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Approximate Smoothings of Locally Lipschitz Functionals (*).

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Sunto. – *L'articolo tratta il problema dell'approssimazione di funzionali localmente Lipschitziani. Viene proposto un concetto di approssimazione che si basa sull'idea dell'approssimazione in grafico del gradiente generalizzato. Si prova l'esistenza di tali approssimazioni per funzionali localmente Lipschitziani definiti in domini aperti di* $\mathbb{R}^N$. *Infine, si presenta un procedimento di approssimazione normale regolare di insiemi regolari* (*introdotti in* [13]).

Summary. – *The paper deals with approximation of locally Lipschitz functionals. A concept of approximation, based on the idea of graph approximation of the generalized gradient, is discussed and the existence of such approximations for locally Lipschitz functionals, defined on open domains in* $\mathbb{R}^N$, *is proved. Subsequently, the procedure of a smooth normal approximation of the class of regular sets (containing e.g. convex and/or epi-Lipschitz sets) is presented.*

1. – Introduction.

In the paper we investigate locally Lipschitz functionals defined on open domains in the real N-dimensional Euclidean space $\mathbb{R}^N$ and their (Clarke generalized) differentiability properties in the context of approximation theory. Our purpose is twofold. First, we determine whether given a locally Lipschitz $f: U \to \mathbb{R}$, where $U \subset \mathbb{R}^N$ is open, it is possible to approximate (in an appropriate way) its Clarke generalized gradient ∂f (see Section 2 for details concerning notation and preliminaries) by gradients of C^1-smooth functions. Our interest in the question is motivated by applications to problems of smooth approximations of sets in $\mathbb{R}^N$ and the celebrated Whitney theorem [30] (see also e.g. [22]) saying:

Given a C^1*-smooth function* $f: U \to \mathbb{R}$ *and a continuous function* $\varepsilon : U \to (0, +\infty)$, *there is a* C^∞ (*or even an* $\mathbb{R}$*-analytic*) *function* $g : U \to \mathbb{R}$ *such that*

(i) $|f(x) - g(x)| < \varepsilon(x)$, *for all* $x \in U$;

(ii) $|\nabla f(x) - \nabla g(x)| < \varepsilon(x)$, *for all* $x \in U$.

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It is natural to ask to what extent this result may be generalized for a locally Lipschitz f. We shall also address the problem of the subgradient representation of a set-valued map. Suppose that $\varphi : U \multimap \mathbb{R}^N$ is a set-valued map (with nonempty values). We say that φ has *(Lipschitz) variational structure* if there is a locally Lipschitz function $f : U \to \mathbb{R}$ such that, for each $x \in U$, $\partial f(x)$, the Clarke generalized gradient of f at x, is contained in $\varphi(x)$. We also say that f is a *(Lipschitz) potential of* φ. There are several results providing sufficient conditions for φ to have such a structure (see e.g. [7, Sec. 2.5] for an up to date survey of the problem).

It appears that both problems have much in common and, in fact, may be answered simultaneously. The key to such an answer is provided by the concept of graph approximation of set-valued maps (see for instance, the second author's survey [20], [19] or [21]). Namely, in Section 3, we shall prove

THEOREM. – *An upper semicontinuous set-valued map* $\varphi : U \multimap \mathbb{R}^N$ *with compact convex values has a (Lipschitz) variational structure if and only if, for any* $\varepsilon > 0$, *there is a* C^∞*-function* $g : U \to \mathbb{R}$ *such that* ∇g *is an* ε*-approximation (on the graph) of* φ. *More precisely a locally Lipschitz function* $f : U \to \mathbb{R}$ *is a potential of* φ *if and only if, for any* $\varepsilon > 0$, *there is a* C^∞*-function* $g : U \to \mathbb{R}$ *such that*

(i) $|f(x) - g(x)| < \varepsilon$ *on* U;

(ii) ∇g *is an* ε*-approximation on the graph of* φ.

Such a function g shall be referred to as the *ε-approximate smoothing* of f. It is clear that this result constitutes a locally Lipschitz counterpart of the Whitney theorem.

To explain our view-point let us recall that due to the Rademacher theorem a locally Lipschitz f is almost everywhere (a.e.) differentiable and, for $x \in U$,

$$\partial f(x) = \operatorname{conv}\left\{p \in \mathbb{R}^N \mid p = \lim_{n\to\infty} \nabla f(x_n),\ x_n \to x \text{ and } \forall n\ \nabla f(x_n) \text{ exists}\right\}. \tag{1}$$

The lack of continuity of the a.e. defined gradient implies that in general $\partial f(x)$ is essentially a set and excludes its (uniform) approximability by gradients of smooth functions. For instance, if $\mathbb{R} \ni x \mapsto f(x) := |x|$, then

$$\partial f(x) = \begin{cases} -1 & \text{if } x < 0\,, \\ [-1, 1] & \text{if } x = 0\,, \\ 1 & \text{if } x > 0\,. \end{cases}$$

This example shows that indeed one can expect neither the existence of uniform continuous approximations of ∂f nor their required particular, i.e. gradient like, form. But ∂f admits continuous graph approximations (this holds for any upper semicontinuous set-valued map with closed convex values in view of the Cellina theorem [8]) and it makes sense to ask whether there are graph-approximations of ∂f being gradients of some smooth real functions. It also appears that our approach has much in common with the notion of a *derivate container* introduced by Warga in [26] (comp. [27]) in the context of non-smooth controllability problems – see Remark 3.8 below.

Secondly, in section 4, we shall provide some examples of applications. Actually, as any approximation result, ours has numerous applications; we have recently used it in the context of partial differential equations with discontinuous nonlinearities [14] as well as in the theory of generalized hamiltonian systems. Here we shall study the following issue. If K is a closed subset of $\mathbb{R}^N$, then the distance function d_K is (globally) Lipschitz but not differentiable (e.g. on the boundary bd K of K). Approximate smoothings of d_K on $\mathbb{R}^N \setminus K$ provide appropriate regularizations of K. More generally, assume that $K := \{x \in U | f(x) \leqslant 0\}$ is closed in the open domain U of a locally Lipschitz function f. Taking a properly chosen approximate smoothing g of f in $U \setminus K$, appropriate sublevel sets of g provide a sequence $\{K_n\}_{n=1}^{\infty}$ of outer approximations of K converging to K (in a well-defined sense) and allowing to study the normal (or tangent) cones to K in terms of the limit behavior of normal (or tangent) cones to K_n. Our interest in these issues has been inspired by the recent papers of Cornet and Czarnecki [11, 12] and Benoist [5]. Our methods allow to obtain more general results by simpler means.

Finally, in Appendix we prove a quantitative version of a deformation lemma valid for locally Lipschitz functionals.

2. – Preliminaries.

As usual, $|\cdot|$ and $\langle \cdot, \cdot \rangle$ denote the Euclidean norm and the inner product in $\mathbb{R}^N$, respectively. Let $A \subset \mathbb{R}^N$; for $x \in \mathbb{R}^N$, $d(x, A) = d_A(x) := \inf_{a \in A} |x - a|$. Given $\varepsilon > 0$, $B(A, \varepsilon) := \{x \in \mathbb{R}^N \,|\, d_A(x) < \varepsilon\}$, $D(A, \varepsilon) := \{x \in \mathbb{R}^N \,|\, d_A(x) \leqslant \varepsilon\}$. The *closure*, the *boundary*, the *interior* of A and the *convex hull* of A are denoted by $\mathrm{cl}\, A$, $\mathrm{bd}\, A$, $\mathrm{int}\, A$ and $\mathrm{conv}\, A$, respectively. By $A^\circ := \{x \in \mathbb{R}^N \,|\, \forall a \in A \ \langle x, a \rangle \leqslant 0\}$ we denote the *(negative) polar cone* of A. Clearly $A^{\circ\circ} := (A^\circ)^\circ$ is the smallest closed convex cone containing A, i.e. $A^{\circ\circ} = \mathrm{cl}\left(\bigcup_{\lambda \geqslant 0} \lambda \cdot \mathrm{conv}\, A\right)$. The *distance* between A and $B \subset \mathbb{R}^N$ is defined as $\mathrm{dist}(A, B) := \inf_{a \in A,\, b \in B} |a - b|$. By the *support function of* A we mean the function $\sigma_A \colon \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ given by

$$\sigma_A(v) := \sup_{a \in A} \langle a, v \rangle, \qquad v \in \mathbb{R}^N.$$

It is clear that σ_A is a lower semicontinuous gauge function (i.e. it is subadditive and positively homogeneous); moreover $\operatorname{cl}\operatorname{conv} A = \{x \in \mathbb{R}^N \mid \forall\, v \in \mathbb{R}^N \ \langle x, v\rangle \leq \sigma_A(v)\}$; σ_A is continuous and finite if and only if A is bounded. Any lower semicontinuous gauge σ determines a closed convex (and bounded whenever σ is finite) set $B = \{x \in \mathbb{R}^N \mid \forall\, v \in \mathbb{R}^N \ \langle x, v\rangle \leq \sigma(v)\}$ and then $\sigma_B = \sigma$.

Consider an open $U \subset \mathbb{R}^N$ and a locally Lipschitz function $f\colon U \to \mathbb{R}$. The *directional derivative* (in the sense of Clarke) *of f at $x \in U$ in the direction $v \in \mathbb{R}^N$* is defined by

$$f^\circ(x;\, v) := \limsup_{y\to x,\, h\to 0^+} \frac{f(y+hv)-f(y)}{h}.$$

It is clear that, for all $x \in U$, $v \in \mathbb{R}^N$, $f^\circ(x;\, v)$ is a well-defined real number and the function $v \mapsto f^\circ(x;\, v)$ is a Lipschitz continuous finite gauge (of rank equal to the Lipschitz rank of f around x). Therefore $f^\circ(x;\, \cdot)$ determines the *generalized gradient of f*

$$\partial f(x) := \{p \in \mathbb{R}^N \mid \forall\, v \in \mathbb{R}^N \ \langle p, v\rangle \leq f^\circ(x;\, v)\}$$

and

$$\sigma_{\partial f(x)}(v) = f^\circ(x;\, v), \qquad v \in \mathbb{R}^N.$$

The set-valued map $U \ni x \mapsto \partial f(x)$ has compact convex values and appears to be upper semicontinuous, since the function $(x, v) \mapsto f^\circ(x;\, v)$ is an upper semicontinuous real function.

The Rademacher theorem asserts that f is differentiable almost everywhere in U, i.e. for almost all $x \in U$, $\nabla f(x)$, the gradient of f at x, exists and

$$\langle \nabla f(x), v\rangle \leq f^\circ(x;\, v), \qquad v \in \mathbb{R}^N.$$

If $g : U \to \mathbb{R}$ is C^1-smooth, then it is locally Lipschitz; $g^\circ(x;\, v) = \langle \nabla g(x), v\rangle$ and $\partial g(x) = \{\nabla g(x)\}$ for $x \in U$ and $v \in \mathbb{R}^N$. The reader should consult [9] for other useful properties of f° and ∂f.

Let us note the following two straightforward facts.

LEMMA 2.1. – *Suppose that $f\colon U \to \mathbb{R}$ is locally Lipschitz.*

(i) *If $g : U \to \mathbb{R}$ is a nonnegative C^1-function, then*

$$\partial(fg)(x) = f(x)\, \nabla g(x) + g(x)\, \partial f(x) \qquad (x \in U).$$

(ii) *If $g_1, g_2\colon U \to \mathbb{R}$ are nonnegative C^1-smooth and $g_1(y) + g_2(y) = 1$ for all y from a neighborhood of $x \in U$, then*

$$\partial f(x) = \partial(g_1 f)(x) + \partial(g_2 f)(x). \qquad \blacksquare$$

Assume $A \subset \mathbb{R}^N$ and let $\varphi : A \multimap \mathbb{R}^N$ be a set-valued map (i.e. to each $x \in A$ a nonempty set $\varphi(x) \subset \mathbb{R}^N$ is assigned). Given a neighborhood $\mathcal{U}$ of the *graph* $\operatorname{Graph}(\varphi) := \{(x, y) \in A \times \mathbb{R}^N \mid y \in \varphi(x)\}$ (in $A \times \mathbb{R}^N$), we say that $g : A \to \mathbb{R}^N$ is a *$\mathcal{U}$-approximation (on the graph)* of φ if

$$\operatorname{Graph}(g) \subset \mathcal{U}.$$

Similarly, given a function $\varepsilon : A \to (0, +\infty)$, we say that g is an *ε-approximation (on the graph)* of φ provided

$$\forall x \in A \qquad g(x) \in B(\varphi(B(x, \varepsilon(x)) \cap A), \varepsilon(x)). \tag{2}$$

It is a routine to show that these notions coincide, that is: for any neighborhood $\mathcal{U}$ of $\operatorname{Graph}(\varphi)$, there is a continuous $\varepsilon(\cdot)$ such that an ε-approximation of φ is a $\mathcal{U}$-approximation, and, conversely, given continuous $\varepsilon(\cdot)$, there is a neighborhood $\mathcal{U}$ such that a $\mathcal{U}$-approximation of φ is an ε-approximation. Thus, in the sequel, we shall speak of ε-approximations for continuous functions ε. For more details concerning graph approximation (in a much more general settiing) see [19], [20] or [21].

