Bollettino
Unione Matematica Italiana

Mario Petrich

Some relations on the lattice of varieties of completely regular semigroups

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **5-B** (2002), n.2, p. 265–278.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2002_8_5B_2_265_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2002.

Bollettino U. M. I.
(8) 5-B (2002), 265-278

Some Relations on the Lattice of Varieties of Completely Regular Semigroups

MARIO PETRICH

Sunto. – *Nel reticolo $\mathcal{L}(\mathcal{CR})$ delle varietà dei semigruppi completamente regolari, considerati come algebre con la moltiplicazione binaria e l'inversione unaria tra i sottogruppi massimali, si studiano le relazioni K_l, K, K_r, T_l, T, T_r, C e L. Qui K denota la relazione nucleo, T la relazione traccia, T_l e T_r le relazioni traccia sinistra e destra rispettivamente, $K_p = K \cap T_p$ per $p \in \{l, r\}$, C la relazione core ed L la relazione locale. Viene data una definizione alternativa per ciascuna di queste relazioni P nella forma*

$$\mathcal{U}\ P\ \mathcal{V} \Leftrightarrow \mathcal{U} \cap \tilde{P} = \mathcal{V} \cap \tilde{P} \qquad (\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})),$$

per alcune sottoclassi $\tilde{P}$ di $\mathcal{CR}$. Si caratterizzano inoltre le intersezioni di queste relazioni ed alcuni dei loro join nel reticolo delle equivalenze su $\mathcal{L}(\mathcal{CR})$.

Summary. – *On the lattice $\mathcal{L}(\mathcal{CR})$ of varieties of completely regular semigroups considered as algebras with the binary multiplication and unary inversion within maximal subgroups, we study the relations K_l, K, K_r, T_l, T, T_r, C and L. Here K is the kernel relation, T is the trace relation, T_l and T_r are the left and the right trace relations, respectively, $K_p = K \cap T_p$ for $p \in \{l, r\}$, C is the core relation and L is the local relation. We give an alternative definition for each of these relations P of the form*

$$\mathcal{U}\ P\ \mathcal{V} \Leftrightarrow \mathcal{U} \cap \tilde{P} = \mathcal{V} \cap \tilde{P} \qquad (\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})),$$

for some subclasses $\tilde{P}$ of $\mathcal{CR}$. We also characterize the intersections of these relations and some joins within the lattice of equivalence relations on $\mathcal{L}(\mathcal{CR})$.

1. – Introduction and summary.

Completely regular semigroups (that is those semigroups which are unions of their subgroups) are considered here as algebras S with the binary operation of multiplication and the unary operation of inversion $a \to a^{-1}$, where a^{-1} is the inverse of a in the maximal subgroup of S to which a belongs. The class $\mathcal{CR}$ of all such algebras, henceforth called completely regular semigroups, forms a variety. For any subvariety $\mathcal{V}$ of $\mathcal{CR}$, we denote by $\mathcal{L}(\mathcal{V})$ the lattice of subvarieties of $\mathcal{V}$.

The global study of $\mathcal{L}(\mathcal{CR})$ is generally based on consideration of certain relations or operators enjoying some useful properties. Firstly we may transfer the kernel and the (left, right) trace on the lattice of fully invariant congruences on a free completely regular semigroup of countably infinite rank to $\mathcal{L}(\mathcal{CR})$ via the usual antiisomorphism, thereby obtaining the relations K, T_l, T and T_r on $\mathcal{L}(\mathcal{CR})$. These relations may be defined directly on $\mathcal{L}(\mathcal{CR})$ in view of some known results. Secondly the core operator C and the local operator L on $\mathcal{L}(\mathcal{CR})$ can be used to define the corresponding relations, denoted here again by C and L, respectively. All of these relations can be defined directly using the intersection with certain subclasses of $\mathcal{CR}$.

The purpose of this note is to establish the following results. In Theorem 1, we prove that K can be given by

$$\mathcal{U}\ K\ \mathcal{V} \Leftrightarrow \mathcal{U} \cap \widetilde{K} = \mathcal{V} \cap \widetilde{K} \qquad (\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR}))$$

for a well specified subclass $\widetilde{K}$ of $\mathcal{CR}$. Partly from known results, we deduce in Theorem 2 that the same form is valid for the relations T_l, T, T_r, C and L for suitable subclasses $\widetilde{T}_l$, $\widetilde{T}$, $\ldots$ of $\mathcal{CR}$. We also characterize the upper ends for some $\mathcal{V}P$, the P-class containing $\mathcal{V}$, where P denotes any of the above relations. In all cases, we prove that the lower end of the class $\mathcal{V}P$ is of the form $\langle \mathcal{V} \cap \widetilde{P} \rangle$, the variety generated by $\mathcal{V} \cap \widetilde{P}$. In Theorem 3 setting $K_p = K \cap T_p$ for $p \in \{l, r\}$, we construct the joins of some of these relations.

We follow the standard terminology and notation which can be found in the usual texts on semigroups.

2. – The kernel relation.

Let S be a regular semigroup, $E(S)$ its set of idempotents and $\mathcal{C}(S)$ its congruence lattice. For any $\varrho \in \mathcal{C}(S)$, the set

$$\ker \varrho = \{a \in S \mid a\ \varrho\ e \text{ for some } e \in E(S)\}$$

is the *kernel* of ϱ. On $\mathcal{C}(S)$ we define the *kernel relation* K by

$$\lambda\ K\ \varrho \quad \text{if} \quad \ker \lambda = \ker \varrho\,.$$

Then K is a complete $\cap$-congruence on $\mathcal{C}(S)$.

For S a free completely regular semigroup on an infinite set X, we denote by ζ the usual antiisomorphism of $\mathcal{L}(\mathcal{CR})$ onto the lattice of fully invariant congruences on S, writing $\zeta : \mathcal{V} \to \zeta_{\mathcal{V}}$. The relation K on $\mathcal{C}(S)$ can now be transferred from $\mathcal{C}(S)$ to $\mathcal{L}(\mathcal{CR})$ by defining a relation, again denoted by K, on

$\mathcal{L}(CR)$ by

$$\mathcal{U}\ K\ \mathcal{V} \quad \text{if} \quad \zeta_{\mathcal{U}}\ K\ \zeta_{\mathcal{V}}.$$

Then K is a complete congruence on $\mathcal{L}(\mathcal{CR})$, see ([3], Theorem 11).

We denote by $\mathcal{G}$, $\mathcal{CS}$, $\mathcal{R}e\mathcal{G}$, $\mathcal{NBG}$ and $\mathcal{O}$ the varieties of groups, completely simple semigroups, rectangular groups, normal bands of groups and orthogroups, respectively.

