Bollettino Unione Matematica Italiana

Wolfgang Woess

Erratum: Heat diffusion on homogeneous trees. Note on a paper by G. Medolla and A. G. Setti: Long time heat diffusion on homogeneous trees

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **5-B** (2002), n.1, p. 259–260.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2002_8_5B_1_259_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2002.

Bollettino U. M. I.
(8) 5-B (2002), 259-260

Erratum

Heat Diffusion on Homogeneous Trees (Note on a Paper by Medolla and Setti)

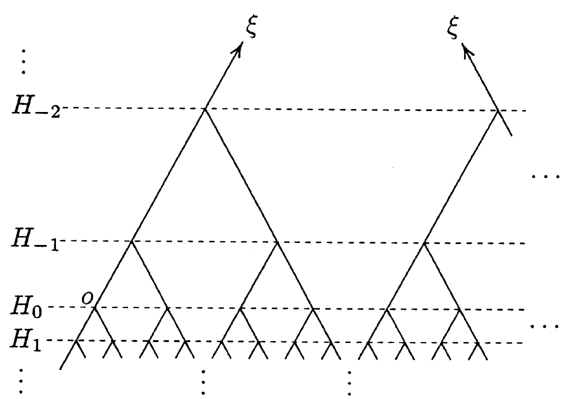
WOLFGANG WOESS

(Bollettino UMI (8) 4-B (2001), 703-709)

Due to a technical problem, some symbols have disappeared in printing.

Considering the distances $d(x, o)$ corresponds to computing with polar coordinates. We now change to «horospheric coordinates». We choose and fix a point ξ in $\mathbb{T}$, and define $\widehat{\mathbb{T}}^* = \widehat{\mathbb{T}} \setminus \{\xi\}$ and $\partial^* \mathbb{T} = \partial \mathbb{T} \setminus \{\xi\}$. If $\eta \in \partial^* \mathbb{T}$ then there is a unique two-sided infinite path $[\ldots, x_{-1}, x_0, x_1, x_2, \ldots]$, denoted $\overline{\xi\eta}$, such that $[x_0, x_1, x_2, \ldots]$ represents the end η and $[x_0, x_{-1}, x_{-2}, \ldots]$ represents ξ. Again, given $v, w \in \widehat{\mathbb{T}}^*$, their *confluent* $v \sqcap w$ *with respect to* ξ is defined by $\overline{v\xi} \cap \overline{w\xi} = \overline{(v \sqcap w)\, \xi}$. Again, this is a vertex, unless $v = w \in \partial^* \mathbb{T}$.

$\mathbb{T}_2$ *in horocyclic layers*



For $x \in \mathbb{T}$, its *height* with respect to ξ is $\mathrm{hor}(x) = d(x, x \sqcap o) - d(x \sqcap o, o)$. This is an integer (not necessesarily positive). The level sets $H_k = \{x \in \mathbb{T} : \mathrm{hor}(x) = k\}$ are the *horocycles.*

LEMMA 2. – *The process* $Y_t = \mathrm{hor}(X_t)$ *is an integer-valued Markov process with the following properties*:

(i) independent increments: *whenever* $0 \leq t_0 < t_1 < \ldots < t_r$, *the random variables* $Y_{t_1} - Y_{t_0}, \ldots, Y_{t_r} - Y_{t_{r-1}}$ *are independent.*

(ii) additivity: *For* $s < t$, *the distribution of* $Y_t - Y_s$ *coincides with that of* Y_{t-s}.

(iii) Y_t *has expected value* $\mathfrak{m}t$ *and variance* t.

PROOF. – Recall that an automorphism of $\mathbb{T}$ is a self-isometry of $\mathbb{T}$ with respect to the graph metric. The action of each automorphism extends continuously to $\partial\mathbb{T}$. Let Γ be the group of automorphisms that fix the end ξ. Then $\mathcal{H}_t$ is Γ-invariant, that is, $h_t(\gamma x, \gamma y) = h_t(x, y)$ for every $\gamma \in \Gamma$. Also, $\mathrm{hor}(\gamma y) - \mathrm{hor}(\gamma x) = \mathrm{hor}(y) - \mathrm{hor}(x)$, and the map $\gamma \mapsto \mathrm{hor}(\gamma x) - \mathrm{hor}(x)$ is independent of $x \in \mathbb{T}$ and a homomorphism $\Gamma \to (\mathbb{Z}, +)$. Therefore, the probability $\Pr[\mathrm{hor}(X_t) = l \mid X_s = x]$ depends only on $t - s$ and $\mathrm{hor}(x)$. From here, (i) and (ii) are straightforward.

(iii) is a computational exercise. As a hint, let μ be the the distribution of the integer-valued random variable $\mathrm{hor}(Z_1)$, given that $Z_0 = 0$. That is, μ has support $\{-1, 1\}$, with $\mu(1) = q/(q+1)$ and $\mu(-1) = 1/(q+1)$. Then the distribution of Y_t is

$$\mu_t = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \mu^{(k)},$$

where $\mu^{(k)}$ is the k-th convolution power of μ. ■

Thus, $(Y_t)_{t \geq 0}$ is a continuous-time analogue of a sum of i.i.d. integer valued random variables, and the characteristic function φ_t of Y_t (the Fourier transform of μ_t) satisfies $\varphi_t = \varphi_1^t$. We have the law of large numbers and central limit theorem as stated in Theorem 1, with Y_t in the place of $d(X_t, o)$. Thus, the following Lemma provides the last step in our proof of Theorem 1.

LEMMA 3. – *As* $t \to \infty$, *the process* $d(X_t, o) - Y_t$ *converges almost surely to the almost surely finite random variable* $2d(o, X_\infty \sqcap o)$.

PROOF. – Let $\eta \in \partial^* \mathbb{T}$, and suppose that $x_t \in \mathbb{T}$ and $(x_t)_{t \geq 0}$ converges to η in the topology of $\partial\mathbb{T}$, when $t \to \infty$. Then there must be t_0 such that $x_t \sqcap o = \eta \sqcap o$ for all $t \geq t_0$. (In particular, $\mathrm{hor}(x_t) \to \infty$.) But then

$$d(x_t, o) = \mathrm{hor}(x_t) + 2d(o, \eta \sqcap o) \qquad \text{for all } t \geq t_0.$$

Now observe that $\Pr[X_\infty = \xi] = 0$, since the distribution ν of X_∞ on $\mathbb{T}$ is equidistribution, that is,

$$\nu(\{\eta \in \partial\mathbb{T} : \theta(\eta, \xi) < e^{-n}\}) = \nu(\{\eta \in \partial\mathbb{T} : d(\eta \wedge \xi, o) \geq n+1\}) = \frac{1}{(q+1)\, q^n}$$

for $n \in \mathbb{N}$. This means that $X_\infty \in \partial^* \mathbb{T}$ almost surely, and given $\omega \in \Omega$ such that $X_\infty(\omega) \in \partial^* \mathbb{T}$, we can apply the above argument to $x_t = X_t(\omega)$.

As a matter of fact, we have shown that almost surely, there is a (random) t_0 such that $d(X_t, o) = Y_t + 2d(o, X_\infty \sqcap o)$ for all $t \geq t_0$. ■