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M. Benchohra, S. K. Ntouyas

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Neutral Functional Differential and Integrodifferential Inclusions in Banach Spaces.

M. BENCHOHRA - S. K. NTOUYAS

Sunto. – *In questo lavoro studiamo l'esistenza di soluzioni deboli su un intervallo compatto di problemi con valore iniziale per inclusioni funzionali neutre differenziali e integrodifferenziali in spazi di Banach. I risultati sono ottenuti usando un teorema di punto fisso per mappe condensanti dovuto a Martelli.*

1. – Introduction.

In this paper we prove the existence of mild solutions, defined on a compact interval, for neutral functional differential and integrodifferential inclusion. In Section 3 we study the neutral functional differential inclusion of the form

$$\frac{d}{dt}[y(t)-f(t, y_t)] \in Ay(t)+F(t, y_t), \qquad \text{a. e.} \quad t \in J=[0, b], \tag{1}$$

$$y_0=\phi, \tag{2}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in E, $F: J \times C(J_0, E) \rightarrow 2^E$ ($J_0=[-r, 0]$) is a bounded, closed, convex multivalued map, $f: J \times C(J_0, E) \rightarrow E$, $\phi \in C(J_0, E)$, and E a real Banach space with the norm $|\cdot|$.

For any continuous function y defined on the interval $J_1=[-r, b]$ and any $t \in J$, we denote by y_t the element of $C(J_0, E)$ defined by

$$y_t(\theta)=y(t+\theta), \qquad \theta \in J_0.$$

Here $y_t(.)$ represents the history of the state from time $t-r$, up to the present time t.

Section 4 is devoted to the study of the existence of mild solutions for neutral functional integrodifferential inclusion of the form

$$\frac{d}{dt}[y(t)-f(t, y_t)] \in Ay(t)+\int_0^t K(t, s) F(s, y_s)\, ds, \qquad t \in J=[0, b], \tag{3}$$

$$y_0=\phi, \tag{4}$$

where A, F, f, ϕ are as in the problem (1)-(2) and $K : D \to \mathbb{R}$, $D = \{(t, s) \in J \times J : t \geq s\}$.

Equations of the type (1)-(2) or (3)-(4) arise in many areas of applied mathematics and such equations have received much attention in recent years. We refer to the books of Erbe, Qingai and Zhang [3], Hale [4] and Henderson [5], the papers of Ntouyas [11], [12], Ntouyas, Sficas and Tsamatos [13] and the references cited therein.

This paper is motivated by the recent papers of Hernandez and Henriquez [6], [7] where quasi-linear neutral functional differential equations are studied, with the aid of semigroup theory and the Sadovski fixed point theorem. Here we prove the existence of mild solutions for the problems (1)-(2) and (3)-(4), relied on a fixed point theorem for condensing maps due to Martelli [10].

2. – Preliminaries.

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

$C(J, E)$ is the Banach space of continuous functions from J into E with the norm

$$\|y\|_\infty := \sup\{\,|y(t)| : t \in J\}.$$

$B(E)$ denotes the Banach space of bounded linear operators from E into E.

A measurable function $y : J \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral see Yosida [16]).

$L^1(J, E)$ denotes the Banach space of continuous functions $y : J \to E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|\,dt \qquad \text{for all } y \in L^1(J, E).$$

Let $(X, \|.\|)$ be a Banach space. A multivalued map $G : X \to 2^X$ is convex (closed) valued, if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \bigcup_{x \in B} G(x)$ is bounded in X, for any bounded set B of X (i.e. $\sup_{x \in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$).

G is called upper semicontinuous (u.s.c.) on X, if for each $x_* \in X$, the set $G(x_*)$ is a nonempty, closed subset of X, and if for each open set B of X containing $G(x_*)$, there exists an open neighbourhood V of x_* such that $G(V) \subseteq B$.

G is said to be completely continuous, if $G(B)$ is relatively compact, for every bounded subset $B \subseteq X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \to x_*$, $y_n \to y_*$, $y_n \in Gx_n$ imply $y_* \in Gx_*$).