Let ε denote the equality relation on any set. For any regular semigroup S, a congruence ϱ on S is *idempotent pure if* $\ker \varrho = E(S)$. We denote by τ the greatest idempotent pure congruence on any regular semigroup S; if $\tau = \varepsilon$, S is said to be *E-disjunctive.* We write $S = (Y; S_\alpha)$ to indicate that the completely regular semigroup S is a semilattice Y of its completely simple components S_α.

For $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, let

$$\mathcal{V}^L = \{S \in \mathcal{CR} \mid eSe \in \mathcal{V} \text{ for all } e \in E(S)\};$$

we say that the semigroups in $\mathcal{V}^L$ are *locally* (*in*) $\mathcal{V}$. In particular $\mathcal{O}^L$ denotes the variety of all locally orthodox completely regular semigroups. Denote by $\mathcal{D}$ the class of all E-disjunctive completely regular semigroups and let

$$\widetilde{K} = \mathcal{G} \cup (\mathcal{CS} \setminus \mathcal{R}e\mathcal{G}) \cup (\mathcal{D} \cap (\mathcal{CR} \setminus \mathcal{O}^L)).$$

We are now ready for our first result.

THEOREM 1. – *For any* $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$, *we have*

$$\begin{aligned}
&\mathcal{U} \cap \widetilde{K} = \mathcal{V} \cap \widetilde{K} \\
\Leftrightarrow &\begin{cases} \mathcal{U} \cap \mathcal{G} = \mathcal{V} \cap \mathcal{G} & \text{if } \mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{O}) \\ \mathcal{U} \cap \mathcal{CS} = \mathcal{V} \cap \mathcal{CS} & \text{if } \mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{O}^L) \setminus \mathcal{L}(\mathcal{O}) \\ \mathcal{U} \cap \mathcal{D} = \mathcal{V} \cap \mathcal{D} & \text{if } \mathcal{U}, \mathcal{V} \notin \mathcal{L}(\mathcal{O}^L) \end{cases} \\
\Leftrightarrow &\ \mathcal{U}\ K\ \mathcal{V}.
\end{aligned}$$

PROOF. – Assume first that $\mathcal{U} \cap \widetilde{K} = \mathcal{V} \cap \widetilde{K}$. We consider tree cases.

Case: $\mathcal{U} \in \mathcal{L}(\mathcal{O})$. Then $\mathcal{U} \cap \widetilde{K} = \mathcal{U} \cap \mathcal{G}$ whence

$$\mathcal{V} \cap \widetilde{K} = (\mathcal{V} \cap \mathcal{G}) \cup (\mathcal{V} \cap (\mathcal{CS} \setminus \mathcal{R}e\mathcal{G})) \cup (\mathcal{V} \cap \mathcal{D} \cap (\mathcal{CR} \setminus \mathcal{O}^L)) \subseteq \mathcal{G}$$

which implies that

$$\mathcal{V} \cap (\mathcal{CS} \setminus \mathcal{R}e\mathcal{G}) = \emptyset. \tag{1}$$

Let $S = (Y; S_\alpha) \in \mathcal{V}$. Then $S_\alpha \in \mathcal{V} \cap \mathcal{CS}$ which by (1) implies that $S_\alpha \in \mathcal{R}e\,\mathcal{G}$. By ([5], Lemma 1), we get that $S \in \mathcal{O}$. Hence $\mathcal{V} \subseteq \mathcal{O}$. But then $\mathcal{V} \cap \widetilde{K} = \mathcal{V} \cap \mathcal{G}$ and thus $\mathcal{U} \cap \mathcal{G} = \mathcal{V} \cap \mathcal{G}$.

Case: $\mathcal{U} \in \mathcal{L}(\mathcal{O}^L) \setminus \mathcal{L}(\mathcal{O})$. Then $\mathcal{U} \cap \widetilde{K} = \mathcal{U} \cap (\mathcal{CS} \setminus \mathcal{R}e\mathcal{G})$ so that $\mathcal{V} \cap \widetilde{K} \subseteq \mathcal{CS}$ and thus

$$(2) \qquad \mathcal{V} \cap \mathcal{D} \cap (\mathcal{CR} \setminus \mathcal{O}^L) = \emptyset\,.$$

Let $S \in \mathcal{V}$. In view of ([8], Lemma 3.1(ii)), we get $S/\tau \in \mathcal{D}$ and hence $S/\tau \in \mathcal{V} \cap \mathcal{D}$. By (2), we have that $S/\tau \in \mathcal{O}^L$ which by ([10], Corollary 7.4(iii)) gives $(S/\tau)/\tau \in \mathcal{N}\mathcal{B}\mathcal{G}$. By ([8], Lemma 3.1(ii)) again we have $(S/\tau) \cong (S/\tau)/\tau$ so that $S/\tau \in \mathcal{N}\mathcal{B}\mathcal{G}$. But again by ([10], Corollary 7.4(iii)), we obtain that $S \in \mathcal{O}^L$. Therefore $\mathcal{V} \subseteq \mathcal{O}^L$ and hence

$$(3) \qquad \mathcal{U} \cap (\mathcal{G} \cup (\mathcal{CS} \setminus \mathcal{R}e\,\mathcal{G})) = \mathcal{U} \cap \widetilde{K} = \mathcal{V} \cap \widetilde{K} = \mathcal{V} \cap (\mathcal{G} \cup (\mathcal{CS} \setminus \mathcal{R}e\,\mathcal{G}))\,.$$

Since $\mathcal{U} \notin \mathcal{L}(\mathcal{O})$, there exists $S \in \mathcal{U}$ such that $S \notin \mathcal{O}$. Hence $S = (Y; S_\alpha)$ and $S_\alpha \notin \mathcal{R}e\,\mathcal{G}$ for some $\alpha \in Y$ in view of ([5], Lemma 1). It follows that $S_\alpha/\mathcal{H}$ is a rectangular band which is neither a left nor a right zero semigroup. Thus $\mathcal{R}\mathcal{B} \subseteq \mathcal{U}$. Also $S_\alpha \in \mathcal{U} \cap (\mathcal{CS} \setminus \mathcal{R}e\,\mathcal{G})$ and thus, by (3), we have $S_\alpha \in \mathcal{V}$ which as above leads to $\mathcal{R}\mathcal{B} \subseteq \mathcal{V}$.