In order to obtain a more analytic characterization of graph-approximations we shall need the following notation. For $A \subset \mathbb{R}^N$, let

$$|||A||| := \sup_{u \in D(0,1)} \inf_{a \in A} \langle a, u \rangle = -\inf_{u \in D(0,1)} \sigma_A(u).$$

It is easy to see that $|||A||| = |||\operatorname{conv} A||| = |||\operatorname{cl}\operatorname{conv} A|||$. Since $D(0, 1)$ is compact convex, in view of the well-known min-max equality (see e.g. [1, Th. 8.1]),

$$|||A||| = \inf_{a \in \operatorname{conv} A} \sup_{u \in D(0,1)} \langle a, u \rangle = \inf_{a \in \operatorname{conv} A} |a|. \tag{3}$$

Thereore $|||A||| = 0$ if and only if $0 \in \operatorname{cl}\operatorname{conv} A$.

In particular, for $f : U \to \mathbb{R}$ as above and $x \in U$,

$$|||\partial f(x)||| = -\inf_{u \in D(0,1)} f^\circ(x; u).$$

LEMMA 2.2. – *Let $U \subset \mathbb{R}^N$, $\varphi : U \multimap \mathbb{R}^N$ with closed convex values, $g : U \to \mathbb{R}^N$ and $\varepsilon : U \to (0, +\infty)$. The following conditions are equivalent:*

(i) *g is an ε-approximation of φ;*

(ii) *for any $x \in U$, there is $\bar{x} \in U$ with $|x - \bar{x}| < \varepsilon(x)$ such that*

$$|||g(x) - \varphi(\bar{x})||| < \varepsilon(x);$$

(iii) *for any $x \in U$, there is $\bar{x} \in U$ with $|x - \bar{x}| < \varepsilon(x)$ such that*

$$\forall u \in D(0, 1) \qquad \langle g(x), u \rangle - \sigma_{\varphi(\bar{x})}(u) < \varepsilon(x).$$

PROOF. – By (2), g is an ε-approximation of φ if and only if, for each $x \in U$, there is $\bar{x} \in B(x, \varepsilon(x))$ and $\bar{y} \in \varphi(\bar{x})$ such that $|g(x) - \bar{y}| < \varepsilon(x)$, i.e. $|||g(x) - \varphi(\bar{x})||| < \varepsilon(x)$ in view of (3).

On the other hand

$$|||g(x) - \varphi(\bar{x})||| = \sup_{u \in D(0,1)} (\langle g(x), u\rangle - \sup_{y \in \varphi(\bar{x})} \langle y, u\rangle);$$

thus (ii) and (iii) are equivalent. ■

3. – Approximate smoothing.

We shall start with the «if» part of Theorem stated in Introduction. Let, as above, $U \subset \mathbb{R}^N$ be open.

THEOREM 3.1. – *Let $\varphi : U \multimap \mathbb{R}^N$ be an upper semicontinuous set-valued map with compact convex values. If, for any (constant) $\varepsilon > 0$, φ admits ε-approximation of the form $\nabla g : U \to \mathbb{R}^N$, where $g : U \to \mathbb{R}$ is a C^1-smooth function, then φ has (Lipschitz) variational structure.*

PROOF. – For any integer $n \geq 1$, let $g_n : U \to \mathbb{R}$ be a C^1-smooth function such that ∇g_n is a $(1/n)$-approximation of φ. Since φ, being upper semicontinuous with compact values, is locally bounded (i.e. for any $x \in U$ and any bounded neighborhood $V_x \subset U$ of x, there is $c_x > 0$ such that $\sup_{z \in \varphi(y)} |z| \leq c_x$ for $y \in V_x$), we get by (2) that the family $\{\nabla g_n\}$ is also (uniformly) locally bounded.

Without loss of generality, we may assume that U is connected. Choose $x_0 \in U$; replacing g_n by $g_n(\cdot) - g_n(x_0)$, we may also assume that $g_n(x_0) = 0$ for all n. This, the local boundedness of the family $\{\nabla g_n\}$ and the mean value theorem imply that, for any $x \in U$, the set $\{g_n(x)\}$ is bounded and the family $\{g_n\}$ is equicontinuous. Therefore, by the (generalized) Ascoli-Arzela theorem (see e.g. [15, Cor. 0.4.12]), we may assume that g_n converges almost uniformly (i.e. uniformly on compact subsets of U) to a continuous function $f : U \to \mathbb{R}$. It is clear that f is locally Lipschitz.

Fix $x \in U$, $u \in D(0, 1)$ and $\varepsilon > 0$. Choose $\eta > 0$ such that $D(x, 3\eta) \subset U$ and

$$\forall z \in B(x, 3\eta) \qquad \sigma_{\varphi(z)}(u) < \sigma_{\varphi(x)}(u) + \varepsilon\,. \tag{4}$$

Such a choice is possible in view of the upper semicontinuity of φ.

Take an arbitrary $y \in U$ with $|y - x| < \eta$ and $0 < h < \eta$. By the mean value theorem, for each n, there is $\theta_n \in (0, 1)$ such that

$$\frac{g_n(y + hu) - g_n(y)}{h} = \langle \nabla g_n(y + \theta_n hu), u\rangle. \tag{5}$$

By Lemma 2.2, there is $\overline{y}_n \in U$ (depending on y and h) with $|y+\theta_n hu-\overline{y}_n| < \frac{1}{n}$ such that

$$\langle \nabla g_n(y+\theta_n hu),\, u\rangle \leqslant \sigma_{\varphi(\overline{y}_n)}(u) + \frac{1}{n}\,. \tag{6}$$

Clearly $|x-\overline{y}_n| \leqslant 2\eta + \frac{1}{n}$; hence, for sufficiently large n, say $n \geqslant n_0$, $|x-\overline{y}_n| < 3\eta$ and $\frac{1}{n} < \varepsilon$. By (5), (6) and (4), for $n \geqslant n_0$,

$$\frac{g_n(y+hu)-g_n(y)}{h} < \sigma_{\varphi(x)}(u) + 2\varepsilon\,. \tag{7}$$

Since $g_n \to f$ uniformly on $D(x, 3\eta)$, by (7)

$$\frac{f(y+hu)-f(y)}{h} \leqslant \sigma_{\varphi(x)}(u) + 2\varepsilon\,.$$

Therefore, for any $u \in D(0, 1)$,

$$\sigma_{\partial f(x)}(u) = f^\circ(x; u) = \inf_{\eta>0} \sup_{y\in B(x,\eta),\, h\in(0,\eta)} \frac{f(y+hu)-f(y)}{h} \leqslant \sigma_{\varphi(x)}(u)\,.$$

This implies that $\sigma_{\partial f(x)} \leqslant \sigma_{\varphi(x)}$ on $\mathbb{R}^N$, i.e. $\partial f(x) \subset \varphi(x)$ for all $x \in U$. ■

REMARK 3.2. – For the sake of completeness we shall discuss a different, more synthetic proof of Theorem 3.1. As above the sequence (g_n) converges almost uniformly to f and the sequence (∇g_n) is locally bounded. By the Dunford-Pettis theorem $\nabla g_n \rightharpoonup h \in L^1_{\mathrm{loc}}(U, \mathbb{R}^N)$ (weakly). Hence, for any test function $k : U \to \mathbb{R}$ (i.e. a C^∞-function with compact support),

$$\int_U f(x)\,\nabla k(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_U g_n(x)\,\nabla k(x)\,\mathrm{d}x =$$
$$- \lim_{n\to\infty} \int_U k(x)\,\nabla g_n(x)\,\mathrm{d}x = -\int_U k(x)\,h(x)\,\mathrm{d}x\,,$$

i.e. h is the weak gradient of f. Since f is locally Lipschitz, $f \in W^{1,\infty}_{\mathrm{loc}}(U)$ and its weak gradient $h = \nabla f$ a.e. on U (where ∇f is the ordinary gradient existing a.e. in view of the Rademacher theorem) – see e.g. [16, Sec. 5.8]. By the Convergence Theorem (see e.g. [2, Th. 3.2.6]), $\nabla f(x) \in \varphi(x)$ for almost all $x \in U$. Hence, by (1), in view of the upper semicontinuity of φ and because values of φ are convex, $\partial f(x) \subset \varphi(x)$ for all $x \in U$. ■

In the rest of this section we shall deal with the converse of Theorem 3.1. The argument we are going to present is standard and follows the steps of the Whitney theorem's proof with necessary adjustments provided by below Lemma 3.4.

Let $\omega : \mathbb{R}^N \to \mathbb{R}$ be a C^∞-mollifier, i.e. a nonnegative function with the support

$$\operatorname{supp} \omega \subset D(0, 1)$$

and such that

$$\int_{\mathbb{R}^N} \omega(x)\, \mathrm{d}x = 1 .$$

Suppose $f : \mathbb{R}^N \to \mathbb{R}$ is a locally Lipschitz function with compact support; then f is globally Lipschitz. For any $\lambda > 0$, we define the *λ-regularization* of f putting

$$g_\lambda(x) := \int_{\mathbb{R}^N} f(x - \lambda z)\, \omega(z)\, \mathrm{d}z . \tag{8}$$

The following properties are self-evident.

PROPOSITION 3.3. – *Let $L > 0$ be the Lipschitz rank of f. If g_λ is a regularization of f, then*

(i) *g_λ is C^∞-smooth;*

(ii) *g_λ is a uniform λL-approximation of f;*

(iii) $\operatorname{supp} g_\lambda \subset D(\operatorname{supp}\, f, \lambda)$. ■

The next lemma plays a key role in our smoothing procedure.

LEMMA 3.4. – *Let $f_1, \ldots, f_n : \mathbb{R}^N \to \mathbb{R}$ be Lipschitz continuous functions of common rank L and compact supports. For any $\varepsilon > 0$, there is $\mu > 0$ such that, for all $x \in \mathbb{R}^N$, there exists $\overline{x} \in B(x, \varepsilon)$ such that*

$$\forall\, u \in D(0, 1) \quad \sup_{z \in B(x, \mu),\, h \in (0, \mu)} \frac{f_i(z + hu) - f_i(z)}{h} < f_i^\circ(\overline{x}; u) + \frac{\varepsilon}{2} \tag{9}$$

for each $i = 1, \ldots, n$.

