**Bollettino**
**Unione Matematica Italiana**

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*Bollettino dell'Unione Matematica Italiana, Serie 8*, Vol. **4-B** (2001), n.3, p. 731–736.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2001_8_4B_3_731_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2001.

Bollettino U. M. I.
(8) 4-B (2001), 731-736

# On Locally Finite Groups and the Centralizers of Automorphisms (*).

PAVEL SHUMYATSKY

**Sunto.** – *Sia $p$ un primo, e $A$ un gruppo abeliano elementare di ordine $p^2$ che agisce sul $p'$-gruppo localmente finito $G$. Supponiamo che esista un intero positivo $m$ tale che* $[C_G(a), \underbrace{C_G(b), \ldots, C_G(b)}_{m}] = 1$ *per ogni $a$, $b \in A^\#$. In questo articolo si dimostra che $G$ è nilpotente, con classe di nilpotenza limitata da una funzione che dipende solo da $p$ e $m$.*

## 1. – Introduction.

Let $G$ be a group admitting an action of a group $A$. We denote by $C_G(A)$ the set $C_G(A) = \{x \in G;\ x^a = x \text{ for any } a \in A\}$, the centralizer of $A$ in $G$ (the fixed-point group). This paper deals with the situation when $A$ is an elementary abelian $p$-group, and $G$ is a (locally) finite $p'$-group. Let $A^\#$ denote the set of non-identity elements of $A$. Assume that $G$ is finite and that $C_G(a)$ is nilpotent for any $a \in A^\#$. J. N. Ward showed that if $A$ has rank at least 2 then $G$ is metanilpotent [9], and that if $A$ has rank at least 3 then $G$ is nilpotent [10]. Later the author found some extensions of these results to infinite groups [7, 8]. In this paper we obtain a sufficient condition for the group $G$ to be nilpotent of bounded class.

THEOREM A. – *Let $p$ be a prime, $G$ a locally finite $p'$-group acted on by an elementary abelian group $A$ of order $p^2$. Assume that there exists a positive integer $m$ such that* $[C_G(a), \underbrace{C_G(b), \ldots, C_G(b)}_{m}] = 1$ *for any $a$, $b \in A^\#$. Then $G$ is nilpotent and the class of $G$ is bounded by a function depending only on $p$ and $m$.*

As an immediate consequence of the above theorem we obtain

COROLLARY B. – *Let $p$ be a prime, $G$ a locally finite $p'$-group acted on by an elementary abelian group $A$ of order $p^2$. Assume that there exists a positi-*

(*) Supported by FAPDF and CNPq-Brazil.

*ve integer $m$ such that $\langle C_G(a), C_G(b)\rangle$, the subgroup of $G$ generated by $C_G(a)$ and $C_G(b)$, is nilpotent of class at most $m$ for any $a, b \in A^{\#}$. Then $G$ is nilpotent and the class of $G$ is bounded by a function depending only on $p$ and $m$.*

The proof of the above results uses the associated Lie rings. In particular we will need the following proposition which may have some independent interest.

PROPOSITION C. – *Let $L$ be a Lie ring such that $L = pL$. Let $A$ be an elementary abelian group of order $p^2$ acting on $L$ by automorphisms. Assume that there exists a positive integer $m$ such that $[C_L(a), \underbrace{C_L(b), \ldots, C_L(b)}_{m}] = 0$ for any $a, b \in A^{\#}$. Then $L$ is nilpotent and the class of $L$ is bounded by a function depending only on $p$ and $m$.*

## 2. – Preliminaries.

The next lemma is well-known (see [2, 6.2.2, 6.2.4] for the proof).

LEMMA 2.1. – *Let $A$ be a finite $p$-group acting on a finite $p'$-group $G$.*

1. *If $N$ is an $A$-invariant normal subgroup of $G$ then $C_{G/N}(A) = C_G(A)N/N$;*
2. *If $A$ is an elementary abelian group of order $p^2$ then $G = \langle C_G(a); a \in A^{\#}\rangle$.*

Similar facts (with basically the same proof) hold for Lie rings.

LEMMA 2.2. – *Let $A$ be a finite $p$-group acting on a Lie ring $L$.*

1. *If $N$ is an $A$-invariant ideal of $L$ such that $pN = N$ then $C_{L/N}(A) = (C_L(A) + N)/N$;*
2. *If $A$ is an elementary abelian group of order $p^2$, and if $pL = L$, then $L = \sum_{a \in A^{\#}} C_G(a)$.*

A well-known theorem of Kreknin [6] says that if a Lie ring $L$ admits a fixed-point-free automorphism of finite order $n$ then $L$ is soluble and the derived length of $L$ is bounded by a function of $n$. We will require the following extension of this result [5].