G has a fixed point if there is $x \in X$ such that $x \in Gx$.

In the following $BCC(X)$ denotes the set of all nonempty bounded, closed and convex subsets of X.

A multivalued map $G : J \to BCC(E)$ is said to be measurable if for each $x \in E$ the function $Y : J \to \mathbb{R}$ defined by

$$Y(t) = d(x, G(t)) = \inf\{|x - z| : z \in G(t)\}$$

belongs to $L^1(J, \mathbb{R})$. For more details on multivalued maps see the books of Deimling [2] and Hu and Papageorgiou [8].

An upper semi-continuous map $G : X \to 2^X$ is said to be condensing [1] if for any subset $B \subseteq X$ with $\alpha(B) \neq 0$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompacteness [1].

We remark that a completely continuous multivalued map is the easiest example of a condensing map.

Our existence results will be proved using the following fixed point result.

LEMMA 2.1 [10]. – *Let X be a Banach space and $N : X \to BCC(X)$ a condensing map. If the set*

$$\Omega := \{y \in X : \lambda y \in N(y) \ \text{for some} \ \lambda > 1\}$$

is bounded, then N has a fixed point.

3. – Existence results for neutral functional differential inclusions.

In order to define the concept of mild solution for (1)-(2), by comparison with the abstract Cauchy problem

$$y'(t) = Ay(t) + h(t)$$

whose properties are well-known [15], we associate (1)-(2) to the integral equation

$$\begin{aligned} y(t) = T(t)[\phi(0) - f(0, \phi)] + f(t, y_t) + \int_0^t AT(t-s)\, f(s, y_s)\, ds \\ + \int_0^t T(t-s)\, g(s)\, ds, \quad t \in [0, b] \end{aligned} \tag{5}$$

where

$$g \in S_{F,y} = \{g \in L^1(J, E) : g(t) \in F(t, y_t) \ \text{for a.e.} \ t \in J\}.$$

DEFINITION 3.1. – *A function* $x : (-r, b) \to E$, $b > 0$ *is called a mild solution of the Cauchy problem* (1)-(2) *if* $y_0 = \phi$, *the restriction of* $y(\cdot)$ *to the interval* $[0, b)$ *is continuous and for each* $0 \leq t < b$ *the function* $AT(t-s) f(s, y_s)$, $s \in [0, t)$, *is integrable and the integral equation* (5) *is satisfied.*

Assume that:

(H1) A is the infinitesimal generator of a compact semigroup of bounded linear operators $T(t)$ in E such that

$$|T(t)| \leq M_1, \quad \text{for some } M_1 \geq 1 \text{ and } |AT(t)| \leq M_2,\ M_2 \geq 0\,.$$

(H2) there exists constants $0 \leq c_1 < 1$ and $c_2 \geq 0$ such that

$$|f(t, u)| \leq c_1 \|u\| + c_2, \qquad t \in J,\ u \in C(J_0, E);$$

(H3) $F : J \times C(J_0, E) \to BCC(E); (t, u) \mapsto F(t, u)$ is measurable with respect to t for each $u \in C(J_0, E)$, u.s.c. with respect to u for each $t \in J$ and for each fixed $u \in C(J_0, E)$ the set

$$S_{F,u} = \{ g \in L^1(J, E) : g(t) \in F(t, u) \text{ for a.e. } t \in J \}$$

is nonempty;

(H4) $\|F(t, u)\| := \sup\{ |v| : v \in F(t, u) \} \leq p(t)\, \psi(\|u\|)$ for almost all $t \in J$ and all $u \in C(J_0, E)$, where $p \in L^1(J, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \to (0, \infty)$ is continuous and increasing with

$$\int_0^b \widehat{m}(s)\, ds < \int_c^\infty \frac{d\tau}{\tau + \psi(\tau)}\,;$$

where

$$c = \frac{1}{1-c_1} \{ [M_1(1+c_1)\|\phi\| + c_2] + c_2 + M_2 c_2 b \}$$

and

$$\widehat{m}(t) = \left\{ \frac{1}{1-c_1} M_2 c_1,\ \frac{1}{1-c_1} M_1 p(t) \right\}.$$

