $\sqrt{t}$ *tende all'infinito, dove t è il tempo. Qui si spiega come un risultato più preciso si ottiene come corollario della legge dei grandi numeri e del teorema del limite centrale per la passeggiata aleatoria sull'albero. Inoltre, si dà una dimostrazione breve e diretta di questi teoremi per la diffusione del calore stessa.*

1. – Introduction.

The homogeneous tree $\mathbb{T} = \mathbb{T}_q$ $(q \geq 2)$ is the unique connected, infinite graph without circuits where each vertex has $q+1$ neighbours. Equipped with its graph metric d (where $d(x, y)$ is the number of edges on the unique shortest path between x and $y \in \mathbb{T}$), it is in many respects a perfect discrete analogue of the hyperbolic plane with the Poincaré metric. The *discrete Laplacian* on $\mathbb{T}$ is the operator $\mathcal{L} = I - P$, where I denotes the identity operator and $P = (p(x, y))_{x, y \in \mathbb{T}}$ is defined by

$$p(x, y) = \frac{1}{q+1}, \quad \text{if } d(x, y) = 1, \quad \text{and } p(x, y) = 0, \quad \text{otherwise},$$

and acts on functions $f: \mathbb{T} \to \mathbb{R}$ by $Pf(x) = \sum_y p(x, y) f(y)$. (We use matrix notation for our linear operators, so that $p(x, y) = P\delta_y(x)$.) The associated heat operator $\mathcal{H}_t = (h_t(x, y))_{x, y \in \mathbb{T}}$ is

$$\mathcal{H}_t = \exp(-t\mathcal{L}) = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} P^k, \qquad t \geq 0 .$$

It is very well known and easily seen that the elements $p^{(k)}(x, y)$ of P^k depend only on k and the distance $d(x, y)$, and analogously or $h_t(x, y)$.

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Together with P, also $\mathcal{H}_t$ is stochastic (all row sums equal one). We set $\mathfrak{m} = (q-1)/(q+1)$ and $\sigma^2 = 4q/(q+1)^2$.

Medolla and Setti [MS] use the asymptotic estimate of Cowling, Meda and Setti [CMS] and honest analytic computations to show that for each $x \in \mathbb{T}$

$$\sum_{y:\ |d(x,y) - \mathfrak{m}t| \leqslant r(t)} h_t(x,y) \to 1 \qquad \text{as } t \to \infty,$$

whenever $r(\cdot)$ is a positive function satisfying $\lim\limits_{t\to\infty} r(t)\, t^{-1/2} = \infty$.

The purpose of this note is to explain the following stronger statement.

THEOREM 1. – *For every* $a, b \in \mathbb{R}$, $a < b$,

$$\sum_{y:\ a\sqrt{t} \leqslant d(x,y) - \mathfrak{m}t \leqslant b\sqrt{t}} h_t(x,y) \to \frac{1}{\sqrt{2\pi}} \int_a^b e^{-u^2/2}\, du \qquad \textit{as } t \to \infty.$$

This is a central limit theorem for the *heat diffusion process* $(X_t)_{t \geqslant 0}$ on $\mathbb{T}$: defined on a suitable probability space $(\Omega, \mathcal{A}, \Pr)$, this is the $\mathbb{T}$-valued continuous-time Markov process whose transition semigroup is $\{\mathcal{H}_t : t \geqslant 0\}$. That is, $\Pr[X_{s+t} = y \,|\, X_s = x] = h_t(x,y)$ for $x, y \in \mathbb{T}$. Theorem 1 says that $d(X_t, X_0)$ is asymptotically normal with mean $t\mathfrak{m}$ and variance t, when $t \to \infty$.

The results of [MS] and Theorem 1 can be understood as discrete-space-analogues of properties of heat diffusion on Riemannian symmetric spaces, compare with Anker and Setti [AS] and Babillot [B]. The present «probabilistic» note is addressed to analysts who might (or should) want to know how Theorem 1 (and, as a corollary, the result of [MS]) is proved probabilistically. (Being «probabilistic» does not mean that the proofs themselves are true only almost surely!)

The transition matrix P governs a discrete-time $\mathbb{T}$-valued Markov chain $(Z_n)_{n \in \mathbb{N}_0}$ with $\Pr[Z_{n+1} = y \,|\, Z_n = x] = p(x,y)$. This is the *simple random walk* on $\mathbb{T}$. It has been studied extensively; in particular, the following law of large numbers and central limit theorem are well known.