THEOREM 2.3. – *Let a Lie ring $L$ admit an automorphism $\phi$ of finite order $n$ such that $[L, \underbrace{C_L(\phi), \ldots, C_L(\phi)}_{m}] = 0$. Assume that $nL = L$. Then $L$ is soluble with derived length at most $(m+1)^{n-1} + \log_2 m$.*

We will also require a Lie-theoretic analogue of the famous criterion of Ph. Hall for a group to be nilpotent [3]: if $G$ is a group having a normal subgroup $N$ such that both $N$ and $G/N'$ are nilpotent then $G$ is nilpotent and the class of $G$ is bounded in terms of the classes of $N$ and $G/N'$. The corresponding Lie-theoretic result was established in [1].

THEOREM 2.4. – *If a Lie ring $L$ has an ideal $N$ such that both $N$ and $L/N'$ are nilpotent then $L$ is nilpotent and the class of $L$ is bounded in terms of the classes of $N$ and $L/N'$.*

## 3. – Main results.

Our first goal is to establish Proposition C. It will be convenient to start with the case where $L$ is metabelian.

LEMMA 3.1. – *Let $L$ be a metabelian Lie ring such that $L = pL$. Let $A$ be an elementary abelian group of order $p^2$ acting on $L$ by automorphisms. Assume that there exists a positive integer $m$ such that $[C_L(a), \underbrace{C_L(b), \dots, C_L(b)}_{m}] = 0$ for any $a, b \in A^{\#}$. Then $L$ is nilpotent and the class of $L$ is at most $(p+1)(m+1)$.*

PROOF. – Let $A_1, \dots, A_{p+1}$ be the cyclic subgroups of $A$, and for $i = 1, 2, \dots, p+1$ we set $C_i = C_L(A_i)$. Let $M$ be the commutator subring of $L$, $M_i = C_i \cap M$, $N_i = M + C_i$. Lemma 2 tells us that $M = \sum_j M_j$ and $L = \sum_j C_j$. We observe that the $N_i$ are ideals and, since $L = \sum_j N_j$, it is sufficient to show that each $N_i$ is nilpotent of class at most m+1. Let $\gamma_k(N_i)$ stand for the $k$-th term of the lower central series of $N_i$. We have

$$\gamma_{m+2}(N_i) = [\underbrace{N_i, \dots, N_i}_{m+2}] \leq [M, \underbrace{C_i, \dots, C_i}_{m}] =$$

$$\left[\sum_j M_j, \underbrace{C_i, \dots, C_i}_{m}\right] = \sum_j [M_j, \underbrace{C_i, \dots, C_i}_{m}] = 0$$

as $[M_j, \underbrace{C_i, \dots, C_i}_{m}] = 0$ for any $i, j$. The lemma follows. ■

PROPOSITION C. – *Let $L$ be a Lie ring such that $L = pL$. Let $A$ be an elementary abelian group of order $p^2$ acting on $L$ by automorphisms. Assume that there exists a positive integer $m$ such that $[C_L(a),$*

$\underbrace{C_L(b), \ldots, C_L(b)}_{m}] = 0$ *for any* $a, b \in A^{\#}$. *Then* $L$ *is nilpotent and the class of* $L$ *is bounded by a function depending only on* $p$ *and* $m$.

PROOF. – Let $C_j$ have the same meaning as in the proof of Lemma 3.1. Since $L = \sum_j C_j$, it follows that $[L, \underbrace{C_i, \ldots, C_i}_{m}] = 0$ for any $i$. Indeed,

$$[L, \underbrace{C_i, \ldots, C_i}_{m}] = \left[\sum_j C_j, \underbrace{C_i, \ldots, C_i}_{m}\right] = \sum_j [C_j, \underbrace{C_i, \ldots, C_i}_{m}] = 0 .$$

Now Theorem 2.3 tells us that $L$ is soluble and the derived length $d$ of $L$ is at most $(m+1)^{p-1} + log_2 m$. We will use induction on $d$ to show that $L$ is nilpotent and that the nilpotency class of $L$ is bounded by a function of $d, m, p$.