Next let $S \in \mathcal{U} \cap \mathcal{CS}$. If $S \in \mathcal{G} \cup (\mathcal{CS} \setminus \mathcal{R}e\,\mathcal{G})$, then by (3), we get $S \in \mathcal{V}$. If $S \in \mathcal{R}e\,\mathcal{G}$, then $S \cong B \times G$ for a rectangular band B and a group G. Hence $G \in \mathcal{U} \cap \mathcal{G}$ so by (3), we have $G \in \mathcal{V}$. We have seen above that $\mathcal{R}\mathcal{B} \subseteq \mathcal{V}$ whence $B \times G \in \mathcal{V}$. Therefore $S \in \mathcal{V}$ which proves that $\mathcal{U} \cap \mathcal{CS} \subseteq \mathcal{V}$. A symmetrical argument can be used to prove that $\mathcal{V} \cap \mathcal{CS} \subseteq \mathcal{U}$. Consequently $\mathcal{U} \cap \mathcal{CS} = \mathcal{V} \cap \mathcal{CS}$.

Case: $\mathcal{U} \notin \mathcal{L}(\mathcal{O}^L)$. By the above two cases we get immediately that also $\mathcal{V} \notin \mathcal{L}(\mathcal{O}^L)$. As in the preceding case, we have $\mathcal{R}\mathcal{B} \subseteq \mathcal{U} \cap \mathcal{V}$. Let $S \in \mathcal{U} \cap \mathcal{D}$. If $S \in \widetilde{K}$, then by hypothesis, we have that $S \in \mathcal{V}$. Let $S \in \mathcal{O}^L$. Since $S \in \mathcal{D}$, as above we conclude that $S \in \mathcal{CS}$. If $S \in \mathcal{R}e\,\mathcal{G}$, then as above we get $S \cong B \times G$ with $G \in \mathcal{U} \cap \widetilde{K}$ whence $G \in \mathcal{V}$ and also $B \in \mathcal{V}$ since $\mathcal{R}\mathcal{B} \subseteq \mathcal{V}$ and thus $S \in \mathcal{V}$. Therefore $S \in \mathcal{V}$ in all cases and we conclude that $\mathcal{U} \cap \mathcal{D} \subseteq \mathcal{V}$. By symmetry, we have also that $\mathcal{V} \cap \mathcal{D} \subseteq \mathcal{U}$ which implies that $\mathcal{U} \cap \mathcal{D} = \mathcal{V} \cap \mathcal{D}$.

We have proved the direct implication of the first equivalence in the statement of the theorem. For the reverse implication, we assume the relevant conditions and argue as follows.

If $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{O})$ and $\mathcal{U} \cap \mathcal{G} = \mathcal{V} \cap \mathcal{G}$, then clearly $\mathcal{U} \cap \widetilde{K} = \mathcal{V} \cap \widetilde{K}$. If $\mathcal{U}$, $\mathcal{V} \in \mathcal{L}(\mathcal{O}^L) \setminus \mathcal{L}(\mathcal{O})$ and $\mathcal{U} \cap \mathcal{CS} = \mathcal{V} \cap \mathcal{CS}$, then

$$\mathcal{U} \cap \mathcal{G} = \mathcal{V} \cap \mathcal{G}, \qquad \mathcal{U} \cap (\mathcal{CS} \setminus \mathcal{R}e\,\mathcal{G}) = \mathcal{V} \cap (\mathcal{CS} \setminus \mathcal{R}e\,\mathcal{G})$$

and thus $\mathcal{U} \cap \widetilde{K} = \mathcal{V} \cap \widetilde{K}$. If $\mathcal{U}, \mathcal{V} \notin \mathcal{L}(\mathcal{O}^L)$ and $\mathcal{U} \cap \mathcal{D} = \mathcal{V} \cap \mathcal{D}$, then

$$\mathcal{U} \cap \mathcal{D} \cap (\mathcal{CS} \setminus \mathcal{O}^L) = \mathcal{V} \cap \mathcal{D} \cap (\mathcal{CS} \setminus \mathcal{O}^L)$$

and thus $\mathcal{U} \cap \widetilde{K} = \mathcal{V} \cap \widetilde{K}$.

This completes the proof of the first equivalence. For the second equivalence, we proceed as follows. First

$$(\mathcal{V} \cap \mathcal{D}) \cap \mathcal{D} \subseteq \mathcal{V} \cap \mathcal{D} \subseteq \langle \mathcal{V} \cap \mathcal{D} \rangle \cap \mathcal{D}$$

and equality prevails throughout. It follows that

$$\langle \mathcal{U} \cap \mathcal{D} \rangle = \langle \mathcal{V} \cap \mathcal{D} \rangle \Leftrightarrow \mathcal{U} \cap \mathcal{D} = \mathcal{V} \cap \mathcal{D}.$$

Thus in the statement of the theorem, we can write

$$\langle \mathcal{U} \cap \mathcal{D} \rangle = \langle \mathcal{V} \cap \mathcal{D} \rangle \quad \text{if} \quad \mathcal{U}, \mathcal{V} \notin \mathcal{L}(\mathcal{O}^L)$$

instead of

$$\mathcal{U} \cap \mathcal{D} = \mathcal{V} \cap \mathcal{D} \quad \text{if} \quad \mathcal{U}, \mathcal{V} \notin \mathcal{L}(\mathcal{O}^L).$$

Now ([8], Theorem 5.8) shows that the second characterization is equivalent to $\mathcal{U}_K = \mathcal{V}_K$ which is evidently equivalent to $\mathcal{U}\ K\ \mathcal{V}$. This establishes the second equivalence.

3. – Further relations.

For any semigroup S and any equivalence relation θ on S, let θ^0 denote the greatest congruence on S contained in θ. Now let S also be regular. For $\varrho \in \mathcal{C}(S)$, the relation

$$tr\ \varrho = \varrho|_{E(S)}$$

is the *trace* of ϱ. We define the *trace relation* T on $\mathcal{C}(S)$ by

$$\lambda\ T\ \varrho \quad \text{if} \quad tr\ \lambda = tr\ \varrho\ .$$

We also define the *left* and the *right traces* of $\varrho \in \mathcal{C}(S)$ by

$$ltr\, \varrho = tr\ (\varrho \vee \mathcal{L})^0, \qquad rtr\, \varrho = tr\ (\varrho \vee \mathcal{R})^0,$$

respectively, where the join is taken within equivalence relations on S. We now define the *left* and *right trace relations* T_l and T_r on $\mathcal{C}(S)$ by

$$\lambda\, T_l\ \varrho \quad \text{if} \quad ltr\ \lambda = ltr\, \varrho\,, \qquad \lambda\ T_r\ \varrho \quad \text{if} \quad rtr\ \lambda = rtr\, \varrho\ ,$$

respectively. All three relations T_l, T and T_r are complete congruences on $\mathcal{C}(S)$, see ([4], Corollary 4.3). As in the case of the kernel relation, the above relations on a free completely regular semigroup S on a infinite set induce relations, again denoted by the same letters, on $\mathcal{L}(\mathcal{CR})$ given by

$$\mathcal{U}\ R\ \mathcal{V} \quad \text{if} \quad \zeta_{\mathcal{U}}\ R\ \zeta_{\mathcal{V}} \qquad (R \in \{T_l, T, T_r\}).$$

All these relations are complete congruences on $\mathcal{L}(\mathcal{CR})$. Set

$$\widetilde{T} = \{S \in \mathcal{CR} \mid \mathcal{H}^0 = \varepsilon\},$$
$$\widetilde{T}_p = \{S \in \mathcal{CR} \mid \mathcal{P}^0 = \varepsilon\} \qquad (p \in \{l, r\}).$$

These are the classes of (left, right) fundamental completely regular semigroups.