THEOREM 2 (Sawyer [S], 1978; Sawyer and Steger [SS], 1987).

$$\frac{d(Z_n, Z_0)}{n} \to \mathfrak{m} \quad \textit{almost surely}, \quad \textit{and} \quad \frac{d(Z_n, Z_0) - n\mathfrak{m}}{\sigma\sqrt{n}} \to N(0,1) \ \textit{in law},$$

where $N(0,1)$ *is the standard normal distribution.*

Compare also with result stated by Levit and Molchanov [LM] without proof. Since the proof of Theorem 2 in [S] and [SS] may appear more complicated than it really is in the case of simple random walk (because they cover a much more general situation), in § 3 a direct and short proof of Theorem 1 will be given. This may also contribute to shed more light on those features of the underlying structure that are responsible for the result.

Before that, let us briefly explain.

2. – How Theorem 1 can be deduced from Theorem 2.

The diffusion process (X_t) can be constructed from the random walk (Z_n) in the following way: the walker, instead of moving randomly from vertex to vertex in discrete time (one step per time unit), waits at each point for an exponential (mean and variance 1) random time before performing the next step. That is, we consider a sequence of independent and identically distributed (i.i.d.) non-negative random variables $(T_n)_{n\in\mathbb{N}}$, also independent of (Z_n), with $\Pr[T_n \geq u] = e^{-u}$ and set $\tau_n = T_1 + \ldots + T_n$, $\tau_0 = 0$. Now, given $t > 0$, define the random number $n(t) = \max\{n : \tau_n \leq t\}$. Then we obtain (a model of) the diffusion process by setting $X_t = Z_{n(t)}$.

These facts are well known in the theory of Markov processes, see e.g. Resnick [7], § 5.10.

In particular, $\tau_n/n \to 1$ almost surely by the law of large numbers, when $t \to \infty$. Since $\tau_{n(t)} \leq t < \tau_{n(t)+1}$, we also get that $\tau_{n(t)}/t \to 1$ and $n(t) \to \infty$ almost surely, and the law of large numbers for (Z_n) implies

$$\frac{d(X_t, X_0)}{t} = \frac{d(Z_{n(t)}, Z_0)}{n(t)} \frac{n(t)}{\tau_{n(t)}} \frac{\tau_{n(t)}}{t} \to \mathfrak{m} \text{ almost surely}.$$

Next, as Z_n and τ_n are independent, the central limit theorem for $d(Z_n, Z_0)$ [Theorem 2] and the one for τ_n [the classical CLT] imply

$$\frac{d(Z_n, Z_0) - \mathfrak{m}\tau_n}{\sqrt{n}} = \sigma \frac{d(Z_n, Z_0) - \mathfrak{m}n}{\sigma\sqrt{n}} - \mathfrak{m}\frac{\tau_n - n}{\sqrt{n}} \to N(0, \sigma^2 + \mathfrak{m}^2) = N(0, 1)$$

in law, when $n \to \infty$. Now decompose

$$\frac{d(X_t, X_0) - t\mathfrak{m}}{\sqrt{t}} = \frac{\sqrt{n(t)}}{\sqrt{t}} \left(\sigma \frac{d(Z_{n(t)}, Z_0) - \mathfrak{m}n(t)}{\sigma\sqrt{n(t)}} - \mathfrak{m}\frac{\tau_{n(t)} - n(t)}{\sqrt{n(t)}} - \mathfrak{m}\frac{t - \tau_{n(t)}}{\sqrt{n(t)}} \right).$$

Since $0 \leq t - \tau_{n(t)} < T_{n(t)+1}$, which has finite second moment, we get that $(t - \tau_{n(t)})/\sqrt{n(t)} \to 0$ almost surely, and since $n(t)/t \to 1$ almost surely, the central limit theorem for $d(X_t, X_0)$ follows.

3. – A direct proof of Theorem 1.

The diffusion process (X_t) has two basic and well known properties:

(1) it is of *nearest neighbour* type: if $x, y \in \mathbb{T}$ and $[x = x_0, x_1, \ldots, x_k = y]$ is the unique shortest path in $\mathbb{T}$ connecting x and y, then the diffusion starting at x must pass almost surely through each x_i in order to reach y.

(2) it is transient: $d(X_t, X_0) \to \infty$ almost surely, when $t \to \infty$. This follows from the fact that the Green kernel $\int_0^\infty h_t(x, y)\, dt$ is finite.

As a consequence, it converges to the *boundary* (space of ends) of $\mathbb{T}$. We briefly recall the construction of the latter. A *ray* is an infinite path $[x_0, x_1, x_2, \ldots]$ of successive neighbours in $\mathbb{T}$ without repeated vertices (each finite sub-path is shortest). Two rays are called *equivalent*, if they differ only by finitely many vertices. An *end* of $\mathbb{T}$ is an equivalence class of rays. We write $\partial\mathbb{T}$ for the space of all ends. Let $\widehat{\mathbb{T}} = \mathbb{T} \cup \partial\mathbb{T}$. If $x \in \mathbb{T}$ and $\xi \in \widehat{\mathbb{T}}$ then there is a unique ray representing ξ that starts at x, denoted by $\overline{x\xi}$. Analogously, if $y \in \mathbb{T}$, then $\overline{xy}$ denotes the shortest path from x to y. We now choose a reference vertex $o \in \mathbb{T}$.