(H5) the function f is completely continuous and for any bounded set $A \subseteq C(J_1, E)$ the set $\{ t \to f(t, y_t) : y \in A \}$ is equicontinuous in $C(J, E)$;

(H6) for each bounded $B \subset C(J_1, E)$, $y \in B$ and $t \in J$ the set

$$\left\{ \int_0^t T(t-s)\, g(s)\, ds : g \in S_{F,y} \right\}$$

is relatively compact.

REMARK 3.2 (i). – *If* $\dim E < \infty$, *then for each* $u \in C(J_0, E)$, $S_{F,u} \neq \emptyset$ (*see Lasota and Opial* [9]).

(ii) $S_{F,u}$ *is nonempty if and only if the function* $Y : J \to \mathbb{R}$ *defined by*

$$Y(t) := \inf\{|v| : v \in F(t, u)\}$$

belongs to $L^1(J, \mathbb{R})$ (*see Papageorgiou* [14]).

The following Lemma is crucial in the proof of our existence results.

LEMMA 3.3 [9]. – *Let I be a compact real interval and X be a Banach space. Let F be a multivalued map satisfying* (H3) *and let* Γ *be a linear continuous mapping from* $L^1(I, X)$ *to* $C(I, X)$, *then the operator*

$$\Gamma \circ S_F : C(I, X) \to BCC(C(I, X)), \quad y \mapsto (\Gamma \circ S_F)(y) := \Gamma(S_{F,y})$$

is a closed graph operator in $C(I, X) \times C(I, X)$.

Now, we are in a position to state and prove our main theorem for this section

THEOREM 3.4. – *Assume that hypotheses* (H1)-(H6) *hold. Then the IVP* (1)-(2) *has at least one mild solution on* J_1.

PROOF. – Let $C(J_1, E)$ be the Banach space of continuous functions from J_1 into E endowed with the sup-norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in [-r, b]\}, \text{ for } y \in C(J_1, E).$$

Transform the problem into a fixed point problem. Consider the multivalued map, $N : C(J_1, E) \to 2^{C(J_1, E)}$ defined by:

For each $h \in Ny$ set

$$h(t) = \begin{cases} \phi(t), & \text{if } t \in J_0 \\ T(t)[\phi(0) - f(0, \phi)] + f(t, y_t) + \int_0^t AT(t-s)\, f(s, y_s)\, ds & \\ \quad + \int_0^t T(t-s)\, g(s)\, ds, & \text{if } t \in J. \end{cases}$$

REMARK 3.5. – *It is clear that the fixed points of N are solutions to* (1)-(2).

We shall show that N is a completely continuous multivalued map, u.s.c. with convex closed values. The proof will be given in several steps.

Step 1: *Ny is convex for each $y \in C(J_1, E)$.*

Indeed, if h_1, h_2 belong to Ny, then there exist $g_1, g_2 \in S_{F,y}$ such that for each $t \in J$ we have

$$h_1(t) = T(t)[\phi(0) - f(0,\phi)] + f(t, y_t) + \int_0^t AT(t-s)\, f(s, y_s)\, ds + \int_0^t T(t-s)\, g_1(s)\, ds\,,$$

and

$$h_2(t) = T(t)[\phi(0) - f(0,\phi)] + f(t, y_t) + \int_0^t AT(t-s)\, f(s, y_s)\, ds + \int_0^t T(t-s)\, g_2(s)\, ds\,.$$

Let $0 \leqslant k \leqslant 1$. Then for each $t \in J$ we have

$$(kh_1 + (1-k)h_2)(t) = T(t)[\phi(0) - f(0,\phi)] + f(t, y_t) + \int_0^t AT(t-s)\, f(s, y_s)\, ds$$
$$+ \int_0^t T(t-s)[kg_1(s) + (1-k)\, g_2(s)]\, ds\,.$$