For any regular semigroup S, $C(S)$ denotes the *core* of S, that is the subsemigroup of S generated by $E(S)$. For $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, let

$$\mathcal{V}^C = \{S \in \mathcal{CR} \mid C(S) \in \mathcal{V}\}.$$

We can now define the *core relation* C and the *local relation* L on $\mathcal{L}(\mathcal{CR})$ by

$$\mathcal{U}\ C\ \mathcal{V} \quad \text{if} \quad \mathcal{U}^C = \mathcal{V}^C, \qquad \mathcal{U}\ L\ \mathcal{V} \quad \text{if} \quad \mathcal{U}^{\mathrm{L}} = \mathcal{V}^{\mathrm{L}},$$

respectively. Denote by $\widetilde{C}$ the class of idempotent generated completely regular semigroups and by $\widetilde{L}$ the class of completely regular monoids. Both relations C and L are complete congruences on $\mathcal{L}(\mathcal{CR})$, see ([9], Theorems 3.1(ii) and 5.1(ii)).

Since the classes of all the above relations P are intervals, we may write them as

$$\mathcal{V}\mathcal{P} = [\mathcal{V}_P, \mathcal{V}^P] \qquad (\mathcal{V} \in \mathcal{L}(\mathcal{CR})).$$

For $P = C$, the above two usages of $\mathcal{V}^P$ are consistent in view of ([9], Theorem 3.1(ii)). For $P = L$, the notation $\mathcal{O}^L$ and the two usages of $\mathcal{V}^L$ are consistent by ([9], Theorem 5.1(ii)). For $P \in \{K, T_l, T, T_r\}$, $\mathcal{V}^P$ can be expressed by means of the Malcev product, see ([2], Proposition 7.2(ii)) and ([8], Theorems 6.2 and 8.2). We shall characterize $\mathcal{V}_P$ as well as $\mathcal{V}^P$, the latter with the exception of K, in our second theorem.

In order to handle the upper ends of some of the relations under study, we first introduce the following notation. For $S \in \mathcal{CR}$, let Hom (S) be the class of all homomorphic images of S and Sub (S) be the class of all completely regular semigroups isomorphic to some subsemigroup of S. For $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{CR}$, define

$$(\mathcal{X} : \mathcal{Y}) = \{S \in \mathcal{CR} \mid \operatorname{Hom}(S) \cap \mathcal{Y} \subseteq \mathcal{X}\},$$

$$[\mathcal{X} : \mathcal{Y}] = \{S \in \mathcal{CR} \mid \operatorname{Sub}(S) \cap \mathcal{Y} \subseteq \mathcal{X}\},$$

$$\langle \mathcal{X} : \mathcal{Y} \rangle = \{S \in \mathcal{CR} \mid \langle S \rangle \cap \mathcal{Y} \subseteq \mathcal{X}\}.$$

We shall also need the following auxiliar results.

LEMMA 1. – *Let* $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$ *and* $\mathcal{C} \subseteq \mathcal{CR}$.

(i) $\mathcal{V} \subseteq \langle \mathcal{V} : \mathcal{C} \rangle \subseteq (\mathcal{V} : \mathcal{C}) \subseteq [\mathcal{V} : \mathcal{C}]$.

(ii) $\mathcal{V} \cap \mathcal{C} = \mathcal{X} \cap \mathcal{C} \qquad (\mathcal{X} \in \{(\mathcal{V} : \mathcal{C}), [\mathcal{V} : \mathcal{C}], \langle \mathcal{V} : \mathcal{C} \rangle\})$.

(iii) $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$, $\mathcal{U} \cap \mathcal{C} = \mathcal{V} \cap \mathcal{C} \Rightarrow \mathcal{U} \subseteq \langle \mathcal{V} : \mathcal{C} \rangle$.

(iv) $(\mathcal{V} : \mathcal{C})$ *is closed under taking homomorphic images,* $[\mathcal{V} : \mathcal{C}]$ *is closed under taking completely regular subsemigroups and* $\langle \mathcal{V} : \mathcal{C} \rangle$ *under both.*

PROOF. – (i) This follows directly from the definition.

(ii) For $\mathcal{X} \in \{(\mathcal{V} : \mathcal{C}), [\mathcal{V} : \mathcal{C}]\}$ the inclusions

$$\mathcal{V} \cap \mathcal{C} \subseteq \langle \mathcal{V} : \mathcal{C} \rangle \cap \mathcal{C} \subseteq \mathcal{X} \cap \mathcal{C}$$

follow from part (i); further, if $S \in \mathcal{X} \cap \mathcal{C}$, then $\{S\} = \{S\} \cap \mathcal{C} \subseteq \mathcal{V}$ and thus $\mathcal{X} \cap \mathcal{C} \subseteq \mathcal{V} \cap \mathcal{C}$.

(iii) Assume the antecedent of part (iii) and let $S \in \mathcal{U}$ and $T \in \langle S \rangle \cap \mathcal{C}$. Then $T \in \langle S \rangle \cap \mathcal{U}$ so that $T \in \mathcal{U} \cap \mathcal{C} = \mathcal{V} \cap \mathcal{C} \subseteq \mathcal{V}$ whence $S \in \langle \mathcal{V} : \mathcal{C} \rangle$. It follows that $\mathcal{U} \subseteq \langle \mathcal{V} : \mathcal{C} \rangle$.

(iv) This is obvious. ■

For any $\mathcal{C} \subseteq \mathcal{CR}$, we define a relation $\overline{\mathcal{C}}$ by

$$\mathcal{U} \,\overline{\mathcal{C}}\, \mathcal{V} \quad \text{if} \quad \mathcal{U} \cap \mathcal{C} = \mathcal{V} \cap \mathcal{C} \qquad (\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR}))\,.$$

Clearly $\overline{\mathcal{C}}$ is a complete $\cap$-congruence on $\mathcal{L}(\mathcal{CR})$.