PROOF. – Fix $y \in \mathbb{R}^N$. By the very definition of the directional derivative, for all $u \in D(0, 1)$, there is $0 < \eta = \eta(y, u) < \dfrac{\varepsilon}{8L}$ such that

$$\sup_{z \in B(y, 2\eta),\, h \in (0, \eta)} \frac{f_i(z + hu) - f_i(z)}{h} < f_i^\circ(y; u) + \frac{\varepsilon}{8} \tag{10}$$

for all $i = 1, \ldots, n$. By compactness, we can choose $u_1, \ldots, u_m \in D(0, 1)$ such that $D(0, 1)$ is covered by $\{B(u_j, \eta(y, u_j))\}_{j=1}^{m}$. Put

$$\eta(y) := \min\{\varepsilon, \eta(y, u_1), \ldots, \eta(y, u_m)\}.$$

Then, for each $i = 1, 2, \ldots, n$,

$$\forall u \in D(0, 1) \qquad \sup_{z \in B(y, 2\eta(y)),\, h \in (0, \eta(y))} \frac{f_i(z + hu) - f_i(z)}{h} < f_i^\circ(y; u) + \frac{\varepsilon}{2}. \tag{11}$$

Indeed, take any $u \in D(0, 1)$, $z \in B(y, 2\eta(y))$ and $h \in (0, \eta(y))$. For some $j \in \{1, \ldots, m\}$ we have $|u - u_j| < \eta(y, u_j)$. Using (10), the Lipschitz continuity of f_i and $f_i^\circ(y; \cdot)$, we get for each i,

$$\begin{aligned} \frac{f_i(z + hu) - f_i(z)}{h} &= \frac{f_i(z + hu_j) - f_i(z)}{h} + \frac{f_i(z + hu) - f_i(z + hu_j)}{h} \\ &< f_i^\circ(y; u_j) + \frac{\varepsilon}{8} + L\eta(y, u_j) < f_i^\circ(y; u_j) + \frac{\varepsilon}{4} \\ &\leqslant f_i^\circ(y; u) + L|u - u_j| + \frac{\varepsilon}{4} < f_i^\circ(y; u) + \frac{\varepsilon}{2}, \end{aligned}$$

which proves (11).

Let $K := \bigcup_{i=1}^{n} \operatorname{supp} f_i$. There are $y_1, \ldots, y_p \in K$ such that K is contained in $V := \bigcup_{j=1}^{p} B(y_j, \eta(y_j))$. The compactness of K also implies the existence of $\eta_0 > 0$ so small that $B(K, \eta_0) \subset V$. We put

$$\mu := \min\left\{\frac{\eta_0}{2}, \eta(y_1), \ldots, \eta(y_p)\right\}.$$

We shall verify (9). Take $x \in \mathbb{R}^M$. Two cases are distinguished. If $x \notin V$, then $B(x, 2\mu) \cap K = \emptyset$, which means that, for $z \in B(x, \mu)$, $0 < h < \mu$ and $u \in D(0, 1)$, one has

$$f_i(z + hu) = 0 = f_i(z),$$

for all i, because $|z + hu - x| \leqslant |z - x| + \mu|u| < 2\mu$. Since $f_i^\circ(x; u) = 0$, condition (9) holds with $\bar{x} := x$. If $x \in V$, then there is $y_j \in K$ such that $|x - y_j| < \eta(y_j)$. Take $z \in B(x, \mu)$, $h \in (0, \mu)$, $u \in D(0, 1)$, then $|z - y_j| < \mu + \eta(y_j) \leqslant 2\eta(y_j)$ and $0 < h < \eta(y_j)$. Hence putting $\bar{x} := y_j$ and using (11) one obtains (9) again. ■

As a consequence we derive the local version of our result.

PROPOSITION 3.5. – *Let f_i ($i = 1, \ldots, n$) be as in Lemma 3.4. Then, for any $\varepsilon > 0$, there exists $\lambda_0 > 0$ such that the following conditions are satisfied:*

(i) *for any* $\lambda \leqslant \lambda_0$, λ*-regularization* g^i_λ *(of* f_i*) is a uniform* ε*-approximation of* f_i, *for all* $i = 1, \ldots, n$;

(ii) *for any* $x \in \mathbb{R}^N$, *there exists* $\overline{x} \in B(x, \varepsilon)$ *such that for all* $\lambda \leqslant \lambda_0$ *and each* $i = 1, \ldots, n$

$$||| \nabla g^i_\lambda(x) - \partial f_i(\overline{x}) ||| < \varepsilon .$$

PROOF. – We apply Lemma 3.4 and put $\lambda_0 := \min \left\{ \mu, \frac{\varepsilon}{L} \right\} > 0$. Take $\lambda \leqslant \lambda_0$. In view of Proposition 3.3 (ii), property (i) holds.

To proceed with (ii), fix a point x and $u \in D(0, 1)$. There is $\delta = \delta(\lambda, x, u) < \mu$ such that, for all i, we have

$$\forall h \in (0, \delta) \quad \langle \nabla g^i_\lambda(x), u \rangle < \frac{g^i_\lambda(x + hu) - g^i_\lambda(x)}{h} + \frac{\varepsilon}{2} . \tag{12}$$

By (9) we get, for $h \in (0, \delta)$,

$$\frac{g^i_\lambda(x + hu) - g^i_\lambda(x)}{h} =$$

$$\frac{1}{h} \int_{D(0, 1)} [f_i(x - \lambda z + hu) - f_i(x - \lambda z)] \, \omega(z) \, \mathrm{d}z < f^\circ_i(\overline{x}; u) + \frac{\varepsilon}{2} .$$

This inequality along with (12) gives $\langle \nabla g^i_\lambda(x), u \rangle < f^\circ_i(\overline{x}; u) + \varepsilon$.
Since u was arbitrary, this completes the proof in view of Lemma 2.2. ■

REMARK 3.6. – By inspection of the above proofs, it is easy to see that we can achieve that, given an arbitary neighborhood W of $\bigcup_{i=1}^{n} \operatorname{supp} f_i$, if $x \in W$, then the existing point $\overline{x}$ lies actually in $W \cap B(x, \varepsilon)$. ■

Now we are in a position to prove the main result of this section.

THEOREM 3.7. – *Suppose* $f : U \to \mathbb{R}$, *where* $U \subset \mathbb{R}^N$ *is open, is a locally Lipschitz function. For any continuous* $\varepsilon : U \to (0, +\infty)$, *there is a* C^∞*-function* $g : U \to \mathbb{R}$ *such that*

(i) *for all* $x \in U$, $|f(x) - g(x)| < \varepsilon(x)$;

(ii) ∇g *is an* ε*-approximation (on the graph) of* $\partial f(\cdot)$.
In other words g *is an* ε*-approximate smoothing of* f.

PROOF. – Let $\{K_n\}_{n=0}^{\infty}$ be an increasing family of compact sets such that

$$K_0 = \emptyset; \qquad \forall n \geq 1 \quad \emptyset \neq K_n \subset \operatorname{int} K_{n+1} \quad \text{and} \quad U = \bigcup_{n=1}^{\infty} K_n. \tag{13}$$

For any $n \geq 1$, let $P_n := \operatorname{int} K_{n+1} \setminus K_{n-1}$. In view of (13), $\{P_n\}$ is an open covering of U; let $\{\varphi_n\}_{n=1}^{\infty}$ be a C^{∞}-smooth partitition of unity subordinated to it, i.e. supp $\varphi_n \subset P_n$ for any $n \geq 1$.

Clearly, functions $\varphi_n \cdot f$ $(n \geq 1)$ are (globally) Lipschitz and may be considered as defined on $\mathbb{R}^N$. In view of the continuity of ε, the numbers

$$\varepsilon_1 := \min \left\{ \frac{1}{2} \min_{x \in K_2} \varepsilon(x),\ \operatorname{dist}(\operatorname{supp} \varphi_3, K_2),\ \operatorname{dist}(\operatorname{supp} \varphi_2, K_1) \right\};$$

$$\varepsilon_n := \min \left\{ \frac{1}{2} \min_{x \in K_{n+1}} \varepsilon(x),\ \operatorname{dist}(\operatorname{supp} \varphi_{n-1}, \complement K_n),\ \operatorname{dist}(\operatorname{supp} \varphi_{n+2}, K_{n+1}) \right\} \quad (n \geq 2)$$

(where $\complement$ denotes the complement) are positive. Applying Proposition 3.5, for each $n \geq 1$, we find $\lambda_n > 0$ such that, for any $0 < \lambda \leq \lambda_n$, λ-regularizations g_λ^n and g_λ^{n+1} of $\varphi_n \cdot f$ and $\varphi_{n+1} \cdot f$, respectively, are appropriate uniform $(\varepsilon_n/2)$-approximations and, for any $x \in P_n \cup P_{n+1}$, there is $\overline{x} \in B(x, \varepsilon_n) \cap (P_n \cup P_{n+1})$ (see Remark) satisfying

$$|||\nabla g_\lambda^n(x) - \partial(\varphi_n f)(\overline{x})||| < \varepsilon_n/2 \tag{14}$$

and

$$|||\nabla g_\lambda^{n+1}(x) - \partial(\varphi_{n+1} f)(\overline{x})||| < \varepsilon_n/2 \tag{15}$$

for all $0 < \lambda \leq \lambda_n$. Moreover, taking into account Proposition 3.3 and decreasing λ_n if necessary, we may assume that

$$\operatorname{supp} g_\lambda^n \subset P_n \quad \text{and} \quad \operatorname{supp} g_\lambda^{n+1} \subset P_{n+1} \tag{16}$$

for all $0 < \lambda \leq \lambda_n$. Furthermore, without loss of generality, one can also assume that the sequence $\{\lambda_n\}_{n=1}^{\infty}$ is nonincreasing.

Finally, define $g : U \to \mathbb{R}$ by the formula

$$g(x) = \sum_{n=1}^{\infty} g_{\lambda_n}^n(x).$$

Evidently, g is well-defined and C^{∞}-smooth since, by (16), at most two terms may be nonzero simultaneously. It remains to show that g is an ε-approximate smoothing of f. For a fixed $x \in U$, there is a unique $n \geq 0$ such that $x \in K_{n+1} \setminus K_n$. If $n = 0$, then $g(y) = g_{\lambda_1}^1(y)$ on a neighborhood of x and the claim follows from (14) and the definition of ε_1. If $n \geq 1$, then, by (16), $g(y) =$

$g^n_{\lambda_n}(y) + g^{n+1}_{\lambda_{n+1}}(y)$ for y from a neighborhood of x. This yields

$$\nabla g(x) = \nabla g^n_{\lambda_n}(x) + \nabla g^{n+1}_{\lambda_{n+1}}(x). \tag{17}$$

By the choice of numbers ε_n, we gather that, for all $y \in B(x, \varepsilon_n)$,

$$\varphi_n(y) + \varphi_{n+1}(y) = 1 . \tag{18}$$

Therefore

$$|g(x) - f(x)| \leqslant |g^n_{\lambda_n}(x) - \varphi_n(x) f(x)| + |g^{n+1}_{\lambda_{n+1}}(x) - \varphi_{n+1}(x) f(x)| < \varepsilon_n \leqslant \varepsilon(x).$$

By (17), (18) and Lemma 2.1,

$$|||\nabla g(x) - \partial f(\bar{x})||| \leqslant |||\nabla g^n_{\lambda_n}(x) - \partial(\varphi_n f)(\bar{x})||| +$$

$$|||\nabla g^{n+1}_{\lambda_{n+1}}(x) - \partial(\varphi_{n+1} f)(\bar{x})||| \leqslant \varepsilon_n \leqslant \varepsilon(x),$$

which, in virtue of Lemma 2.2, ends the proof. ■

REMARK 3.8. – We would like to thank the referee for turning our attention to the contribution of J. Warga concerning the so-called derivate containers. In [26] Warga studies a Lipschitz functional $f: U \to \mathbb{R}$ and says that a family of compact subsets $\{\Lambda^\varepsilon f(x) \mid x \in U, \varepsilon > 0\}$ of $\mathbb{R}^N$ is a *derivate container* for f if $\Lambda^\eta f(x) \subset \Lambda^\varepsilon f(x)$ $(\eta < \varepsilon)$ and, for every compact subset $U^* \subset U$, there exists a sequence $\{f_i\}$ of C^1-fuctions defined on a neighborhood of U^* such that $\lim_{i\to\infty} f_i = f$ uniformly on U^* and, for any $x \in U^*$ and $\varepsilon > 0$, there are i_0, $\delta > 0$ (depending on ε and U^*) such that $\nabla f_i(y) \in \Lambda^\varepsilon f(x)$ provided $i \geqslant i_0$ and $|y - x| < \delta$. He also shows [26, Th. 2.5], that the family $\partial^\varepsilon f(x) :=$ cl conv$\{\nabla f(y) \mid |y - x| \leqslant \varepsilon, \nabla f(y)$ exists$\}$ $(x \in U, \varepsilon > 0)$ is a derivate container. It is not difficult to see that the generalized gradient $\partial f(x) = \bigcap_{\varepsilon > 0} \partial^\varepsilon f(x)$ for each $x \in U$. Having these facts, one may prove the local version of the «only if» part of Theorem from Section 1 – comp. Proposition 3.5. It seems however that the presented self-contained setting and the consequent use of the language of graph-approximations is more convenient. ■

COROLLARY 3.9. – *Let $f: U \to \mathbb{R}$, where $U \subset \mathbb{R}^N$ is open, be a locally Lipschitz function. Suppose $c \in \mathbb{R}$ and $f^{-1}(c) \neq \emptyset$. There exists a locally Lipschitz function $g: U \to \mathbb{R}$ such that*:

(i) $g^{-1}(c) = f^{-1}(c)$ *and* $(f(x) - c)(g(x) - c) \geqslant 0$ *for all* $x \in U$;

(ii) g *is* C^∞*-smooth on* $U \setminus f^{-1}(c)$;

(iii) *for all* $x \in f^{-1}(c)$, $\partial g(x) \subset \partial f(x)$.