Since $S_{F,y}$ is convex (because F has convex values) then

$$kh_1 + (1-k)\, h_2 \in Ny\,.$$

Using (H5) it suffices to show that the operator $N_1\colon C(J_1, E) \to 2^{C(J_1, E)}$ defined by:

For each $h \in N_1 y$ set

$$h(t) = \begin{cases} \phi(t), & \text{if } t \in J_0 \\ \displaystyle\int_0^t AT(t-s)\, f(s, y_s)\, ds + \int_0^t T(t-s)\, g(s)\, ds\,, & \text{if } t \in J \end{cases}$$

is completely continuous.

Step 2: *N_1 maps bounded sets into bounded sets in $C(J_1, E)$.*

Indeed, it is enough to show that there exists a positive constant l such that for each $h \in N_1 y$, $y \in B_q = \{y \in C(J_1, E) \colon \|y\|_\infty \leqslant q\}$ one has $\|h\|_\infty \leqslant l$.

If $h \in N_1 y$, then there exists $g \in S_{F,y}$ such that for each $t \in J$ we have

$$h(t) = \int_0^t AT(t-s)\, f(s, y_s)\, ds + \int_0^t T(t-s)\, g(s)\, ds\,.$$

By (H1) and (H4) we have for each $t \in J$

$$\|h(t)\| \leqslant \int_0^t \|AT(t-s)\, f(s,\, y_s)\|\, ds + \int_0^t \|T(t-s)\, g(s)\| ds$$

$$\leqslant M_2\, b(c_1\, q + c_2) + M_1 \sup_{y \in [0,\, q]} \psi(y) \left(\int_0^t p(s)\, ds \right).$$

Then for each $h \in N(B_r)$ we have

$$\|h\|_\infty \leqslant M_2\, b(c_1\, q + c_2) + M_1 \sup_{y \in [0,\, q]} \psi(y) \left(\int_0^b p(s)\, ds \right) := l\,.$$

Step 3: *N_1 maps bounded sets into equicontinuous sets of* $C(J_1,\, E)$.

Let $t_1,\, t_2 \in J,\, t_1 < t_2$ and $B_q = \{ y \in C(J_1,\, E) : \|y\|_\infty \leqslant q\}$ be a bounded set of $C(J_1,\, E)$.

For each $y \in B_q$ and $h \in N_1 y$, there exists $g \in S_{F,\, y}$ such that

$$h(t) = \int_0^t AT(t-s)\, f(s,\, y_s)\, ds + \int_0^t T(t-s)\, g(s)\, ds\,, \qquad t \in J\,.$$

Thus

$$\begin{aligned}\|h(t_2) - h(t_1)\| \leqslant & \left\| \int_0^{t_2} [AT(t_2 - s) - AT(t_1 - s)]\, f(s,\, y_s)\, ds \right\| \\ & + \left\| \int_{t_1}^{t_2} AT(t_1 - s)\, f(s,\, y_s)\, ds \right\| \\ & + \left\| \int_0^{t_2} [T(t_2 - s) - T(t_1 - s)]\, g(s)\, ds \right\| \\ & + \left\| \int_{t_1}^{t_2} T(t_1 - s)\, g(s)\, ds \right\| \\ \leqslant & \int_0^{t_2} \|A[T(t_2 - s) - T(t_1 - s)]\| (c_1 \|y_s\| + c_2)\, ds \\ & + \int_{t_1}^{t_2} \|AT(t_1 - s)\| (c_1 \|y_s\| + c_2)\, ds\end{aligned}$$

$$+\int_0^{t_2} \|T(t_2-s)-T(t_1-s)\| \, \|g(s)\| \, ds$$

$$+\int_{t_1}^{t_2} \|T(t_1-s)\| \, \|g(s)\| \, ds \, .$$

As $t_2 \to t_1$ the right-hand side of the above inequality tends to zero.