If P is any relation on $\mathcal{L}(\mathcal{CR})$, for $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, we denote by $\mathcal{V}_P$ the least and by $\mathcal{V}^P$ the greatest element of the class $\mathcal{V}P$ if these exist. The next lemma provides some simple properties of these concepts.

LEMMA 2. – *Let* $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$ *and* $\mathcal{C} \subseteq \mathcal{CR}$.

(i) $\mathcal{V}_{\overline{\mathcal{C}}} = \langle \mathcal{V} \cap \mathcal{C} \rangle$.

(ii) If $(\mathcal{V} : \mathcal{C}) \in \mathcal{L}(\mathcal{CR})$, *then* $\mathcal{V}^{\overline{\mathcal{C}}} = (\mathcal{V} : \mathcal{C}) = \langle \mathcal{V} : \mathcal{C} \rangle$.

(iii) If $[\mathcal{V} : \mathcal{C}] \in \mathcal{L}(\mathcal{CR})$, *then* $\mathcal{V}^{\overline{\mathcal{C}}} = [\mathcal{V} : \mathcal{C}] = \langle \mathcal{V} : \mathcal{C} \rangle$.

(iv) $\mathcal{V}^{\overline{\mathcal{C}}} = \langle \mathcal{V} : \mathcal{C} \rangle \Leftrightarrow \langle \mathcal{V} : \mathcal{C} \rangle \in \mathcal{L}(\mathcal{CR})$

$\Leftrightarrow \langle \mathcal{V} : \mathcal{C} \rangle$ *is closed under direct products.*

PROOF. – (i) First

$$\langle \mathcal{V} \cap \mathcal{C} \rangle \cap \mathcal{C} \subseteq \mathcal{V} \cap \mathcal{C} \subseteq \langle \mathcal{V} \cap \mathcal{C} \rangle \cap \mathcal{C}$$

and equality prevails throughout so that $\langle \mathcal{V} \cap \mathcal{C} \rangle \,\overline{\mathcal{C}}\, \mathcal{V}$. If $\mathcal{U} \in \mathcal{L}(\mathcal{CR})$ is such that $\mathcal{U} \cap \mathcal{C} = \mathcal{V} \cap \mathcal{C}$, then

$$\langle \mathcal{V} \cap \mathcal{C} \rangle = \langle \mathcal{U} \cap \mathcal{C} \rangle \subseteq \mathcal{U}\,.$$

(ii) (iii) Let $\mathcal{X} \in \{(\mathcal{V} : \mathcal{C}), [\mathcal{V} : \mathcal{C}]\}$ and assume that $\mathcal{X} \in \mathcal{L}(\mathcal{CR})$. By Lemma 1(ii), we have $\mathcal{X} \cap \mathcal{C} = \mathcal{V} \cap \mathcal{C}$ which by Lemma 1(iii) implies that $\mathcal{X} \subseteq \langle \mathcal{V} : \mathcal{C} \rangle$.

Hence Lemma 1(i) implies that $\mathcal{X} = \langle \mathcal{V} : \mathcal{C} \rangle$. But then $\langle \mathcal{V} : \mathcal{C} \rangle \in \mathcal{L}(\mathcal{CR})$ which by Lemma 1(iii) yields that $\mathcal{V}^{\overline{\mathcal{C}}} = \langle \mathcal{V} : \mathcal{C} \rangle$.

(iv) The only nontrivial implication follows directly from Lemma 1(iii)(iv). ■

We now return to our specific relations.

COROLLARY. – *For any* $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, *we have* $\mathcal{V}^{\overline{\mathcal{G}}} = [\mathcal{V} : \mathcal{G}] = \langle \mathcal{V} : \mathcal{G} \rangle$.

PROOF. – It is well known that $[\mathcal{V} : \mathcal{G}]$ is a variety. Now apply Lemma 2(ii). ■

The next lemma is known.

LEMMA 3. – *Let S be a regular semigroup. Then $\mathcal{H}^0$ (respectively $\mathcal{L}^0$, $\mathcal{R}^0$) is the least fundamental (respectively left, right fundamental) congruence on S.*

PROOF. – Let ϱ be a fundamental congruence on S and let $a \,\mathcal{H}^0 b$. Then $xay \,\mathcal{H}\, xby$ for all $x, y \in S^1$ whence $(x\varrho)(a\varrho)(y\varrho) \,\mathcal{H}\, (x\varrho)(b\varrho)(y\varrho)$ which implies that $a\varrho \,\mathcal{H}^0\, b\varrho$ whence $a\varrho = b\varrho$ in view of the hypothesis. Hence $a \,\varrho\, b$ and $\mathcal{H}^0 \subseteq \varrho$. This proves the statement for $\mathcal{H}^0$. The same proof is valid for $\mathcal{L}^0$ and $\mathcal{R}^0$. ■

LEMMA 4. – *Let* $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$.
(i) $\mathcal{V}^P = (\mathcal{V} : \widetilde{P}) = \langle \mathcal{V} : \widetilde{P} \rangle \qquad (P \in \{T, T_l, T_r\})$.
(ii) $\mathcal{V}^P = [\mathcal{V} : \widetilde{P}] = \langle \mathcal{V} : \widetilde{P} \rangle \qquad (P \in \{C, L\})$.

PROOF. – (i) By ([8], Theorem 6.2), we have

$$\mathcal{V}^T = \{S \in \mathcal{CR} \mid S/\mathcal{H}^0 \in \mathcal{V}\}$$

whence $(\mathcal{V} : \widetilde{T}) \subseteq \mathcal{V}^T$ since $S/\mathcal{H}^0 \in \widetilde{T}$. Conversely, let $S \in \mathcal{V}^T$ and let $Q \in \mathrm{Hom}(S) \cap \widetilde{T}$. Then $Q \cong S/\varrho$ for a fundamental congruence ϱ. By Lemma 3, we have $\varrho \supseteq \mathcal{H}^0$ which implies that S/ϱ is a homomorphic image of $S/\mathcal{H}^0$. By the above, $S/\mathcal{H}^0 \in \mathcal{V}$ and hence also $S/\varrho \in \mathcal{V}$. Thus $Q \in \mathcal{V}$ which proves that $S \in (\mathcal{V} : \widetilde{T})$. Therefore $\mathcal{V}^T \subseteq (\mathcal{V} : \widetilde{T})$ and equality prevails. Hence $(\mathcal{V} : \widetilde{T}) \in \mathcal{L}(\mathcal{CR})$ and Lemma 2(ii) yields that also $\mathcal{V}^T = \langle \mathcal{V} : \widetilde{T} \rangle$.