PROOF. – For simplicity of notation assume that $c = 0$. Let $\varepsilon: U \to (0, +\infty)$ be given by $\varepsilon(x) = \min\{|f(x)|^2, |f(x)|/2\}$. Clearly $\varepsilon(x) > 0$ if $x \in$

$U \setminus f^{-1}(0)$. Put $f_E = f|f^{-1}(0, +\infty)$ and $f_I = f|f^{-1}(-\infty, 0)$. In view of Theorem 3.7, there are ε-approximate smoothings $g_E : f^{-1}(0, +\infty) \to \mathbb{R}$ and g_I: $f^{-1}(-\infty, 0) \to \mathbb{R}$ of f_E and f_I, respectively. The function $g : U \to \mathbb{R}$ is given by

$$g(x) = \begin{cases} g_E(x) & \text{if } f(x) > 0 \\ 0 & \text{if } f(x) = 0 \\ g_I(x) & \textit{if } f(x) < 0 \,. \end{cases}$$

It is not difficult to show that g is locally Lipschitz, conditions (i), (ii) are satisfied and

$$\forall x \in U \qquad |f(x) - g(x)| \leqslant \varepsilon(x)\,. \tag{19}$$

To check (iii), suppose first that $x \in f^{-1}(0)$ and that $\nabla g(x)$ exists. For any $u \in \mathbb{R}^N$ and $h > 0$, we have

$$\frac{g(x+hu)}{h} = \frac{g(x+hu) - f(x+hu)}{h} + \frac{f(x+hu)}{h} \leqslant$$

$$\frac{|f(x+hu)|^2}{h} + \frac{f(x+hu)}{h} \leqslant L_x^2 h |u|^2 + \frac{f(x+hu) - f(x)}{h}$$

where L_x is a Lipschitz rank of f around x. Hence

$$\langle \nabla g(x), u \rangle \leqslant f^\circ(x; u)\,,$$

i.e. $\nabla g(x) \in \partial f(x)$. Now suppose that g is not differentiable at $x \in f^{-1}(0)$. Recall that, in virtue of (1),

$$\partial g(x) = \operatorname{conv}\left\{ z = \lim_{n \to \infty} \nabla g(x_n) \,|\, x_n \to x,\ x_n \in \Omega_g \right\}$$

where Ω_g is the set (of full measure) on which ∇g exists. For any $z = \lim \nabla g(x_n) \in \partial g(x)$ ($x_n \to x$ and $x_n \in \Omega_g$), there are two possibilities: almost all x_n belong to $f^{-1}(0)$ or, for a subsequence (still denoted by $\{x_n\}$) one has $f(x_n) \neq 0$. In the first case $z \in \partial f(x)$ since $\nabla g(x_n) \in \partial f(x_n)$ and the graph of ∂f is closed. Otherwise, for any n, there is $\bar{x}_n \in B(x_n, \varepsilon(x_n))$ such that $\nabla g(x_n) \in B(\partial f(\bar{x}_n), \varepsilon(x_n))$. Clearly $\bar{x}_n \to x$; hence, again by the closedness of the graph of ∂f, we gather that $z \in \partial f(x)$. Hence $\partial g(x) \subset \partial f(x)$ since the latter set is convex. ■

4. – Approximate smoothings of sets.

In this section we shall apply results of Section 3 to the problem of regularization of sets in $\mathbb{R}^N$ and their approximation by smooth sets.

Suppose U is an open subset of $\mathbb{R}^N$. Let $K \subset U$ be closed in U. By the *normal cone to K at $x \in K$* we understand the cone

$$N_K(x) := \partial d_K(x)^{\circ\circ}.$$

This cone admits a more intrinsic description by means of the *tangent cone to K at x* (in the sense of Clarke)

$$C_K(x) := \left\{ u \in \mathbb{R}^N \mid \lim_{y \to x,\, y \in K,\, h \to 0^+} \frac{d_K(y + hu)}{h} = 0 \right\};$$

namely

$$N_K(x) = C_K(x)^\circ = \{ p \in \mathbb{R}^N \mid \forall\, u \in C_K(x) \quad \langle p, u \rangle \leqslant 0 \}.$$

It is clear that if $x \in \operatorname{int} K$, then $N_K(x) = \{0\}$.

Here and below, by a *smooth set* in U we understand a closed (in U) set K for which $\operatorname{bd}_U K$, *the boundary with respect to* U, is an $(N-1)$-dimensional C^∞-smooth manifold, that is, for any $x \in \operatorname{bd}_U K$, there is a neighborhood $V \subset U$ of x and a C^∞-function $g : V \to \mathbb{R}$ such that 0 is a regular value of g and $\operatorname{bd}_U K \cap V = g^{-1}(0)$. In this case

$$N_K(x) = \{ \lambda \nabla g(x) \mid \lambda \geqslant 0 \}.$$

Let us first observe that the very notion of *regularization* and/or *approximation* of a set requires a careful description. If one wants just to approximate a given (closed in U) set $K \subset U$ in the sense of the existence of a smooth set $\widetilde{K}$ in an arbitrary neighborhood V of K (but such that $U \setminus V \neq \emptyset$), then the problem is easy. It is enough to take a nonnegative C^∞-function $\eta : U \to \mathbb{R}$ equal to 0 on K and 1 on the complement $U \setminus V$ of U (such functions always exist). The Sard theorem implies the existence of a regular value $\varepsilon \in (0, 1)$ of η. The set $\widetilde{K} := \eta^{-1}[0, \varepsilon]$ is a desired smooth set contained in V. In this way, putting $\widetilde{K}_n := \eta^{-1}[0, \varepsilon_n]$ where (ε_n) is a sequence of regular values tending to 0, we obtain the sequence $\{\widetilde{K}_n\}$ of closed (in U) smooth neighborhoods of K such that $K = \bigcap_{n=1}^{\infty} \widetilde{K}_n$ (it is clear that if the boundary of K with respect to U is not compact, then the family $\{\widetilde{K}_n\}$ is not necessarily cofinal in the family of all neighborhoods of K).

However, such approximations seem to be useless if one requires that a set $\widetilde{K}$ should have the same geometry and/or topology as K. For instance, without additional assumptions on K and η one cannot expect that $\widetilde{K}$ is homeomorphic to K. The geometrical properties of K, which should be inherited by its approximations, may be – for instance – stated in the language of tangent (or normal) cones to K and the accuracy of approximations studied in terms of their relations with tangent (or normal) cones to approximating sets. Some abstract variants of such approximations are discussed in [12].

Regular sets.

In the present paper we shall deal with sets of the form

$$K = \{x \in U | f(x) \leqslant 0\} \tag{S}$$

where $U \subset \mathbb{R}^N$ is open and $f: U \to \mathbb{R}$ is a locally Lipschitz function. We say that K is *represented* by f. This, of course, does not restrict generality since any closed set is represented by its distance function being (globally) Lipschitz.

Let

$$Z = Z(f) := \{x \in U | 0 \in \partial f(x)\}$$

be the set of *critical points* of f.

REMARK 4.1. – Since the domain U of f is open, neither K nor Z must be closed in $\mathbb{R}^N$; these sets are, however, closed in U. It is easy to see that $K = \operatorname{int} K \cup f^{-1}(0)$ and $\operatorname{bd}_U K$, the boundary of K with respect to U, equals to $\operatorname{bd} K \cap U = K \setminus \operatorname{int} K \subset \operatorname{bd} f^{-1}(0) \cap U \subset f^{-1}(0)$. If

$$f^{-1}(0) \cap Z = \emptyset\,, \tag{20}$$

then $\operatorname{bd}_U K = f^{-1}(0)$. ■

Above mentioned additional conditions concerning K and implying the existence of appropriate «good» approximations will now be stated in terms of the functional constraint f. After [13] and [4] we recall the key definition.

DEFINITION 4.2. – Let K be of the form (S).

(i) We say that K *is regular* if the set $Z \setminus K$ is closed in U.

(ii) If, for any $x \in \operatorname{bd}_U K$,

$$\liminf_{y \to x,\, y \in U \setminus K} |||\partial f(y)||| > 0\,, \tag{21}$$

then K is said to be *strictly regular*.

REMARK 4.3. – It is to be emphasized that (strict) regularity is not a property of a set alone; it strongly depends on a function representing it. For instance, any closed set $K \subset \mathbb{R}^N$ admits a C^∞-function $f: \mathbb{R}^N \to \mathbb{R}$ such that $K = f^{-1}(0)$ but, in general, above conditions (i), (ii) are not satisfied. Therefore, when speaking of (strict) regularity of $K \subset \mathbb{R}^N$ we have to consider a particular (locally Lipschitz) function $f: U \to \mathbb{R}$ representing it.

(1) If K is regular, then there is an open set $V \subset U$ conatining K such that, for $x \in V \backslash K$, $0 \notin \partial f(x)$ (e.g. we may take $V := U \backslash (Z \backslash K)$). Thus, the regularity of K means that f has no critical points in $V \backslash K$. We emphasize also that no assumptions concerning the behavior of ∂f on K are stated.

(2) It is clear that any strictly regular set is regular for (21) implies the existence of V as in (1) above [1].

(3) Assume that K is of the form (S) and consider the following condition:

($*$) *there is a neighborhood* $V \subset U$ *of* K *such that* $\inf_{y \in V \backslash K} |||\partial f(y)||| > 0.$

Evidently ($*$) implies that K is strictly regular [2]. Conversely, if $\mathrm{bd}_U K$ is compact and K is strictly regular, then condition ($*$) is satisfied. Hence (at least for sets of the form (S) with compact boundary with respect to U) one may think of strictly regular sets in terms of condition ($*$).

(4) If condition (20) holds, then K is strictly regular (since the function $y \to |||\partial f(y)|||$ is lower semicontinuous) and has nonempty interior. The same argument shows that the set $U \backslash \mathrm{int}\, K$ (represented by $-f$) is strictly regular.

(5) At last observe that, in general, neither regular nor strictly regular sets are required to have nonempty interiors. ■

We shall now provide some examples of (strictly) regular sets.

EXAMPLE 4.4.

(1) Any closed convex subset of $\mathbb{R}^N$ (represented by d_K) is strictly regular (see [13]).

(2) Any proximate retract (see [23], [17]) is strictly regular. Recall that a closed set $K \subset \mathbb{R}^N$ is a *proximate retract* if there is an open neighborhood V of K and a retraction $r : V \to K$ such that $|r(x) - x| = d_K(x)$ for all $x \in V$. An easy computation shows that in this case $|||\partial d_K(x)||| \geqslant 1$ on $V \backslash K$. Thus K, represented by d_K, is indeed strictly regular.

(3) In particular *proximally smooth sets* from [10] are proximate retracts and, thus, strictly regular.

(1) Direct argument: let $\{x_n\}$ be a sequence in $Z \backslash K$ such that $x_n \to x \in U$. Clearly $x \in Z$; if $x \in K$, then $x \in \mathrm{bd}_U K$ and $0 = \lim_{n \to \infty} |||\partial f(x_n)||| \geqslant \liminf_{y \to x,\, y \notin K} |||\partial f(y)||| > 0$, a contradiction.