The equicontinuity for the cases $t_1 < t_2 \leqslant 0$ and $t_1 \leqslant 0 \leqslant t_2$ are obvious.

As a consequence of Step 2, Step 3, (H5) and (H6) together with the Ascoli-Arzela theorem we can conclude that $N : C(J_1, E) \to 2^{C(J_1, E)}$ is a compact multivalued map, and therefore, a condensing map.

Step 4: *N has a closed graph.*

Let $y_n \to y_*$, $h_n \in Ny_n$, and $h_n \to h_*$. We shall prove that $h_* \in Ny_*$. $h_n \in Ny_n$ means that there exists $g_n \in S_{F, y_n}$ such that

$$h_n(t) = T(t)[\phi(0) - f(0, \phi)] + f(t, y_{nt}) + \int_0^t AT(t-s) \, f(t, y_{ns}) \, ds$$

$$+\int_0^t T(t-s) \, g_n(s) \, ds \, , \qquad t \in J \, .$$

We must prove that there exists $g_* \in S_{F, y_*}$ such that

$$h_*(t) = T(t)[\phi(0) - f(0, \phi)] + f(t, y_{*t}) + \int_0^t AT(t-s) \, f(t, y_{*s}) \, ds$$

$$+\int_0^t T(t-s) \, g_*(s) \, ds, \qquad t \in J \, .$$

Since f is continuous we have that

$$\Bigg\| \left(h_n - T(t)[\phi(0) - f(0, \phi)] - f(t, y_{nt}) - \int_0^t AT(t-s) \, f(t, y_{ns}) \, ds \right)$$

$$- \left(h_* - T(t)[\phi(0) - f(0, \phi)] - f(t, y_{*t}) - \int_0^t AT(t-s) \, f(t, y_{*s}) \, ds \right) \Bigg\|_\infty \to 0 \, ,$$

as $n \to \infty$.

Consider the linear continuous operator

$$\Gamma : L^1(J, E) \to C(J, E)$$

$$g \mapsto \Gamma(g)(t) = \int_0^t T(t-s)\, g(s)\, ds\,.$$

From Lemma 3.3, it follows that $\Gamma \circ S_F$ is a closed graph operator.

Moreover, we have that

$$h_n(t) - T(t)[\phi(0) - f(0, \phi)] - f(t, y_{nt}) - \int_0^t AT(t-s)\, f(t, y_{ns})\, ds \in \Gamma(S_{F, y_n})\,.$$

Since $y_n \to y_*$, it follows from Lemma 3.3 that

$$h_*(t) - T(t)[\phi(0) - f(0, \phi)] - f(t, y_{*t}) - \int_0^t AT(t-s)\, f(t, y_{*s})\, ds$$

$$= \int_0^t T(t-s)\, g_*(s)\, ds$$

for some $g_* \in S_{F, y_*}$.

Therefore N is a completely continuous multivalued map, u.s.c. with convex closed values. In order to prove that N has a fixed point, we need one more step.

Step 5: *The set*

$$\Omega := \{ y \in C(J_1, E) : \lambda y \in Ny,\ \lambda > 1 \}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in Ny$ for some $\lambda > 1$. Thus there exists $g \in S_{F, y}$ such that

$$y(t) = \lambda^{-1} T(t)[\phi(0) - f(0, \phi)] + \lambda^{-1} f(t, y_t) + \lambda^{-1} \int_0^t AT(t-s)\, f(s, y_s)\, ds$$

$$+ \lambda^{-1} \int_0^t T(t-s)\, g(s)\, ds, \quad t \in J\,.$$

This implies by (H1), (H2) and (H4) that for each $t \in J$ we have

$$\|y(t)\| \leqslant M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_1\|y_t\| + c_2 + M_2\int_0^t (c_1\|y_s\| + c_2)\, ds$$

$$+M_1\int_0^t p(s)\, \psi(\|y_s\|)\, ds\,.$$

We consider the function μ defined by

$$\mu(t) = \sup\{\,|y(s)|\, :\, -r \leqslant s \leqslant t\}\,, \qquad 0 \leqslant t \leqslant b\,.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the previous inequality we have for $t \in J$