The same type of argument is valid for T_P using ([8], Theorem 8.2) and its dual.

(ii) Since $C(S) \in \mathrm{Sub}(S) \cap \widetilde{C}$, we get $[\mathcal{V} : \widetilde{C}] \subseteq \mathcal{V}^C$. Conversely, $C(S)$ contains all idempotent generated subsemigroups of S whence $\mathcal{V}^C \subseteq [\mathcal{V} : \widetilde{C}]$ and equality prevails. Hence $[\mathcal{V} : \widetilde{C}] \in \mathcal{L}(\mathcal{CR})$ and Lemma 2(iii) yields that also $\mathcal{V}^C = \langle \mathcal{V} : \widetilde{C} \rangle$.

Since $eSe \in Sub(S) \cap \widetilde{L}$ for any $e \in E(S)$, we get $[\mathcal{V} : \widetilde{L}] \subseteq \mathcal{V}^L$. Conversely, if M is a submonoid of S, then for its identity e, we have $M \subseteq eSe$ whence $\mathcal{V}^L \subseteq [\mathcal{V} : \widetilde{L}]$ and equality prevails. Hence $[\mathcal{V} : \widetilde{L}] \in \mathcal{L}(\mathcal{CR})$ and Lemma 2(iii) yields that also $\mathcal{V}^L = \langle \mathcal{V} : \widetilde{L} \rangle$. ■

We are now ready for our second theorem.

THEOREM 2. – *Let* $P \in \{K, T_l, T, T_r, C, L\}$ *and* $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$.
(i) $\mathcal{U}\, P\, \mathcal{V} \Leftrightarrow \mathcal{U} \cap \widetilde{P} = \mathcal{V} \cap \widetilde{P}$.
(ii) $\mathcal{V}_P = \langle \mathcal{V} \cap \widetilde{P} \rangle$.
(iii) $\mathcal{V}_P = \langle \mathcal{V} : \widetilde{P} \rangle$ *except for* $P = K$.

PROOF. – (i) For $P = K$, this forms part of Theorem 1; for $P = T_l$, this was proved in ([8], Corollary 8.3); for $P = T$, this was proved in ([8], Corollary 6.3); the case $P = T_r$ is dual to the case $P = T_l$.

Let $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$. Suppose first that $\mathcal{U}\, C\, \mathcal{V}$ so that $\mathcal{U}^C = \mathcal{V}^C$ and let $S \in \mathcal{U} \cap \widetilde{C}$. Then $C(S) = S \in \mathcal{U}$ and thus $S \in \mathcal{U}^C$ whence $S \in \mathcal{V}^C$. But then $S = C(S) \in \mathcal{V}$ which implies that $\mathcal{U} \cap \widetilde{C} \subseteq \mathcal{V}$. By symmetry, we also have $\mathcal{V} \cap \widetilde{C} \subseteq \mathcal{U}$ whence $\mathcal{U} \cap \widetilde{C} = \mathcal{V} \cap \widetilde{C}$. Conversely, suppose that $\mathcal{U} \cap \widetilde{C} = \mathcal{V} \cap \widetilde{C}$ and let $S \in \mathcal{U}^C$. Then $C(S) \in \mathcal{U} \cap \widetilde{C}$ and thus $C(S) \in \mathcal{V}$ whence $S \in \mathcal{V}^C$. Therefore $\mathcal{U}^C \subseteq \mathcal{V}^C$ and equality follows by symmetry. This proves the assertion for the case $P = C$.

Next suppose that $\mathcal{U}\, L\, \mathcal{V}$ so that $\mathcal{U}^L = \mathcal{V}^L$ and let $S \in \mathcal{U} \cap \widetilde{L}$. Then for any $e \in E(S)$, we have $eSe \in \mathcal{U}$ and thus $S \in \mathcal{U}^L$. The hypothesis implies that $S \in \mathcal{V}^L$. Since S is a monoid, we get $S \in \mathcal{V}$. Hence $\mathcal{U} \cap \widetilde{L} \subseteq \mathcal{V}$ and by symmetry also $\mathcal{V} \cap \widetilde{L} \subseteq \mathcal{U}$. Therefore $\mathcal{U} \cap \widetilde{L} = \mathcal{V} \cap \widetilde{L}$. Conversely, assume that $\mathcal{U} \cap \widetilde{L} = \mathcal{V} \cap \widetilde{L}$ and let $S \in \mathcal{U}^L$. Then for any $e \in E(S)$, $eSe \in \mathcal{U} \cap \widetilde{L}$ so that $eSe \in \mathcal{V}$ which implies $S \in \mathcal{V}^L$. Therefore $\mathcal{U}^L \subseteq \mathcal{V}^L$ and equality follows by symmetry. This proves the assertion for the case $P = L$.

(ii) This a direct consequence of part (i) and Lemma 2(i).

(iii) This forms part of Lemma 4. ■

For K we only have

$$\mathcal{V} \subseteq (\mathcal{V} : \mathcal{D}) \subseteq (\mathcal{V} \vee \mathcal{S} : \mathcal{D}) \subseteq \mathcal{V}^K,$$

as it is easily verified.

By ([6], Proposition 9.1), for any completely regular semigroup S and λ, $\varrho \in \mathcal{C}(S)$, we have

$$\lambda\, K\, \varrho \Leftrightarrow \lambda \cap \mathcal{H} = \varrho \cap \mathcal{H},$$
$$\lambda\, T\, \varrho \Leftrightarrow \lambda \vee \mathcal{H} = \varrho \vee \mathcal{H},$$
$$\lambda\, T_p\, \varrho \Leftrightarrow \lambda \cap \mathcal{P} = \varrho \cap \mathcal{P} \quad (p \in \{l, r\}).$$

Now letting S be a free completely regular semigroup on a countably infinite set and $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$, we get

$$\mathcal{U} \cap \tilde{T} = \mathcal{V} \cap \tilde{T} \Leftrightarrow \mathcal{U}\, T\, \mathcal{V} \Leftrightarrow \zeta_{\mathcal{U}}\, T\, \zeta_{\mathcal{V}} \Leftrightarrow \zeta_{\mathcal{U}} \vee \mathcal{H} = \zeta_{\mathcal{V}} \vee \mathcal{H}.$$

Extending ζ by setting $\zeta_{\tilde{T}} = \mathcal{H}$, we get

$$\mathcal{U} \cap \tilde{T} = \mathcal{V} \cap \tilde{T} \Leftrightarrow \mathcal{U}\, T\, \mathcal{V} \Leftrightarrow \zeta_{\mathcal{U}} \vee \zeta_{\tilde{T}} = \zeta_{\mathcal{V}} \vee \zeta_{\tilde{T}}. \tag{4}$$

We obtained a formula which remains valid if we substitute $\tilde{T}$ by a variety. An analogous discussion is valid for T_p, namely letting $\zeta_{\tilde{T}_p} = \mathcal{P}$ for $p \in \{l, r\}$, in formula (4) we may substitute T by T_p.