(2) Actually condition ($*$) was used in [13] to define strictly regular sets; in [13] the infinite-dimensional situation was considered and condition (ii) of Definition 4.2 was not sufficient for our purposes.

(4) Any orientable $(N-1)$-dimensional closed C^1-manifold M in $\mathbb{R}^N$ is strictly regular. Indeed, then $M = g^{-1}(0)$ where g is a C^1 function defined on a neighborhood W of M and 0 is a regular value of g. Letting $f(x) = |g(x)|$ for $x \in W$, we see that M is represented by f and $\liminf_{y \to x,\, y \notin M} |\nabla f(y)| > 0$. Note that such manifolds are not necessarily proximate retracts unless they are of class C^2.

Similarly the graph of a C^1-function $g : U_1 \to \mathbb{R}^{N_2}$, where $U_1 \subset \mathbb{R}^{N_1}$ is open ($N = N_1 + N_2$); it is represented by $f : U = U_1 \times \mathbb{R}^{N_2} \to \mathbb{R}$ given by $f(x, y) = |g(x) - y|$, $(x, y) \in U$.

(5) To see a set which is regular but not strictly regular consider $K := S_1 \cup S_{-1}$ where $S_k := \{z = (x, y) \in \mathbb{R}^2 \mid (x-k)^2 + y^2 = 1\}$.

(6) The so-called «Warsaw sinusoid», i.e. the set $K := \{(x, y) \in \mathbb{R}^2 \mid x \neq 0,\ y = \sin(1/x)\} \cup \{0\} \times [-1, 1]$ is not regular.

Smooth approximation of regular sets.

First let us introduce some auxiliary concepts. Given a sequence $\{A_n\}_{n=1}^{\infty}$ of subsets of $\mathbb{R}^N$, the *upper limit of* (A_n), $\operatorname{Limsup}_{n \to \infty} A_n$, is the set of cluster points of sequences $\{y_n\}$ such that $y_n \in A_n$. Similarly, given a set-valued map φ from $A \subset \mathbb{R}^N$ into the family of (nonempty) subsets of $\mathbb{R}^N$, the *upper limit of* φ *at the cluster point* x_0 *of* A is defined as

$$\operatorname*{Limsup}_{x \to x_0} \varphi(x) := \{y \in \mathbb{R}^N \mid \liminf_{x \to x_0} d(y, \varphi(x)) = 0\}.$$

REMARK 4.5. – If K, represented by a locally Lipschitz $f : U \to \mathbb{R}$, is strictly regular, then for each $x \in \mathrm{bd}_U K$,

$$\operatorname*{Limsup}_{y \to x,\ y \notin K} \partial f(y)^{\circ\circ} \subset \partial f(x)^{\circ\circ}. \tag{22}$$

Indeed: let $q \in \operatorname{Limsup}_{y \to x, y \notin K} \partial f(y)^{\circ\circ}$. There are sequences $y_n \to x$, $y_n \notin K$ and $q_n \to q$, $q_n \in \partial f(y_n)^{\circ\circ}$, $n \geq 1$. By (21), there is $m > 0$ such that $|||\partial f(y_n)||| \geq m$ for almost all n; thus $q_n = \lambda_n p_n$ where $p_n \in \partial f(y_n)$ and $\lambda_n \geq 0$. Clearly the sequence $\{\lambda_n\}$ is bounded and (for a subsequence) $\lambda_n \to \lambda \geq 0$. Since (again for a subsequence) $p_n \to p \in \partial f(x)$, we infer that $q = \lambda p \in \partial f(x)^{\circ\circ}$. ■

We shall need another concept of a limit for sequences of sets. Recall that the (extended) Hausdorff distance between two closed nonempty subsets A, B of $\mathbb{R}^N$ is defined by

$$\mathcal{H}(A, B) := \max\left\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\right\},$$

which may be equal to $+\infty$ when A or B is unbounded. Given a sequence $\{A_n\}_{n=1}^{\infty}$ of closed subsets of $\mathbb{R}^N$ and a closed $A\subset\mathbb{R}^N$ we write

$$A = \mathcal{H} - \lim_{n\to\infty} A_n$$

if and only if $\lim_{n\to\infty} \mathcal{H}(A_n, A) = 0$. In this case, for any $\varepsilon > 0$, $A\subset B(A_n, \varepsilon)$ and $A_n \subset B(A, \varepsilon)$ for almost all $n\geq 1$. Observe also that if $A = \mathcal{H} - \lim_{n\to\infty} A_n$, then $A = \operatorname{Limsup}_{n\to\infty} A_n = \operatorname{Lim\,inf}_{n\to\infty} A_n$ (see e.g. [3] for the notion of the *lower limit* Lim inf).

Let us also note the following simple facts.

LEMMA 4.6. – *Let $A, B\subset\mathbb{R}^N$ be compact convex and $C = A^{\circ\circ}$.*

(i) *If $0\notin A$, then $C = \bigcup_{\lambda\geq 0} \lambda A$ and C is pointed* (3).

(ii) *The closed convex cone spanned by $C\cup B$, i.e. the cone $[C\cup B]^{\circ\circ}$ equals to* $\operatorname{cl}\left\{\bigcup_{\lambda\geq 0} \lambda \operatorname{conv}[A\cup B]\right\}$.

(iii) *If $0\notin \operatorname{conv}[A\cup B]$, then the cone $[C\cup B]^{\circ\circ}$ is pointed; conversely, if $[C\cup B]^{\circ\circ}$ is pointed, $0\notin A$ and $0\notin B$, then $0\notin\operatorname{conv}[A\cup B]$.* ■

Now we state the main result of this section.

THEOREM 4.7. – *Let K, represented by a locally Lipschitz function $f\colon U\to\mathbb{R}$, be regular and suppose that $\operatorname{bd}_U K$ is compact. Then:*

(i) *there is a sequence $\{K_n\}_{n=1}^{\infty}$ of closed (in U) smooth sets with compact boundaries (with respect to U) and such that, for all $n\geq 1$,*

$$K = \bigcap_{n=1}^{\infty} K_n, \qquad K\subset \operatorname{int} K_{n+1}\subset K_{n+1}\subset \operatorname{int} K_n, \tag{23}$$

$$K = \mathcal{H} - \lim_{n\to\infty} K_n \tag{24}$$

and K_n is bi-Lipschitz homeomorphic to K_{n+1} (4). *If K is strictly regular, then for any $n\geq 1$, K is a strong deformation retract of K_n* (5).

(ii) *For all $n\geq 1$ and all $x\in\operatorname{bd}_U K_n$, the closed convex cone spanned by $N_{K_n}(x)\cup\partial f(x)$ is pointed;*

(3) Recall that a closed cone L is *pointed* if $L\cap(-L) = \{0\}$.
(4) Meaning that there is a Lipschitz homeomorphism $H_n\colon K_n\to K_{n+1}$ with the inverse H_n^{-1} being also Lipschitz.
(5) I.e. There is a continuous homotopy $H: K_n\times[0, 1]\to K_n$ such that $H(0, x) = x$, $H(1, x)\in K$ for $x\in K_n$ and $H(t, x) = x$ for $t\in[0, 1]$, $x\in K$.

(iii) *for any sequence* $\{x_n\}_{n=1}^{\infty}$ *such that* $x_n \in K_n$ *and* $x_n \to x \in \mathrm{bd}_U K$,

$$\operatorname*{Limsup}_{n\to\infty} N_{K_n}(x_n) \subset \operatorname*{Limsup}_{y\to x,\, y\notin K} \partial f(y)^{\circ\circ}. \tag{25}$$

PROOF. – I. The compactness of $\mathrm{bd}_U K$ implies that in place of the open neighborhood V of K (from Remark (1)) one may take $V := B(K, 2\delta)$ where $\delta > 0$ is small enough to ensure that $\mathrm{cl}\, V \subset U$ and f has no critical points in $V \backslash K$ and, if K is strictly regular, then

$$\inf_{x\in V\backslash K} |||\,\partial f(x)\,||| =: m > 0 \tag{26}$$

(comp. Remark 4.3 (3)). Clearly $\mathrm{cl}\, V \backslash \mathrm{int}\, K$ is compact. Let

$$W := B(K, \delta) \qquad \text{and} \qquad \Omega := W \backslash Z$$

(recall that $Z = \{x \in U \,|\, 0 \in \partial f(x)\}$). Then W and Ω are open, $K \subset W \subset \mathrm{cl}\, W \subset V$ and $W \backslash K \subset \Omega$.

II. Let $x \in \Omega$. Since $0 \notin \partial f(x)$, by the separation theorem, there are $\mu > 0$ and $u \in \mathbb{R}^N$, $|u| = 1$, such that $\sigma_{\partial f(x)}(u) + \mu < -\sqrt{\mu}$. The upper semicontinuity of the function $\Omega \ni x \mapsto \sigma_{\partial f(x)}(u)$ implies the exitence of $r = r_x \in (0, \mu)$ such that $D(x, 2r) \subset U$ and $\sigma_{\partial f(y)}(u) + \mu < -\sqrt{\mu}$ for all $y \in D(x, 2r)$. Hence

$$\mathrm{conv}\, D(\partial f(D(x, 2r)), r) \cap B(0, \sqrt{r}) = \emptyset\,. \tag{27}$$

Let $\{\lambda_j\}_{j\in J}$ be a partition of unity inscribed into the cover $\{B(x, r_x)\}_{x\in\Omega}$, i.e. for each $j \in J$, there is $x_j \in \Omega$ such that $\mathrm{supp}\, \lambda_j \subset B(x_j, r_j)$ where $r_j := r_{x_j}$. Define

$$\varepsilon_1(x) = \sum_{j\in J} \lambda_j(x)\, r_j, \qquad x \in \Omega\,.$$

Clearly $\varepsilon_1 \colon \Omega \to (0, +\infty)$ is continuous. Given $x \in \Omega$, there is $i \in J$ such that $\lambda_i(x) > 0$ (therefore $x \in B(x_i, r_i)$) and $\varepsilon_1(x) \leq r_i$. Hence

$$D(\partial f(D(x, \varepsilon_1(x))), \varepsilon_1(x)) \subset D(\partial f(D(x_i, 2r_i)), r_i)\,.$$

Now let

$$\varepsilon(x) := \min\{\varepsilon_1(x), d_Z(x), \delta, |f(x)|^2, |f(x)|/2\}, \qquad x \in \Omega\,.$$

It is clear that, for any $x \in \Omega$, $B(x, \varepsilon(x)) \subset V \backslash Z$ and

$$\mathrm{conv}\, B(\partial f(B(x, \varepsilon(x))), \varepsilon(x)) \cap B(0, \sqrt{\varepsilon(x)}) = \emptyset\,. \tag{28}$$

Moreover, $\varepsilon(x) > 0$ for $x \in \Omega \backslash f^{-1}(0)$.

If K is strictly regular, then (26) holds and modifying the above construction in an obvious way, we may assume that, for $x \in W \backslash K$,

$$\operatorname{conv} B(\partial f(B(x, \varepsilon(x))), \varepsilon(x)) \cap B(0, m/2) = \emptyset . \tag{29}$$

III. As in the proof of Corollary 3.9, we construct a locally Lipschitz function $g : \Omega \to \mathbb{R}$ such that $g^{-1}(0) = f^{-1}(0) \cap \Omega$, g is an ε-smoothing of f on $\Omega \backslash f^{-1}(0)$, $\partial g(x) \subset \partial f(x)$ for $x \in f^{-1}(0)$ and $f(x)g(x) > 0$ on $\Omega \backslash f^{-1}(0)$. Moreover

$$0 \notin \partial g(x) \quad \text{whenever} \quad x \in \Omega . \tag{30}$$

It is clear for $x \in g^{-1}(0) \cap \Omega$. If $x \in \Omega \backslash g^{-1}(0)$, then

$$\nabla g(x) \in B(\partial f(B(x, \varepsilon(x))), \varepsilon(x)) ; \tag{31}$$

hence, by (28),

$$|\nabla g(x)| > \sqrt{\varepsilon(x)} > 0 . \tag{32}$$

Similarly, by (28),

$$0 \notin \operatorname{conv} [\partial g(x) \cup \partial f(x)] \tag{33}$$

for all $x \in \Omega$.