$$\mu(t) \leqslant M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_1\mu(t) + c_2 + M_2\int_0^{t^*} (c_1\mu(s) + c_2)\, ds$$

$$+M_1\int_0^{t^*} p(s)\, \psi(\mu(s))\, ds$$

$$\leqslant M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_1\mu(t) + c_2 + M_2 c_1\int_0^t \mu(s)\, ds + M_2 c_2 b$$

$$+M_1\int_0^t p(s)\, \psi(\mu(s))\, ds\,.$$

Thus

$$\mu(t) \leqslant \frac{1}{1-c_1}\Bigg\{M_1[(1+c_1)\|\phi\| + c_2] + c_2 + M_2 c\, 2b + M_2 c_1\int_0^t \mu(s)\, ds$$

$$+M_1\int_0^t p(s)\, \psi(\mu(s))\, ds\Bigg\}, \quad t \in J\,.$$

If $t^* \in J_0$ then $\mu(t) = \|\phi\|$ and the previous inequality holds, since $M_1 \geqslant 1$.

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$c = v(0) = \frac{1}{1-c_1}\{M_1[(1+c_1)\|\phi\| + c_2] + c_2 + M_2 c_2 b\}\,,$$

$$\mu(t) \leqslant v(t)\,, \qquad t \in J\,,$$

and

$$v'(t) = \frac{1}{1-c_1} M_2 c_1 \mu(t) + \frac{1}{1-c_1} M_1 p(t)\psi(\mu(t)), \qquad t \in J .$$

Using the nondecreasing character of ψ we get

$$\begin{aligned} v'(t) &\leqslant \frac{1}{1-c_1} M_2 c_1 v(t) + \frac{1}{1-c_1} M_1 p(t)\, \psi(v(t)) \\ &\leqslant \widehat{m}(t)[v(t) + \psi(v(t))], \qquad t \in J . \end{aligned}$$

This implies for each $t \in J$ that

$$\int_{v(0)}^{v(t)} \frac{du}{u + \psi(u)} \leqslant \int_0^b \widehat{m}(s)\, ds < \int_{v(0)}^{\infty} \frac{du}{u+\psi(u)} .$$

This inequality implies that there exists a constant L such that $v(t) \leqslant L$, $t \in J$, and hence $\mu(t) \leqslant L$, $t \in J$. Since for every $t \in J$, $\|y_t\| \leqslant \mu(t)$, we have

$$\|y\|_\infty := \sup\{\,|y(t)|\,:\, -r \leqslant t \leqslant b\} \leqslant L ,$$

where L depends only on b and on the functions p and ψ. This shows that Ω is bounded. Set $X := C(J_1, E)$. As a consequence of Lemma 2.1 we deduce that N has a fixed point which is a solution of (1)-(2).

4. – Existence results for neutral functional integrodifferential inclusions.

In this section we consider the solvability of IVP (3)-(4).

Let us list the following hypotheses:

(H7) for each $t \in J$, $K(t, s)$ is measurable on $[0, t]$ and

$$K(t) = \operatorname{ess\,sup}\{\,|K(t, s)|,\ 0 \leqslant s \leqslant t\},$$

is bounded on J;

(H8) the map $t \mapsto K_t$ is continuous from J to $L^\infty(J, \mathbb{R})$; here $K_t(s) = K(t, s)$;

(H9) $\|F(t, u)\| := \sup\{\,|v|\,:\, v \in F(t, u)\} \leqslant p(t)\, \psi(\|u\|)$ for almost all $t \in J$ and all $u \in C(J_0, E)$, where $p \in L^1(J, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \to (0, \infty)$ is continuous and increasing with

$$b \sup_{t \in J} K(t) \int_0^b \overline{m}(s)\, ds < \int_{\bar{c}}^{\infty} \frac{d\tau}{\tau + \psi(\tau)} ;$$

where

$$\overline{c} = \frac{1}{1-c_1}\{M_1[(1+c_1)\|\phi\|+c_2]+c_2]+M_2 c_2 b\}$$

and

$$\overline{m}(t) = \max\left\{\frac{1}{1-c_1}M_2 c_1,\ \frac{1}{1-c_1}M_1 b \sup_{t\in J} K(t)\right\}.$$