This interpretation does not carry over to K in view of the above equivalence. Thus we have no analogue for K, or C or L.

4. – All meets and some joins.

The meets are of course intersections; they are characterized in Lemma 6 below in a general setting. A few joins are computed in Theorem 3.

If σ is an equivalence relation on a lattice L whose classes are intervals and $a \in L$, we write the σ-class of a as the interval $a\sigma = [a_\sigma, a^\sigma]$.

LEMMA 5. – *Let σ and τ be equivalence relations on a lattice L whose classes are intervals. Then for any $a \in L$, we have $a(\sigma \cap \tau) = [a_\sigma \vee a_\tau, a^\sigma \wedge a^\tau]$.*

PROOF. – For any $x \in L$, we have

$$\begin{aligned} x \in a(\sigma \cap \tau) &\Leftrightarrow x \in a\sigma \cap a\tau \Leftrightarrow a_\sigma \leqslant x \leqslant a^\sigma,\ a_\tau \leqslant x \leqslant a^\tau \\ &\Leftrightarrow a_\sigma \vee a_\tau \leqslant x \leqslant a^\sigma \wedge a^\tau \Leftrightarrow x \in [a_\sigma \vee a_\tau, a^\sigma \wedge a^\tau] \end{aligned}$$

as asserted. ∎

With the above notation, we have $a_{\sigma \cap \tau} = a_\sigma \vee a_\tau$, $a^{\sigma \cap \tau} = a^\sigma \wedge a^\tau$.

LEMMA 6. – *Let X and Y be equivalence relations on $\mathcal{L}(\mathcal{CR})$ for which there exist subclasses $\tilde{X}$ and $\tilde{Y}$ of $\mathcal{L}(\mathcal{CR})$ such that for any $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$,*

$$\mathcal{U}\, X\, \mathcal{V} \Leftrightarrow \mathcal{U} \cap \tilde{X} = \mathcal{V} \cap \tilde{X},$$

$$\mathcal{U}\, Y\, \mathcal{V} \Leftrightarrow \mathcal{U} \cap \tilde{Y} = \mathcal{V} \cap \tilde{Y}.$$

Then for any $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$, *we have*

$$\mathcal{U}\, X \cap Y\, \mathcal{V} \Leftrightarrow \mathcal{U} \cap (\tilde{X} \cup \tilde{Y}) = \mathcal{V} \cap (\tilde{X} \cup \tilde{Y}).$$

PROOF. – Indeed, it follows easily that

$$\begin{aligned} \mathcal{U}\, X \cap Y\, \mathcal{V} &\Leftrightarrow \mathcal{U}\, X\, \mathcal{V}, \qquad \mathcal{U}\, Y\, \mathcal{V} \\ &\Leftrightarrow \mathcal{U} \cap \tilde{X} = \mathcal{V} \cap \tilde{X}, \quad \mathcal{U} \cap \tilde{Y} = \mathcal{V} \cap \tilde{Y} \\ &\Leftrightarrow \mathcal{U} \cap (\tilde{X} \cup \tilde{Y}) = \mathcal{V} \cap (\tilde{X} \cup \tilde{Y}). \qquad \blacksquare \end{aligned}$$

COROLLARY 1. – *With the notation of Lemma* 2, *we have*

$$\mathcal{V}_{X \cap Y} = \langle \mathcal{V} \cap (\tilde{X} \cap \tilde{Y}) \rangle$$

and thus

$$\langle \mathcal{V} \cap \tilde{X} \rangle \vee \langle \mathcal{V} \cap \tilde{Y} \rangle = \langle \mathcal{V} \cap (\tilde{X} \cap \tilde{Y}) \rangle.$$

For $p \in \{l, r\}$, set $K_p = K \cap T_p$.

COROLLARY 2. – *For any* $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathcal{CR})$ *and* $p \in \{l, r\}$, *we have*

$$\mathcal{U}\, K_p\, \mathcal{V} \Leftrightarrow \mathcal{U} \cap (\tilde{K} \cup \tilde{T}_p) = \mathcal{V} \cap (\tilde{K} \cup \tilde{T}_p).$$

We need some further auxiliar statements

LEMMA 7. – *Let P and Q be relations on a lattice L whose classes are intervals and such that $a^{PQ} = a^{QP}$ for all $a \in L$. Then $P \vee Q = PQP = QPQ$.*

PROOF. – If $a\, PQPQ \dots Q\, b$, then

$$a\, P\, x_1\, Q\, x_2\, P \dots x_n\, Q\, b$$

for some $x_i \in L$, whence

$$a^P = x_1^P, \; x_1^Q = x_2^Q, \; \dots, \; x_n^Q = b^Q$$

which yields

$$a^{PQ} = x_1^{PQ}, \; x_1^{QP} = x_2^{QP}, \; \dots, \; x_n^{QP} = b^{QP}$$

and the hypothesis implies that $a^{PQ} = b^{QP}$. But then

$$a\, P\, a^P\, Q\, a^{PQ} = b^{PQ}\, Q\, b^P\, P\, b$$

so that $a\,PQP\,b$. It follows that $P \vee Q = PQP$ and symmetrically $P \vee Q = QPQ$. ■

In the above lemma, it suffices that each class of P and of Q has a greatest element. Also we may reverse the situation by working with least elements in each P- and Q-class. The next lemma is known [9].

LEMMA 8. – *For any* $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$, $P \in \{K_l, K_r\}$ *and* $Q \in \{C, L\}$, *we have* $\mathcal{V}^{PQ} = \mathcal{V}^{QP}$.

PROOF. – We consider only the case $P = K_l$ and $Q = C$; the other cases have essentially the same proofs. Indeed,

$$\begin{aligned}
\mathcal{V}^{K_l C} &= (\mathcal{V}^{K_l})^C \\
&= \{S \in \mathcal{CR} \mid C(S) \in \mathcal{V}^{K_l} = \mathcal{V}^{K \cap T_l} = \mathcal{V}^K \cap \mathcal{V}^{T_l}\} \quad \text{by Lemma 5} \\
&= \{S \in \mathcal{CR} \mid C(S) \in \mathcal{V}^K\} \cap \{S \in \mathcal{CR} \mid C(S) \in \mathcal{V}^{T_l}\} \\
&= (\mathcal{V}^K)^C \cap (\mathcal{V}^{T_l})^C = \mathcal{V}^{KC} \cap \mathcal{V}^{T_l C} \\
&= \mathcal{V}^{CK} \cap \mathcal{V}^{CT_l} \quad \text{by ([6], Lemmas 5.3 and 5.5)} \\
&= (\mathcal{V}^C)^K \cap (\mathcal{V}^C)^{T_l} \\
&= \mathcal{V}^{C(K \cap T_l)} \quad \text{by Lemma 5} \\
&= \mathcal{V}^{CK_l},
\end{aligned}$$

as required. ■

Parts of the following theorem can be found in ([1], Corollaries 3.1 and 3.4) and [11] for existence varieties of regular semigroups.