IV. We shall now redefine g to obtain a locally Lipschitz function $h : W \to \mathbb{R}$ satisfying assumptions of Lemma 5.1 (see Appendix). We put

$$h(x) = \begin{cases} g(x) & \text{if } x \in W \backslash K ; \\ 0 & \text{if } x \in K . \end{cases}$$

Then $h^{-1}(-\infty, 0] = K$. Take $b > 0$ such that $h^{-1}(-\infty, b] \subset W$.

Take a positive integer $n_0 \geq b^{-1}$ and, for any $n \geq 1$, let

$$K_n := h^{-1}(-\infty, (n_0 + n)^{-1}] .$$

Condition (23) is immediate and (24) holds since given an arbitrary $\varepsilon > 0$, there is n_1 such that $K_n \subset B(K, \varepsilon)$ for all $n \geq n_1$.

Since, for any $n \geq 1$, $(n_0 + n)^{-1} > 0$ is, by (30), a regular value of h, K_n is a smooth set (it is closed in W and, its boundary with respect to W or U, $\operatorname{bd}_U K_n = h^{-1}((n + n_0)^{-1})$ is compact and contained in $W \backslash K$).

Let $n \geq 1$. There are now two cases:

(a) K is regular;

(b) K is strictly regular.

In each case there is $0 \leqslant a < (n_0 + n + 1)^{-1}$ such that $g^{-1}(a, b) = h^{-1}(a, b)$ is nonempty,

$$\inf_{x \in h^{-1}(a, b)} |||\partial g(x)||| > 0 \quad \text{and} \quad \operatorname{cl} g^{-1}(a, b) \subset W. \tag{34}$$

Indeed, in case (a), we may take any $a \in (0, (n_0 + n + 1)^{-1})$: then $h^{-1}[a, b] = g^{-1}[a, b]$ is compact and contained in Ω, so by (30) and the lower semicontinuity of $|||\partial g(\cdot)|||$, condition (34) is satisfied. If (b) holds, then in view of (31) and (29), condition (34) is satisfied with $a = 0$.

Clearly $h = g$ and $\partial h = \partial g$ on $h^{-1}(a, b) = g^{-1}(a, b)$; in view of (34) and part (B) of Lemma 5.1 $\left(\text{where } d = \dfrac{1}{n_0 + n},\ c = \dfrac{1}{n_0 + n + 1}\right)$, K_n is homeomorphic to K_{n+1} through a bi-Lipschitz homeomorphism and if K is strictly regular, then K is a strong deformation retract of K_n. This shows assertion (i).

V. Let $x \in \mathrm{bd}_U K_n$, $n \geqslant 1$. By Lemma 4.6 (ii), $[N_{K_n}(x) \cup \partial f(x)]^{\circ\circ}$, the closed convex cone spanned by $N_{K_n}(x) \cup \partial f(x)$, is equal to $\operatorname{cl}\left\{ \bigcup_{\lambda \geqslant 0} \lambda \cdot \operatorname{conv}[\nabla g(x) \cup \partial f(x)] \right\}$. Again by Lemma 4.6 (iii) and (33), this cone is pointed, which proves assertion (ii).

VI. Finally let $x \in \mathrm{bd}_U K$, take a sequence $\{x_n\}$ such that $x_n \in K_n$, $x_n \to x$ and suppose that $q \in \operatorname{Limsup}_{n \to \infty} N_{K_n}(x_n)$. Without loss of generality suppose that $h(x_n) = g(x_n) = \dfrac{1}{n_0 + n}$ (i.e. $x_n \in \mathrm{bd}_U K_n$), $n \geqslant 1$. Again without loss of generality we may assume that there exists a sequence $q_n \in N_{K_n}(x_n)$ such that $q_n \to q$. For any n, by (31), there is $\lambda_n \geqslant 0$ such that

$$q_n = \lambda_n \nabla g(x_n) = \lambda_n p_n + \lambda_n \varepsilon(x_n)\, u_n \tag{35}$$

where $p_n \in \partial f(\overline{x}_n)$, $f(\overline{x}_n) > 0$, for some $\overline{x}_n \in B(x_n, \varepsilon(x_n))$, and $|u_n| \leqslant 1$. By (32), the sequence $\left\{\lambda_n \sqrt{\varepsilon(x_n)}\right\}$ is bounded, which means that $\lambda_n \varepsilon(x_n) \to 0$, i.e.

$$q = \lim_{n \to \infty} \overline{q}_n, \qquad \text{where} \qquad \overline{q}_n = \lambda_n p_n \in \partial f(\overline{x}_n)^{\circ\circ}.$$

Therefore $q \in \operatorname{Limsup}_{y \to x,\, y \notin K} \partial f(y)^{\circ\circ}$.

The proof of part (iii) and thereby of our theorem is completed. ∎

REMARK 4.8. – Let K be as in the above theorem.

(1) If K is strictly regular, then by (22),

$$\operatorname{Limsup}_{n \to \infty} N_{K_n}(x_n) \subset \partial f(x)^{\circ\circ}$$

or, equivalently,

$$\operatorname{Limsup}_{n \to \infty} \operatorname{Graph}(N_{K_n}) \subset \operatorname{Graph}(\partial f(\cdot)^{\circ\circ}).$$

This can be improved: (arguing as in step VI) we see that the sequence (λ_n) is bounded and that (passing to a subsequence if necessary) $\nabla g(x_n) \to z \in \partial g(x)$;

hence, by (35),

$$\operatorname*{Limsup}_{n\to\infty} N_{K_n}(x_n) \subset \partial g(x)^{\circ\circ}.$$

(2) Let K be regular. Take $b'>0$ such that $f^{-1}(-\infty, b'] \subset W$, a sufficiently large $n \geqslant 1$ such that $K \subset K_n \subset f^{-1}(-\infty, b')$ and $a' \in (0, b')$ (or $a'=0$ if K is strictly regular) such that $f^{-1}(-\infty, a'] \subset \operatorname{int} K_n$. Then $\inf\limits_{x\in f^{-1}(a', b')} |||\partial f(x)||| > 0$. Let $\Delta(x) = h(x) - (n_0+n)^{-1}$, $x\in W$, i.e. $\mathrm{bd}_U K_n = \Delta^{-1}(0)$. It is clear that $a' < \inf\limits_{\Delta^{-1}(0)} f \leqslant \sup\limits_{\Delta^{-1}(0)} f < b'$, $\Delta>0$ on $f^{-1}(b')$, $\Delta<0$ on $f^{-1}(a')$ and, since $\partial\Delta = \partial h$ on $\Delta^{-1}(0)$, by (33), $\inf\limits_{x\in\Delta^{-1}(0)} |||\partial\Delta(x)\cup\partial f(x)||| > 0$. Therefore, by part (B) of Lemma 5.1, for any $a' < c \leqslant d < b'$, the sets $f^{-1}(-\infty, c]$ and $f^{-1}(-\infty, d]$ are bi-Lipschitz homeomorphic, K is a strong deformation retract of $f^{-1}(-\infty, d]$ provided K is strictly regular, and, by part (A) of Lemma 5.1, for $c \in \left\{a', \inf\limits_{\Delta^{-1}(0)} f\right\}$, the sets $f^{-1}(-\infty, c]$ and K_n are bi-Lipschitz homeomorphic.

Hence: K_n is bi-Lipschitz homeomorphic to $f^{-1}(-\infty, c]$ for all $n\geqslant 1$ and all, sufficiently small, $c>0$. ■

To proceed further we need the notion of an epi-Lipschitz set. Let $U\subset\mathbb{R}^N$ be open. A set K of the form (S), represented by a locally Lipschitz $f\colon U\to\mathbb{R}$, satisfying (20) is strictly regular and, additionally, it is epi-Lipschitz in the sense of Rockafellar [24]: for any $x\in K$, the Clarke normal cone $N_K(x)$ is pointed. Indeed, in view of [2, Prop. 16, 7.3], for any $x\in \mathrm{bd}_U K$, $N_K(x)\subset\partial f(x)^{\circ\circ} = \bigcup\limits_{\lambda\geqslant 0}\lambda\partial f(x)$. Thus $N_K(x)$ must be pointed.

Conversely, given a closed (in U) epi-Lipschitz set $K\subset U$ (i.e. such that, for all $x\in\mathrm{bd}_U K$, the cone $N_K(x)$ is pointed), we gather that K is strictly regular and condition (20) holds. Namely it is represented by $\Delta_K := d_K - d_{\complement K}\colon U\to\mathbb{R}$ and, as shown by Hiriart-Urruty, $0\notin\partial\Delta_K(x)$, $x\in\mathrm{bd}_U K$. Clearly the interior $\operatorname{int} K$ of an epi-Lipschitz set is nonempty and $\mathrm{bd}_U K = \Delta_K^{-1}(0)$.

The next result explains the relevance of assertion (ii) of Theorem 4.7.

THEOREM 4.9. – *Suppose that K, represented by a locally Lipschitz function $f\colon U\to\mathbb{R}$, is regular and has the compact boundary $\mathrm{bd}_U K$. Assume that there is a family $\{M_n\}_{n=1}^\infty$ of subsets of U, being closed in U, with compact boundaries $\mathrm{bd}_U M_n = M_n\setminus\operatorname{int} M_n$ such that*

$$K = \mathcal{H}-\lim_{n\to\infty} M_n, \tag{36}$$

and, for almost all $n\geqslant 1$,

$$K\subset\operatorname{int} M_n. \tag{37}$$

If,

$$(38)\quad \begin{cases} \textit{for almost all } n \geqslant 1 \textit{ and all } x \in \operatorname{bd}_U M_n, \\ \textit{the closed convex cone spanned by } N_{M_n}(x) \cup \partial f(x) \textit{ is pointed}, \end{cases}$$

then, for almost all n,

(i) M_n *is epi-Lipschitz;*

(ii) M_n *is bi-Lipschitz homeomorphic to* M_{n+1} *and to* K_n (*where* K_n *is the set constructed in Theorem* 4.7);

(iii) K *is a strong deformation retract of* M_n *provided* K *is strictly regular.*

PROOF. – Assumption (38) implies that, for almost all n, say $n \geqslant n_0 \geqslant 1$, and all $x \in \operatorname{bd}_U M_n$, the normal cone $N_{M_n}(x)$ is pointed. Hence, for $n \geqslant n_0$, M_n is an epi-Lipschitz set; thus it is represented by $\Delta_n\colon U \to \mathbb{R}$, $\Delta_n(x) = d(x, M_n) - d(x, \complement M_n)$; condition (20) holds, i.e. $0 / \in \partial\Delta_n(x)$ and

$$N_{M_n}(x) = \bigcup_{\lambda \geqslant 0} \lambda \partial \Delta_n(x)$$

for $x \in \Delta_n^{-1}(0) = \operatorname{bd}_U M_n$. By Lemma 4.6 (iii), $0 \notin \operatorname{conv}[\partial\Delta_n(x) \cup \partial f(x)]$ (on $\Delta_n^{-1}(0)$); since $\Delta_n^{-1}(0)$ is compact, we have

$$(39)\qquad \inf_{x \in \Delta_n^{-1}(0)} |||\, \partial\Delta_n(x) \cup \partial f(x) \,||| > 0$$

provided $n \geqslant n_0$.

There, of course, exists $b > 0$ such that $f^{-1}(-\infty, b] \subset U$ and $0 \notin \partial f(x)$ if $x \in f^{-1}(0, b]$. By (36), there is $n_1 \geqslant n_0$ such that $M_n \subset f^{-1}(-\infty, b)$ for $n \geqslant n_1$. Let $n \geqslant n_1$; by (37) there is $a > 0$ such that $f^{-1}(-\infty, a] \subset \operatorname{int} M_n$ (and $a = 0$ if K is strictly regular). In view of compactness, $a < \inf_{\Delta_n^{-1}(0)} f$, $\sup_{\Delta_n^{-1}(0)} f < b$; moreover $\Delta_n > 0$ on $f^{-1}(b)$ and $\Delta_n < 0$ on $f^{-1}(a)$. Since $\inf_{x \in f^{-1}(a,b)} |||\, \partial f(x) \,||| > 0$, in view of (39) and part (A) of Lemma 5.1, we see that M_n is bi-Lipschitz homeomorphic to $f^{-1}(-\infty, c]$ where $a < c \leqslant \inf_{\Delta_n^{-1}(0)} f$ and K is a strong deformation retract of M_n if K is strictly regular.