(H10) for each bounded $B \subset C(J_1, E)$, $y \in B$ and $t \in J$ the set

$$\left\{\int_0^t T(t-s)\int_0^s K(s,\sigma)\, g(\sigma)\, d\sigma ds : g \in S_{F,y}\right\}$$

is relatively compact.

We define the mild solution for the problem (3)-(4) by the integral equation

$$y(t) = T(t)[\phi(0) - f(0,\phi)] + f(t, y_t) + \int_0^t AT(t-s)\, f(s, y_s)\, ds$$
$$+\int_0^t T(t-s)\int_0^s K(s,u)\, g(u)\, du\, ds, \qquad t \in J \tag{6}$$

where

$$g \in S_{F,y} = \{g \in L^1(J,E) : g(t) \in F(t, y_t) \text{ for a.e. } t \in J\}.$$

DEFINITION 4.1. – *A function $x:(-r,b)\to E$, $b>0$ is called a mild solution of the Cauchy problem* (3)-(4) *if $y_0 = \phi$, the function $AT(t-s)\, h(s, x_s)$, $s\in[0,t)$, is integrable and the integral equation* (6) *is satisfied.*

Now, we are able to state and prove our main theorem.

THEOREM 4.2. – *Assume that hypotheses* (H1), (H2), (H3), (H5), (H7)-(H10) *are satisfied. Then the* IVP (3)-(4) *has at least one mild solution on J_1.*

PROOF. – Let $C(J_1, E)$ be the Banach space of continuous functions from J_1 into E endowed with the sup-norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in [-r, b]\}, \qquad \text{for } y \in C(J_1, E).$$

Transform the problem into a fixed point problem. Consider the multivalued map, $N : C(J_1, E) \to 2^{C(J_1,E)}$ defined by:

For each $h \in Ny$ set

$$h(t) = \begin{cases} \phi(t), & \text{if } t \in J_0 \\ T(t)[\phi(0) - f(0, \phi)] + f(t, y_t) + \int_0^t AT(t-s)\, f(s, y_s)\, ds & \\ \quad + \int_0^t T(t-s) \int_0^s K(s, u)\, g(u)\, du\, ds\,, & \text{if } t \in J\,. \end{cases}$$

REMARK 4.3. – *It is clear that the fixed points of N are solutions to* (3)-(4).

As in Theorem 3.4 we can show (with obvious modifications) that N is a completely continuous multivalued map, u.s.c. with convex closed values, and therefore a condensing map.

Here we repeat only the proof that the set

$$\Omega := \{ y \in C(J_1, E) : \lambda y \in N(y),\ \lambda > 1 \}$$

is bounded.

Let $y \in \Omega$. Then $\lambda y \in Ny$ for some $\lambda > 1$. Thus there exists $g \in S_{F,y}$ such that

$$y(t) = \lambda^{-1} T(t)[\phi(0) - f(0, \phi)] + \lambda^{-1} f(t, y_t) + \lambda^{-1} \int_0^t AT(t-s)\, f(s, y_s)\, ds$$

$$\lambda^{-1} \int_0^t T(t-s) \int_0^s K(s, u)\, g(u) du\, ds\,, \quad t \in J\,.$$