From now on we assume without loss of generality that $X_0 = o$.

For $v, w \in \widehat{\mathbb{T}}$, their *confluent* $v \wedge w$ *with respect to* o is defined by $\overline{ov} \cap \overline{ow} = \overline{o(v \wedge w)}$. This is a vertex, unless $v = w \in \partial\mathbb{T}$. With the new (ultra-)metric

$$\theta(v, w) = \exp(-d(v \wedge w, o)), \quad \text{if } v \neq w, \qquad \text{and } \theta(v, v) = 0,$$

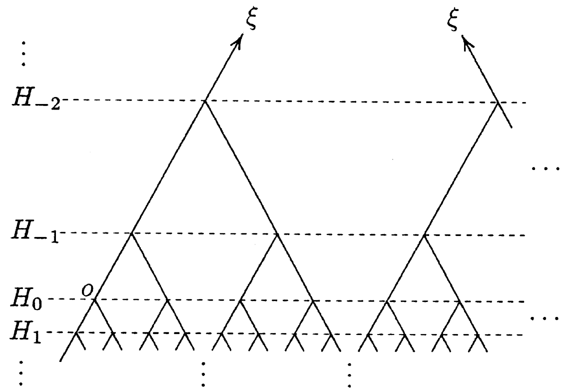
$\widehat{\mathbb{T}}$ becomes a compact space, and $\mathbb{T}$ itself is open and dense.

LEMMA 1. – *There is a $\partial\mathbb{T}$-valued random variable X_∞ such that $X_t \to X_\infty$ almost surely in the topology of $\partial\mathbb{T}$, when $t \to \infty$.*

PROOF. – Let Ω' be the set of elements $\omega \in \Omega$ for which the trajectory $(X_t(\omega))_{t \geq 0}$ satisfies (1) and (2). Then $\Pr(\Omega') = 1$. Let $\omega \in \Omega'$, and suppose that $X_t(\omega)$ has two different accumulation points ξ and η. By (2), both are in $\partial\mathbb{T}$. Let $y = \xi \wedge \eta$. If $t' < t''$ are such that $\theta(X_{t'}(\omega), \xi) < e^{-d(y, o)}$ and $\theta(X_{t''}(\omega), \eta) < e^{-d(y, o)}$ then (using that θ is an ultrametric) $X_{t'}(\omega) \wedge X_{t''}(\omega) = y$. By (1), there must be $s \in [t', t'']$ such that $X_s(\omega) = y$. But t', and hence also s, can be arbitrarly large, in contradiction with (2). ∎

Considering the distances $d(x, o)$ corresponds to computing with polar coordinates. We now change to «horospheric coordinates». We choose and fix a point ξ in $\mathbb{T}$, and define $\widehat{\mathbb{T}}^* = \widehat{\mathbb{T}} \setminus \{\xi\}$ and $\partial^*\mathbb{T} = \partial\mathbb{T} \setminus \{\xi\}$. If $\eta \in \partial^*\mathbb{T}$ then there is a unique two-sided infinite path $[\ldots, x_{-1}, x_0, x_1, x_2, \ldots]$, denoted $\overline{\xi\eta}$, such that $[x_0, x_1, x_2, \ldots]$ represents the end η and $[x_0, x_{-1}, x_{-2}, \ldots]$ represents ξ. Again, given $v, w \in \widehat{\mathbb{T}}^*$, their *confluent* $v \;\; w$ *with respect* *to* ξ is defined by $\overline{v\xi} \cap \overline{w\xi} = \overline{(v \quad w)\,\xi}$. Again, this is a vertex, unless $v = w \in \partial^* \mathbb{T}$.

$\mathbb{T}_2$ *in horocyclic layers*



For $x \in \mathbb{T}$, its *height* with respect to ξ is $\mathrm{hor}(x) = d(x, x \quad o) - d(x \quad o, o)$. This is an integer (not necessesarily positive). The level sets $H_k = \{x \in \mathbb{T} : \mathrm{hor}(x) = k\}$ are the *horocycles.*

LEMMA 2. – *The process* $Y_t = \mathrm{hor}(X_t)$ *is an integer-valued Markov process with the following properties*:

(i) independent increments: *whenever* $0 \leqslant t_0 < t_1 < \ldots < t_r$, *the random variables* $Y_{t_1} - Y_{t_0}, \ldots, Y_{t_r} - Y_{t_{r-1}}$ *are independent.*

(ii) additivity: *For* $s < t$, *the distribution of* $Y_t - Y_s$ *coincides with that of* Y_{t-s}.