Employing Remark 4.8 (2) we complete the proof. ■

REMARK 4.10. – In the setting of Theorem 4.9, instead of (38) assume that, for any sequence $\{x_n\}$ such $x_n \in M_n$ and $x_n \to x \in \operatorname{bd}_U K$,

$$(40)\qquad \operatorname*{Limsup}_{n \to \infty} N_{M_n}(x_n) \subset N_f(x) := \operatorname*{Limsup}_{y \to x,\, y \notin K} \partial f(y)^{\circ\circ}.$$

We claim that

if, for any $x \in \mathrm{bd}_U K$, $N_f(x)$ is pointed (or, in the strict regular situation, the cone $\partial f(x)^{\circ\circ}$ containing $N_f(x)$) is pointed, then condition (38) *is satisfied.*

To see this observe first that, for almost all $n \geq 1$ and $x \in M_n$, the cones $\partial f(x)^{\circ\circ}$ and $N_{M_n}(x)$ are pointed. It is obvious for $\partial f(x)^{\circ\circ}$ – see Lemma 4.6 (i). As concerns the other cone, suppose to the contrary (skipping subsequences) that, for any n, there would exist $x_n \in \mathrm{bd}_U M_n$, $p_n \in N_{M_n}(x_n)$, $|p_n| = 1$ such that the straight line $\{\lambda p_n \mid \lambda \in \mathbb{R}\} \subset N_{M_n}(x_n)$. Clearly (a subsequence) $x_n \to x \in \mathrm{bd}_U K$, $p_n \to p \in N_f(x)$, $|p| = 1$, and at the same time $-p \in \partial f(x)^{\circ\circ}$, a contradiction.

Assume now that our claim does not hold. Again skipping subsequences in order to simplify the notation, there is a sequence $\{x_n\}_{n=1}^{\infty}$, $x_n \in \mathrm{bd}_U M_n$, such that the closed convex cone spanned by $N_{M_n}(x_n) \cup \partial f(x_n)$ (see Lemma 4.6 (ii)) is not pointed. Since both cones $N_{M_n}(x_n)$ and $\partial f(x_n)^{\circ\circ}$ are pointed, it easily follows that, for any $n \geq 1$, there are $p_n \in N_{M_n}(x_n)$, $|p_n| = 1$ and $q_n \in \partial f(x_n)^{\circ\circ}$, $|q_n| = 1$ such that $p_n = -q_n$. It is clear that (for subsequences) $x_n \to x \in \mathrm{bd}_U K$, $p_n \to p \in N_f(x)$, $q_n \to q \in N_f(x)$, $p = -q \neq 0$, i.e. $N_f(x)$ is not pointed, a contradiction.

However, the pointedness of $N_f(x)$ is too strong an assumption. Even in very natural situations it cannot be satisfied (e.g. consider K being a one-point set). ■

In the following statement we recover generalizations of the main results due to Cornet-Czarnecki [12]. We belive that our framework makes it possible to essentially simplify their approach.

COROLLARY 4.2 (comp. [12]). – *Suppose that K, of the form (S), satisfies condition* (20) *(i.e. K is epi-Lipschitz) and the boundary $\mathrm{bd}_U K$ is compact.*

(A) *Then there is a sequence $\{K_n\}$ of closed (in U) smooth sets such that $K = \bigcap_{n=1}^{\infty} K_n$, $K \subset \mathrm{int}\, K_{n+1} \subset K_{n+1} \subset \mathrm{int}\, K_n$, $K = \mathcal{H} - \lim_{n\to\infty} K_n$ and K_n is bi-Lipschitz homeomorphic to K for all $n \geq 1$. Moreover,*

$$\operatorname*{Limsup}_{n\to\infty} \mathrm{Graph}\,(N_{M_n}) \subset \mathrm{Graph}\,(N_K)$$

and, for all $n \geq 1$ and $x \in \mathrm{bd}_U K_n$, the cone $[N_{K_n}(x) \cup \partial f(x)]^{\circ\circ}$ is pointed.

(B) *If there is a sequence $\{M_n\}_{n=1}^{\infty}$ of subsets of U, being closed in U, with compact boundaries $\mathrm{bd}_U M_n$ such that*

$$K = \mathcal{H} - \lim_{n\to\infty} M_n$$

and

$$\operatorname*{Limsup}_{n \to \infty} \operatorname{Graph}(N_{M_n}) \subset \operatorname{Graph}(N_K), \tag{41}$$

then for all n,

(i) M_n *is an epi-Lipschitz set*;

(ii) M_n *is bi-Lipschitz homeomorphic to* K (6).

Observe that above, contrary to Theorem 4.9, we do not assume that $K \subset \operatorname{int} M_n$. Since now the cone $\partial f(x)^{\circ\circ}$ ($x \in \mathrm{bd}_U K$) is pointed, condition ($*$) from the footnote (or (41)) implies that (38) is satisfied.

PROOF. – Again we use the notation of the proof of Theorem 4.7. Let us return to step IV of this proof. Since (20) is satisfied there is $a < 0$ such that $g^{-1}[a, b]$ is compact and contained in Ω; hence (34) holds again.

Take a locally Lipschitz function $\eta : W \to \mathbb{R}$ such that $\eta | Z \equiv 0$ and $\eta | g^{-1}[a, +\infty) \equiv 1$ and define

$$h(x) = \begin{cases} 0 & \text{if } x \in Z \cap W; \\ \eta(x)\, g(x) & \text{if } x \in \Omega. \end{cases}$$

Now we may proceed as in steps IV – VI of the proof of Theorem 4.7 (employing Lemma 5.1) in order to complete the proof of part (A) of the assertion.

As concerns part (B) we see that, by condition ($*$) (or (41)) and Remark 4.10, for almost all $n \geq 1$ and all $x \in \mathrm{bd}_U M_n$, the cone $[N_{M_n}(x) \cup \partial f(x)]^{\circ\circ}$ is pointed and we may reason as in the proof of Theorem 4.9. Since now $a < 0$ we may take $c = 0$, i.e. M_n (with large n) is bi-Lipschitz homeomorphic to $f^{-1}(-\infty, 0] = K$. ■

5. – Appendix – Deformation lemma.

A part of the following lemma is a well-known standard for smooth functions (see e.g. [25] and [29]). A particular case of the version we present has been used in [12] without a proof (with a reference to [6] where only a part of the complete proof is given). For the reader's convenience (and for a future reference) we provide a simple, constructive and quantitative proof of a result slightly surpassing our actual needs.

(6) Since $N_K(x) \subset \partial f(x)^{\circ\circ}$ for $x \in \mathrm{bd}_U K$, condition (41) may be replaced by a slightly weaker one:

$$\operatorname*{Limsup}_{n \to \infty} \operatorname{Graph}(N_{M_n}) \subset \operatorname{Graph}(\partial f(\cdot)^{\circ\circ}). \tag{$*$}$$

LEMMA 5.1. – *Let* $h : W \to \mathbb{R}$, *where* $W \subset \mathbb{R}^N$ *is open, be a locally Lipschitz function such that, for some* $a < b$, *the set* $\mathrm{cl}\, h^{-1}(a, b)$ *is nonempty and contained in* W. *Suppose* $\inf_{x \in h^{-1}(a, b)} |||\partial h(x)||| > 0$ (7).

(A) *Assume further that there is a locally Lipschitz function* $\Delta : W \to \mathbb{R}$ *such that* $\Delta^{-1}(0) \neq \emptyset$ *and*

$$a < \inf_{x \in \Delta^{-1}(0)} h(x) \leqslant \sup_{x \in \Delta^{-1}(0)} h(x) < b\,, \tag{42}$$

$$\Delta > 0 \quad on \quad h^{-1}(b), \qquad \Delta < 0 \quad on \quad h^{-1}(a)\,, \tag{43}$$

$$\inf_{x \in \Delta^{-1}(0)} |||\partial h(x) \cup \partial \Delta(x)||| > 0\,. \tag{44}$$

Then, for any $c \in (a, \inf_{\Delta^{-1}(0)} h]$,

(i) *the sets* $h^c := h^{-1}(-\infty, c]$ *and* $S := \Delta^{-1}(-\infty, 0]$ *are bi-locally Lipschitz homeomorphic*;

(ii) h^a *is a strong deformation retract of* S.

(B) *For any* $c \leqslant d$ *in* (a, b), h^c *and* h^d *are bi-locally Lipschitz homeomorphic and* h^a *is a strong deformation retract of* h^d.

PROOF. – Part (B) follows from (A) if we take $\Delta(x) := h(x) - d$.

To prove (A) put $\Omega := h^{-1}(a, b)$. Clearly Ω is open, $\mathrm{cl}\, \Omega \subset W$ and, by (42), $\Delta^{-1}(0) \subset \Omega$. By (44), for some $m > 0$,

$$\min\left\{ \inf_{x \in h^{-1}(a, b)} |||\partial h(x)|||\,, \inf_{x \in \Delta^{-1}(0)} |||\partial h(x) \cup \partial \Delta(x)||| \right\} > m\,.$$

For any $x \in \Delta^{-1}(0)$, $m < |||\partial h(x) \cup \partial \Delta(x)||| = \sup_{u \in D(0, 1)} \inf_{p \in \partial h(x) \cup \partial \Delta(x)} \langle p, u \rangle$. Hence there is is $u_x \in D(0, 1)$, such that

$$\inf_{p \in \partial h(x)} \langle p, u_x \rangle > m \quad \text{and} \quad \inf_{p \in \partial \Delta(x)} \langle p, u_x \rangle > m\,,$$

i.e.

$$h^\circ(x;\, -u_x) < -m \quad \text{and} \quad \Delta^\circ(x;\, -u_x) < -m\,.$$

Since $h^\circ(\cdot;\, -u_x)$ and $\Delta^\circ(\cdot;\, -u_x)$ are upper semicontinuous, there is a neighborhood N_x of x, $N_x \subset \Omega$ such that, for all $y \in N_x$,

$$h^\circ(y;\, -u_x) < -m \quad \text{and} \quad \Delta^\circ(y;\, -u_x) < -m\,.$$

(7) We shall see that this implies, in particular, that $h^{-1}(\xi) \neq \emptyset$ for any $\xi \in [a, b]$.

Similarly, for any $x \in \Omega \backslash \Delta^{-1}(0)$, there is $u_x \in D(0, 1)$ and a neighborhood $M_x \subset \Omega \backslash \Delta^{-1}(0)$ of x such that

$$h^\circ(y; -u_x) < -m, \qquad y \in M_x.$$

Let $\{\lambda_j\}_{j \in J}$ be a locally Lipschitz partition of unity inscribed into the covering $\{N_x\}_{x \in \Delta^{-1}(0)} \cup \{M_x\}_{x \in \Omega \backslash \Delta^{-1}(0)}$, i.e. for any $j \in J$, $\lambda_j \colon \mathbb{R}^N \to [0, 1]$ is locally Lipschitz and $\operatorname{supp} \lambda_j \subset N_{x_j}$ for some $x_j \in \Delta^{-1}(0)$ or $\operatorname{supp} \lambda_j \subset M_{x_j}$ with $x_j \in \Omega \backslash \Delta^{-1}(0)$. Define $V \colon \Omega \to \mathbb{R}^N$ by the formula

$$V(x) := \sum_{j \in J} \lambda_j(x)\, u_j, \qquad x \in \Omega\,,$$

where $u_j := -u_{x_j}$, $j \in J$. It is clear that V is locally Lipschitz; for any $x \in \Omega$, $|V(x)| \leqslant 1$, and

$$h^\circ(x; V(x)) < -m \quad \text{for} \quad x \in \Omega\,, \tag{45}$$

$$\Delta^\circ(x; V(x)) < -m \quad \text{for} \quad x \in N \tag{46}$$

where N is a neighborhood of $\Delta^{-1}(0)$ (in Ω).