If $d = 2$ then $L$ is metabelian and the required result follows from Lemma 3.1. Assume $d \geqslant 3$ and let $M$ be the metabelian term of the derived series of $L$. The inductive hypothesis is that $L/M'$ is nilpotent and has nilpotency class bounded in terms of $d, m, p$. By Lemma 3.1 $M$ is nilpotent of class at most $(p+1)(m+1)$. Thus, Theorem 2.4 implies that $L$ is nilpotent of class bounded by a function of $d, m, p$. ■

LEMMA 3.2. – *Assume the hypothesis of Theorem A and let G be finite. Then G is nilpotent.*

PROOF. – Assume that $G$ is a counterexample whose order is as small as possible. Let $A_1, \ldots, A_{p+1}$ be the cyclic subgroups of $A$. For any $A$-invariant subgroup $H$ of $G$ we let $H_i$ denote $C_H(A_i)$. Since each $G_i$ is nilpotent, it follows that $G$ is soluble [11]. Let $F = F(G)$ be the Fitting subgroup of $G$. If $F$ is not abelian $G/F'$ is nilpotent by the inductive hypothesis and so the Ph. Hall Criterion cited in the paragraph preceding Theorem 2.4 shows that $G$ is nilpotent, a contradiction. Hence $F$ is abelian and so, by Lemma 2.1, $F = \prod_j F_j$.

Since the order of $G$ is as small as possible, the quotient $G/F$ is nilpotent. It follows that any subgroup of $G$ containing $F$ is subnormal. Since $F$ is generated by all subnormal nilpotent subgroups, it follows that no subgroup properly containing $F$ is nilpotent. Hence any such $A$-invariant subgroup provides a counterexample to the lemma and, using the minimality of $|G|$, we conclude that $G/F$ is abelian and $A$ acts irreducibly on $G/F$. By Lemma 2.1 $G/F$ is generated by the centralizers of $A_i$. These are all $A$-invariant and so some $A_k$ acts on $G/F$ trivially. Lemma 2.1 now shows that $G = FG_k$. Then we have

$$[\underbrace{G, \ldots, G}_{m+2}] \leq [F, \underbrace{G_k, \ldots, G_k}_{m}] = \left[\prod_j F_j, \underbrace{G_k, \ldots, G_k}_{m}\right] = \prod_j [F_j, \underbrace{G_k, \ldots, G_k}_{m}] = 1\,.$$

Thus, $G$ is nilpotent. ■

Now we are ready to conclude the proof of Theorem A.

THEOREM A. – *Let $p$ be a prime, $G$ a locally finite $p'$-group acted on by an elementary abelian group $A$ of order $p^2$. Assume that there exists a positive integer $m$ such that $[C_G(a), \underbrace{C_G(b), \ldots, C_G(b)}_{m}] = 1$ for any $a, b \in A^{\#}$. Then $G$ is nilpotent and the class of $G$ is bounded by a function depending only on $p$ and $m$.*

PROOF. – The usual inverse limit argument along the lines of [4] reduces the theorem to the case where $G$ is finite. So we assume that $G$ is finite and hence, by the previous lemma, nilpotent. The construction associating a Lie ring $L(G)$ with any nilpotent group $G$ is well-known. Let $\gamma_k$ denote the $k$th term of the lower central series of $G$. Set $L_k = \gamma_k/\gamma_{k+1}$ and view $L_k$ as an additive abelian group. Then $L(G) = \oplus_k L_k$. If $x \in \gamma_i$, $y \in \gamma_j$ then, for corresponding elements $x\gamma_{i+1}$, $y\gamma_{j+1}$ of $L(G)$, we set $[x\gamma_{i+1}, y\gamma_{j+1}] = [x, y]\gamma_{i+j+1}$. Thus, we obtain a product operation on the set $\cup_k L_k$. This can be uniquely extended by linearity on the additive abelian group $L(G)$ and, equipped with the product, $L(G)$ becomes a Lie ring. The Lie ring has the same nilpotency class as the group from which it was constructed. In our situation the group $A$ acts naturally on each quotient $\gamma_k/\gamma_{k+1}$ and this action extends uniquely to an action by automorphisms on the Lie ring $L(G)$. Lemma 2.1 shows that if $a \in A$ then $C_L(a)$ is the direct sum of the quotients $C_{\gamma_k}(a)\gamma_{k+1}/\gamma_{k+1}$ and, since $[C_G(a), \underbrace{C_G(b), \ldots, C_G(b)}_{m}] = 1$ for any $a, b \in A^{\#}$, it follows that $[C_L(a), \underbrace{C_L(b), \ldots, C_L(b)}_{m}] = 0$. Finally, we note that $L(G)$ is finite and has the same order as $G$. Therefore $pL(G) = L(G)$ and, by Proposition C, the nilpotency class of $L(G)$ (the same as of $G$) is bounded by a function depending only on $p$ and $m$.

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*Pervenuta in Redazione*
*il 7 dicembre 1999*