(iii) Y_t *has expected value* mt *and variance* t.

PROOF. – Recall that an automorphism of $\mathbb{T}$ is a self-isometry of $\mathbb{T}$ with respect to the graph metric. The action of each automorphism extends continuously to $\partial\mathbb{T}$. Let Γ be the group of automorphisms that fix the end ξ. Then $\mathcal{H}_t$ is Γ-invariant, that is, $h_t(\gamma x, \gamma y) = h_t(x, y)$ for every $\gamma \in \Gamma$. Also, $\mathrm{hor}(\gamma y) - \mathrm{hor}(\gamma x) = \mathrm{hor}(y) - \mathrm{hor}(x)$, and the map $\gamma \mapsto \mathrm{hor}(\gamma x) - \mathrm{hor}(x)$ is independent of $x \in \mathbb{T}$ and a homomorphism $\Gamma \to (\mathbb{Z}, +)$. Therefore, the probability $\Pr[\mathrm{hor}(X_t) = l \,|\, X_s = x]$ depends only on $t - s$ and $\mathrm{hor}(x)$. From here, (i) and (ii) are straightforward.

(iii) is a computational exercise. As a hint, let μ be the the distribution of the integer-valued random variable $\mathrm{hor}(Z_1)$, given that $Z_0 = 0$. That is, μ has support $\{-1, 1\}$, with $\mu(1) = q/(q+1)$ and $\mu(-1) = 1/(q+1)$. Then the distribution of Y_t is

$$\mu_t = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \mu^{(k)},$$

where $\mu^{(k)}$ is the k-th convolution power of μ. ■

Thus, $(Y_t)_{t \geqslant 0}$ is a continuous-time analogue of a sum of i.i.d. integer valued random variables, and the characteristic function φ_t of Y_t (the Fourier transform of μ_t) satisfies $\varphi_t = \varphi_1^t$. We have the law of large numbers and central limit theorem as stated in Theorem 1, with Y_t in the place of $d(X_t, o)$. Thus, the following Lemma provides the last step in our proof of Theorem 1.

LEMMA 3. – *As $t \to \infty$, the process $d(X_t, o) - Y_t$ converges almost surely to the almost surely finite random variable $2d(o, X_\infty \quad o)$.*

PROOF. – Let $\eta \in \partial^* \mathbb{T}$, and suppose that $x_t \in \mathbb{T}$ and $(x_t)_{t \geqslant 0}$ converges to η in the topology of $\partial\mathbb{T}$, when $t \to \infty$. Then there must be t_0 such that $x_t \quad o = \eta \quad o$ for all $t \geqslant t_0$. (In particular, $\mathrm{hor}(x_t) \to \infty$.) But then

$$d(x_t, o) = \mathrm{hor}(x_t) + 2d(o, \eta \quad o) \qquad \text{for all } t \geqslant t_0.$$

Now observe that $\Pr[X_\infty = \xi] = 0$, since the distribution ν of X_∞ on $\mathbb{T}$ is equidistribution, that is,

$$\nu(\{\eta \in \partial\mathbb{T} : \theta(\eta, \xi) < e^{-n}\}) = \nu(\{\eta \in \partial\mathbb{T} : d(\eta \wedge \xi, o) \geqslant n+1\}) = \frac{1}{(q+1)\, q^n}$$

for $n \in \mathbb{N}$. This means that $X_\infty \in \partial^* \mathbb{T}$ almost surely, and given $\omega \in \Omega$ such that $X_\infty(\omega) \in \partial^* \mathbb{T}$, we can apply the above argument to $x_t = X_t(\omega)$.

As a matter of fact, we have shown that almost surely, there is a (random) t_0 such that $d(X_t, o) = Y_t + 2d(o, X_\infty \quad o)$ for all $t \geqslant t_0$. ■

Three final remarks are due here. First, this proof of Theorem 1 is modelled after Cartwright, Kaimanovich and Woess [3], who studied in detail random walks on the group of automorphims of $\mathbb{T}$ that fix a given boundary point. Second, a (weaker) variant of Lemma 3 had already been used by Sawyer [8] in proving the law of large numbers for isotropic random walks on $\mathbb{T}$. Third, Medolla and Setti [MS] also indicate l^p-estimates for the heat kernel when $p \neq 1$: it seems that these estimates do not allow a short probabilistic proof such as the one for the case $p = 1$ outlined here.

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