For any $x \in \Omega$, the Cauchy initial value problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\sigma(t, x) = V(\sigma(t, x)) \\ \sigma(0, x) = x \end{cases}$$

has a unique solution $\sigma(\cdot, x)$ in Ω defined on the maximal interval of existence $J_x := (T_-(x), T_+(x))$ with $-\infty \leqslant T_-(x) < 0 < T_+(x) \leqslant +\infty$. It is well-known (see [18]) that $T_-(\cdot), T_+(\cdot) \colon \Omega \to \mathbb{R} \cup \{\pm\infty\}$ are upper semicontinuous and lower semicontinuous, respectively; $U := \{(t, x) \mid x \in \Omega, t \in J_x\}$ is open and $\sigma \colon U \to \Omega$ is locally Lipschitz.

Fix $x \in \Omega$ and let $\alpha(t, x) = h(\sigma(t, x))$, $t \in J_x$. It is clear that $\alpha_x := \alpha(\cdot, x)$ is locally Lipschitz and, by (45), for almost all $t \in J_x$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha_x(t) \leqslant \alpha_x^\circ(t; 1) \leqslant h^\circ\left(\sigma(t, x); \frac{\mathrm{d}}{\mathrm{d}t}\sigma(t, x)\right) < -m\,.$$

Hence $\alpha(\cdot, x)$ is strictly decreasing (we say that h *decreases along trajectories*) and, for all $t_1, t_2 \in J_x$, $t_1 < t_2$,

$$a - b \leqslant \alpha(t_2, x) - \alpha(t_1, x) = \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\alpha_x(s)\,\mathrm{d}s < -m(t_2 - t_1)\,. \tag{47}$$

Therefore, actually, $|T_{\pm}(x)| \leqslant \frac{b-a}{m}$. Since,

$$|\sigma(t_2, x) - \sigma(t_1, x)| \leqslant \int_{t_1}^{t_2} |V(\sigma(s, x)| \mathrm{d}s \leqslant t_2 - t_1,$$

the (left and right) limits $\sigma(T_{\pm}(x), x) := \lim_{t \to T_{\pm}(x)} \sigma(t, x)$ exist, belong to $\mathrm{cl}\, \Omega \subset W$ and

$$h(\sigma(T_-(x), x)) = b\,, \qquad h(\sigma(T_+(x), x)) = a\,.$$

In this way both functions σ and α are well-defined on $\{(t, x) \mid x \in \Omega,\, t \in \mathrm{cl}\, J_x = [T_-(x), T_+(x)]\}$ and $\alpha([T_-(x), T_+(x)] \times \{x\}) = [a, b]$.

Observe that $T_-(\cdot)$ (resp. $T_+(\cdot)$) is also lower (resp. upper) semicontinuous on Ω. We prove, for example, that T_+ is upper semicontinuous. Suppose to the contrary that there is a sequence $\{x_n\}$ in Ω such that $x_n \to x \in \Omega$ and $T_+(x) + \varepsilon_0 \leqslant T_+(x_n)$ for some $\varepsilon_0 > 0$ and all n. For any $t \in J_x$ and n, by (47) and the upper (resp. lower) semicontinuity of $T_-(\cdot)$ (resp. $T_+(\cdot)$),

$$a - \alpha(t, x_n) = \lim_{\tau \to T_+(x_n)^-} (\alpha(\tau, x_n) - \alpha(t, x_n)) \leqslant -m(T_+(x_n) - t) \leqslant -m\varepsilon_0.$$

Hence, when $n \to \infty$, $a - \alpha(t, x) \leqslant -m\varepsilon_0$ and, when $t \to T_+(x)^-$, $0 \leqslant -m\varepsilon_0$, contradiction. In a similar manner we prove that T_- is lower semicontinuous. Thus $T_{\pm}$ are continuous on Ω.

Fix $x_0 \in \Omega$ and $p < q$ in J_{x_0}. The compactness of the trajectory $\Sigma_{p,q}(x_0) := \{\sigma(t, x_0) \mid t \in [\min\{p, 0\}, \max\{q, 0\}]\}$ entails that both h and V are (globally) Lipschitz of rank $L_0 > 0$ on a neighborhood $\mathcal{U}$ of $\Sigma_{p,q}(x_0)$. The continuity of $T_{\pm}(\cdot)$ and σ implies the existence of a neighborhood $\mathcal{U}_0$ of x_0 such that $[p, q] \subset J_x$ and $\Sigma_{p,q}(x) = \{\sigma(t, x) \mid t \in [\min\{p, 0\}, \max\{q, 0\}]\} \subset \mathcal{U}$ for $x \in \mathcal{U}_0$(8). By the Gronwall inequality (see [18, III, Th. 1.1]), for $x_1, x_2 \in \mathcal{U}_0$, $t \in [p, q]$,

$$|\sigma(t, x_1) - \sigma(t, x_2)| \leqslant M_0 |x_1 - x_2| \tag{48}$$

where $M_0 = \exp(L_0 \max\{|p|, |q|\})$. Hence, for such x_1, x_2 and t,

$$|\alpha(t, x_1) - \alpha(t, x_2)| = |h(\sigma(t, x_1)) - h(\sigma(t, x_2)| \leqslant L_0 M_0 |x_1 - x_2|\,. \tag{49}$$

For any $s \in (a, b)$ and $x \in \Omega$, let $\beta(s, x)$ be a unique time in J_x such that $\alpha(\beta(s, x), x) = s$. Clearly, $\beta(h(x), x) = 0$ and $\beta(\cdot, x)$ is decreasing on (a, b).

The function $\beta : (a, b) \times \Omega \to \mathbb{R}$ is locally Lipschitz. Indeed, take $x_0 \in \Omega$ and

(8) It is not difficult to precisely establish the size of $\mathcal{U}_0$ knowing how large $\mathcal{U}$ is.

$c < d$ in (a, b). Put $p = \beta(d, x_0)$, $q = \beta(c, x_0)$. For any $x_1, x_2 \in \mathcal{U}_0$ ($\mathcal{U}_0$ was determined above) and $s_i \in [c, d]$, putting $t_i = \beta(s_i, x_i)$ $(i = 1, 2)$, by (47) and (49), we get

$$|t_2 - t_1| \leqslant \frac{1}{m} |\alpha(t_2, x_2) - \alpha(t_1, x_2)| \leqslant$$

$$\frac{1}{m}(|s_2 - s_1| + |\alpha(t_1, x_1) - \alpha(t_1, x_2)|) \leqslant \frac{1}{m}(|s_1 - s_2| + L_0 M_0 |x_1 - x_2|).$$

For any $x \in \Omega$, there is a unique time $\tau(x) \in J_x$ such that $\Delta(\sigma(\tau(x), x)) = 0$. The existence follows since, by (43), Δ changes the sign along trajectories. To see the uniqueness suppose that, there are $t, t' \in J_x$, $t < t'$ such that $\Delta(\sigma(t, x)) = 0 = \Delta(\sigma(t', x))$. Since Δ decreases along trajectories contained in N (see (46)), the set of zeros of $\Delta(\sigma(\cdot, x))$ is isolated and, thus, $t_1 := \min\{\xi \in (t, t'] \,|\, \Delta(\sigma(\xi, x)) = 0\}$ is well-defined and $t_1 > t$. The function $[t, t_1] \ni \xi \mapsto \Delta(\sigma(\xi, x))$ is decreasing on intervals $[t, t+\varepsilon)$ and $(t_1 - \varepsilon, t_1]$ $\left(\text{for some small } 0 < \varepsilon < \frac{t_1 - t}{3}\right)$; hence it has a zero in $[t + \varepsilon, t_1 - \varepsilon]$, a contradiction.

The function $\tau : \Omega \to \mathbb{R}$ is continuous. Indeed, let $x_n \to x_0 \in \Omega$. By (42), passing to a subsequence if necessary, $\tau(x_n) \to t_0 \in J_{x_0}$; hence $0 = \Delta(\sigma(\tau(x_n), x_n)) \to \Delta(\sigma(t_0, x_0))$, i.e. $t_0 = \tau(x_0)$.

We show that τ is actually locally Lipschitz. Take $x_0 \in \Omega$; the continuity of σ implies that there is $\varepsilon > 0$ such that $\sigma((\tau(x_0) - \varepsilon, \tau(x_0) + \varepsilon) \times B(x_0, \varepsilon)) \subset N_0$ where $N_0 \subset N$ (see (46)) is a neighborhood of $\sigma(\tau(x_0), x_0)$ on which Δ is Lipschitz with constant $l > 0$. The continuity of τ implies the existence of $0 < \delta < \varepsilon$ such that, for $x \in B(x_0, \delta)$, $|\tau(x) - \tau(x_0)| < \varepsilon$. Take $x_1, x_2 \in B(x_0, \delta)$ and suppose that $\tau(x_1) \leqslant \tau(x_2)$. Then

$$\Delta(\sigma(\tau(x_2), x_2)) - \Delta(\sigma(\tau(x_1), x_2)) = \int_{\tau(x_1)}^{\tau(x_2)} \frac{\mathrm{d}}{\mathrm{d}s} \Delta(\sigma(s, x_2))\, \mathrm{d}s \leqslant -m(\tau(x_2) - \tau(x_1))$$

hence, by (48),

$$\tau(x_2) - \tau(x_1) \leqslant \frac{1}{m}[\Delta(\sigma(\tau(x_1), x_2)) - \Delta(\sigma(\tau(x_2), x_2))] =$$

$$\frac{1}{m}[\Delta(\sigma(\tau(x_1), x_2)) - \Delta(\sigma(\tau(x_1), x_1))] \leqslant$$

$$\frac{l}{m}[\sigma(\tau(x_1), x_2) - \sigma(\tau(x_1), x_1)] \leqslant \frac{l M_0}{m} |x_1 - x_2|.$$

Observe finally that, for any $x \in \varDelta^{-1}(0)$ and $t \in J_x$, $\tau(\sigma(t, x)) = -t$. Moreover, for all $x \in \Omega$, $\varDelta(\sigma(t, x)) > 0$ when $T_-(x) \leqslant t < \tau(x)$ and $\varDelta(\sigma(t, x)) < 0$ for $\tau(x) < t \leqslant T_+(x)$.

Take $a < c \leqslant c_1 := \inf_{\varDelta^{-1}(0)} h$ and an auxiliary number $\varepsilon > 0$ such that $c - \varepsilon > a$. We define $H : S \to W$ and $G : h^c \to W$ by the formulae

$$H(x) = \begin{cases} \sigma\left(\dfrac{\beta(c-\varepsilon, x)\, \tau(\sigma(\beta(c,x),x))}{\tau(\sigma(\beta(c,x),x)) + \beta(c,x) - \beta(c-\varepsilon, x)}, x \right) & \text{if } x \in S \setminus h^{-1}(-\infty, c-\varepsilon] \\ x & \text{if } x \in h^{-1}(-\infty, c-\varepsilon], \end{cases}$$

$$G(x) = \begin{cases} x & \text{if } x \in h^{-1}(-\infty, c-\varepsilon] \\ \sigma\left(\dfrac{\beta(c-\varepsilon, x)\, \tau(\sigma(\beta(c, x), x))}{\beta(c-\varepsilon, x) - \beta(c, x)}, x \right) & \text{if } x \in h^{-1}(c-\varepsilon, c]. \end{cases}$$

It is a routine to check that $h \circ H$ does not increase and $h \circ G$ does not decrease along trajectories; hence $H(S) = h^c$, $G(h^c) = S$; moreover $H \circ G$ is the identity on h^c and $G \circ H$ is the identity on S. Since H and G are locally Lipschitz, the proof of part (i) of (A) is completed.

In a similar manner a homotopy $\chi : S \times [0, 1] \to W$ given by

$$\chi(x, \lambda) = \begin{cases} \sigma(\lambda T_+(x), x) & \text{if } x \in S \setminus h^a,\ \lambda \in [0, 1] \\ x & \text{if } x \in h^a,\ \lambda \in [0, 1] \end{cases}$$

is easily seen to be continuous, maps $S \times [0, 1]$ onto S and provides the required strong deformation retraction of S onto h^a. ∎

The above proof considerably simplifies if $h^{-1}[a, b]$ is compact. In this case the above homeomorphisms appear to be Lipschitz continuous instead of being merely locally Lipschitz.

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