THEOREM 3. – *Let* $P \in \{T_l, T, T_r, K_l, K, K_r, C, L\}$ *and* $Q \in \{C, L\}$. *Then* $P \vee Q = PQP = QPQ$.

PROOF. – For any $\mathcal{V} \in \mathcal{L}(\mathcal{CR})$ and $p \in \{l, r\}$, we have

$$\begin{aligned}
&\mathcal{V}^{T_p C} = \mathcal{V}^{CT_p},\ \mathcal{V}^{TC} = \mathcal{V}^{CT} && \text{by ([7], Lemma 5.5)} \\
&\mathcal{V}^{K_p C} = \mathcal{V}^{CK_p} && \text{by Lemma 8} \\
&\mathcal{V}^{KC} = \mathcal{V}^{CK} && \text{by ([7], Lemma 5.3)} \\
&\mathcal{V}^{T_p L} = \mathcal{V}^{LT_p},\ \mathcal{V}^{TL} = \mathcal{V}^{LT} && \text{by ([7], Lemma 6.5)} \\
&\mathcal{V}^{K_p L} = \mathcal{V}^{LK_p} && \text{by Lemma 8} \\
&\mathcal{V}^{KL} = \mathcal{V}^{LK} && \text{by ([7], Lemma 6.3).}
\end{aligned}$$

For these combinations, the assertion follows by Lemma 7. It remains to consider the case $C \vee L$. Indeed,

$$\begin{aligned}
\mathcal{U}\, CLC\, \mathcal{V} &\Leftrightarrow \mathcal{U}\, C\, \mathcal{X}\, L\, \mathcal{Y}\, C\, \mathcal{V} \quad \text{for some } \mathcal{X}, \mathcal{Y} \\
&\Leftrightarrow \mathcal{U}^C = \mathcal{X}^C,\ \mathcal{X}^L = \mathcal{Y}^L,\ \mathcal{Y}^C = \mathcal{V}^C \\
&\Rightarrow \mathcal{U}^{CLC} = \mathcal{X}^{CLC},\ \mathcal{X}^{LC} = \mathcal{Y}^{LC},\ \mathcal{Y}^{CLC} = \mathcal{V}^{CLC} \\
&\Rightarrow \mathcal{U}^{LC} = \mathcal{X}^{LC} = \mathcal{Y}^{LC} = \mathcal{V}^{LC} \\
&\Rightarrow \mathcal{U}\, L\, \mathcal{U}^L C\, \mathcal{U}^{LC} = \mathcal{V}^{LC} C\, \mathcal{V}^L L\, \mathcal{V} \\
&\Rightarrow \mathcal{U}\, LCL\, \mathcal{V} \Rightarrow CLC \subseteq LCL;
\end{aligned}$$

conversely

$$\begin{aligned}
\mathcal{U}\, LCL\, \mathcal{V} &\Leftrightarrow \mathcal{U}\, L\, \mathcal{X}\, C\, \mathcal{Y}\, L\, \mathcal{V} \quad \text{for some } \mathcal{X}, \mathcal{Y} \\
&\Leftrightarrow \mathcal{U}^L = \mathcal{X}^L,\ \mathcal{X}^C = \mathcal{Y}^C,\ \mathcal{Y}^L = \mathcal{V}^L \\
&\Rightarrow \mathcal{U}^{LCL} = \mathcal{X}^{LCL},\ \mathcal{X}^{CL} = \mathcal{Y}^{CL},\ \mathcal{Y}^{LCL} = \mathcal{V}^{LCL} \\
&\Rightarrow \mathcal{U}^{CL} = \mathcal{X}^{CL} = \mathcal{Y}^{CL} = \mathcal{V}^{CL} \\
&\Rightarrow \mathcal{U}\, L\, \mathcal{U}^C C\, \mathcal{U}^{CL} = \mathcal{V}^{CL} C\, \mathcal{V}^C L\, \mathcal{V} \\
&\Rightarrow \mathcal{U}\, CLC\, \mathcal{V} \Rightarrow LCL \subseteq CLC.
\end{aligned}$$

Therefore $LCL = CLC$ whence $C \vee L = LCL$. ■

REFERENCES

[1] K. AUINGER, *Complete congruences on lattices of (existence) varieties, Semigroups and Languages*, Porto Conf. 1994, World Scientific (1996), 1-10.

[2] P. R. JONES, *Mal'cev product of varieties of completely regular semigroups*, J. Austral. Math. Soc., **42** (1987), 227-246.

[3] F. PASTIJN, *The lattice of completely regular varieties*, J. Austral. Math. Soc., **49** (1990), 24-42.

[4] F. PASTIJN - M. PETRICH, *Congruences on regular semigroups*, Trans. Amer. Math. Soc., **295** (1986), 607-633.

[5] M. PETRICH, *Semigroups certain of whose subsemigroups have identities*, Czechoslovak J. Math., **16** (1966), 186-198.

[6] M. PETRICH, *The Green relations approach to congruences on completely regular semigroups*, Ann. Mat. Pura Appl. (iv), **167** (1994), 117-146.

[7] M. PETRICH - N. R. REILLY, *Semigroups generated by certain operators on varieties of completely regular semigroups*, Pacific J. Math., **132** (1988), 151-175.

[8] M. PETRICH - N. R. REILLY, *Operators related to E-disjunctive and fundamental completely regular semigroups*, J. Algebra, **134** (1990), 1-27.

[9] M. PETRICH - N. R. REILLY, *Operators related to idempotent generated and monoid completely regular semigroups*, J. Austral. Math. Soc., **49** (1990), 1-23.
[10] N. R. REILLY, *Varieties of completely regular semigroups*, J. Austral. Math. Soc., **38** (1985), 372-393.
[11] N. R. REILLY - S. ZHANG, *Commutativity of operators on the lattice of existence varieties*, Monatsh. Math., **123** (1997), 337-364.

Uz garmu, 21420 Bol, Brač, Croatia

Pervenuta in Redazione
the first version arrived in 1999
il 22 gennaio 2002