This implies by (H1), (H2), (H7)-(H9) that for each $t \in J$ we have

$$\begin{aligned} \|y(t)\| &\leq M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_1\|y_t\| + c_2 + M_2 \int_0^t (c_1\|y_s\| + c_2)\, ds \\ &\quad + M_1 \left\| \int_0^t \int_0^s K(s, u)\, g(u)\, du\, ds \right\| \\ &\leq M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_1\|y_t\| + c_2 + M_2 \int_0^t (c_1\|y_s\| + c_2)\, ds \\ &\quad + M_1 \int_0^t \int_0^s |K(s, u)|\, p(u)\, \psi(\|y_u\|)\, du\, ds \end{aligned}$$

$$\leqslant M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_1\|y_t\| + c_2 + M_2\int_0^t (c_1\|y_s\| + c_2)\, ds$$

$$+ M_1 b \sup_{t\in J} K(t) \int_0^t p(s)\, \psi(\|y_s\|)\, ds\,.$$

We consider the function μ defined by

$$\mu(t) = \sup\{\,|y(s)|\,:\, -r \leqslant s \leqslant t\}, \qquad 0 \leqslant t \leqslant b\,.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the previous inequality we have for $t \in J$

$$\mu(t) \leqslant M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_1\mu(t) + c_2 + M_2 c_1\int_0^{t^*} \mu(s)\, ds + M_2 c_2 b$$

$$+ M_1 b \sup_{t\in J} K(t) \int_0^{t^*} p(s)\, \psi(\mu(s))\, ds$$

$$\leqslant M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_1\mu(t) + c_2 + M_2 c_1\int_0^{t} \mu(s)\, ds + M_2 c_2 b$$

$$+ M_1 b \sup_{t\in J} K(t) \int_0^{t} p(s)\, \psi(\mu(s))\, ds\,.$$

Thus

$$\mu(t) \leqslant \frac{1}{1-c_1}\Big\{ M_1[(1+c_1)\|\phi\| + c_2] + c_2 + M_2 c_2 b +$$

$$+ M_2 c_1 \int_0^t \mu(s) ds + M_1 b \sup_{t\in J} K(t) \int_0^t p(s)\, \psi(\mu(s))\, ds \Big\}, \qquad t\in J\,.$$

If $t^* \in J_0$ then $\mu(t) = \|\phi\|$ and the previous inequality holds since $M_1 \geqslant 1$.

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$c = v(0) = \frac{1}{1-c_1}\{M_1[(1+c_1)\|\phi\| + c_2] + c_2 + M_2 c_2 b\},$$

$$\mu(t) \leqslant v(t), \qquad t\in J\,,$$

and

$$v'(t) = \frac{1}{1-c_1} M_2 c_1 \mu(t) + \frac{1}{1-c_1} M_1 b \sup_{t\in J} K(t)\, p(t)\, \psi(\mu(t)), \qquad t \in J\,.$$

Using the nondecreasing character of ψ we get

$$v'(t) \leqslant \frac{1}{1-c_1} M_2 c_1 v(t) + \frac{1}{1-c_1} M_1 b \sup_{t \in J} K(t)\, p(t)\, \psi(v(t))$$

$$\leqslant \overline{m}(t)[v(t) + \psi(v(t))], \quad t \in [0, b].$$

This implies for each $t \in J$ that

$$\int_{v(0)}^{v(t)} \frac{du}{u + \psi(u)} \leqslant \int_0^b \overline{m}(s)\, ds < \int_{v(0)}^{\infty} \frac{du}{u + \psi(u)}.$$

This inequality implies that there exists a constant $\overline{L}$ such that $v(t) \leqslant \overline{L}$, $t \in J$, and hence $\mu(t) \leqslant \overline{L}$, $t \in J$. Since for every $t \in J$, $\|y_t\| \leqslant \mu(t)$, we have

$$\|y\|_\infty := \sup\{\,|y(t)|: -r \leqslant t \leqslant b\} \leqslant \overline{L},$$

where $\overline{L}$ depends only on b and on the functions p and ψ. This shows that Ω is bounded.

Set $X := C(J_1, E)$. As a consequence of Lemma 2.1 we deduce that N has a fixed point which is a solution of (3)-(4).

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M. Benchohra: Département de Mathématiques, Université de Sidi Bel Abbès,
BP 89, 22000 Sidi Bel Abbès, Algérie

S. K. Ntouyas: Department of Mathematics, University of Ioannina,
451 10 Ioannina, Greece

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