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On the Curvature of Moduli Space of Special Lagrangian Submanifolds.

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Sunto. – *Si studia il tensore curvatura della metrica Riemanniana definita in modo naturale sullo spazio dei moduli di una sottovarietà Lagrangiana speciale di una varietà Calabi-Yau. Si ottengono alcune proprietà interessanti, in particolare si dimostra che la curvatura di Ricci è non negativa sotto una opportuna ipotesi che, secondo una congettura di N. Hitchin, è sempre verificata.*

Summary. – *In this paper we study the curvature tensor of the Riemannian metric defined in a natural way on the moduli space of compact special Lagrangian submanifolds of a Calabi-Yau manifold. We state some curvature properties and we prove that the Ricci curvature is non negative under an assumption on the determinant of g.*

1. – Introduction.

Special Lagrangian submanifolds are a field of great interest in theoretical physics and, specially after the pioneer work of Strominger, Yau and Zaslow, [6], they have become a central object in string theory and mirror symmetry.

In [3], McLean proved that the local moduli space, $\mathcal{M}(L)$, of a compact special Lagrangian submanifold L of a Calabi-Yau manifold is smooth, of real dimension equal to the first Betti number of L and the tangent space of $\mathcal{M}(L)$ at L is canonically identified to the space $\mathcal{H}^1(L)$ of harmonic 1-forms on L. In particular a natural Riemannian metric, g, is defined on $\mathcal{M}(L)$ induced by the L^2 norm on harmonic forms, [2], [3], [6].

Let $H^1(L, \mathbb{R})$ be the first De Rham cohomology group of L and $H^1(L, \mathbb{R})^*$ be the dual vector space, in [2], Hitchin proved that $(\mathcal{M}(L), g)$ can be isometrically embedded as a Lagrangian submanifold of $H^1(L, \mathbb{R}) \times H^1(L, \mathbb{R})^*$, with canonical symplectic and semi-Riemannian structures.

Using this approach we compute the full curvature tensor of g and we state some curvature properties. In particular we prove that the Ricci curvature is non negative under an assumption on the determinant of g that Hitchin conjectured to be always true. Furthermore we obtain that the natural Kähler metric on the local moduli space of special Lagrangian submanifolds with unitary line bundles, introduced by Strominger, Yau and Zaslow, is Ricci flat if and only if Hitchin's embedding has d-closed mean curvature form.

2. – Preliminaries.

Let $(X, \tilde{J}, \tilde{g}, \Omega)$ be an n-dimensional complex Calabi-Yau manifold equipped with a complex structure $\tilde{J}$, a Ricci flat Kähler metric $\tilde{g}$ and a covariant constant, nowhere vanishing, holomorphic $(n, 0)-$ form Ω satisfying:

$$\frac{\widetilde{\omega}^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \left(\frac{\sqrt{-1}}{2} \right)^n \Omega \wedge \overline{\Omega}, \tag{1}$$

where $\tilde{\omega}(\cdot, \cdot) = \tilde{g}(\cdot, \tilde{J} \cdot)$ is the Kähler form.

Let $\iota : L \to X$ be a compact special Lagrangian submanifold, that is an n-dimensional real submanifold such that $\iota^*(\tilde{\omega}) = 0$ and $\iota^*(\Im m\Omega) = 0$, in particular $\iota^*(\Re e\Omega) = \mathrm{vol}(L)$, where $\Re e\Omega$ and $\Im m\Omega$ are respectively the real and the imaginary part of Ω and $\mathrm{vol}(L)$ is the volum on L with respect to the restricted metric $g = \iota^*(\tilde{g})$.

R. C. McLean, in [3], proved the following:

THEOREM 1. – *A normal vector field V to a compact special Lagrangian submanifold L is the deformation vector field to a normal deformation through special Lagrangian submanifolds if and only if the corresponding 1-form $(\tilde{J}V)^\flat$ is harmonic. Thus the Zariski tangent space at L to the moduli space, $\mathcal{M}(L)$, of special Lagrangian submanifolds is naturally identified with the space of harmonic 1-forms and, in contrast to the case of complex submanifolds, there are no obstructions in extending a first order special Lagrangian deformation to an actual special Lagrangian deformation.*

In particular $\mathcal{M}(L)$ carries a Riemannian metric g defined as in the following:

$$g(X_1, X_2) = \int_L \langle \theta_1, \theta_2 \rangle d\,\mathrm{vol}(L) \tag{2}$$

where $X_1, X_2 \in T_L\mathcal{M}(L)$ are identified with harmonic 1-forms θ_1, θ_2 and $\langle \cdot, \cdot \rangle$ is the pointwise inner product on 1-forms.

In [3], N. Hitchin proved the following:

THEOREM 2. – *The following facts hold:*

(1) *There exists a C^∞ map $F : \mathcal{M}(L) \to H^1(L, \mathbb{R}) \times H^1(L, \mathbb{R})^*$;*

(2) *$F^*(G) = g$, where G is the natural flat semi-Riemannian metric on $H^1(L, \mathbb{R}) \times H^1(L, \mathbb{R})^*$;*

(3) *F embeds $\mathcal{M}(L)$ as a ω-Lagrangian submanifold, where ω is the natural symplectic structure on $H^1(L, \mathbb{R}) \times H^1(L, \mathbb{R})^*$; in particular locally $\mathcal{M}(L)$ is defined as the image of a section $d\phi : H^1(L, \mathbb{R}) \to T^*(H^1(L, \mathbb{R}))$, for some C^∞ map $\phi : H^1(L, \mathbb{R}) \to \mathbb{R}$.*

Using this approach we are able to compute the full curvature tensor of g and to state some interesting curvature properties.

3. – The geometry of $V \times V^*$.

In this section we illustrate general geometric properties of the product $V \times V^*$ where V is a finite dimensional vector space and V^* is its dual space. In the applications V will be the first De Rham cohomology group $H^1(L; \mathbb{R})$ of the compact special Lagrangian submanifold L.

3.1. *Semi-Riemannian structure.*

Let V be a m-dimensional real vector space and let V^* be the dual vector space, a non degenerate scalar product, G, of positivity index = negativity index = m is canonically defined on $V \times V^*$ by posing:

$$G\left(\begin{pmatrix} v \\ \alpha \end{pmatrix}, \begin{pmatrix} w \\ \beta \end{pmatrix}\right) = \frac{1}{2}(\beta(v) + \alpha(w)) \tag{3}$$

$\forall v, w \in V$, $\forall \alpha, \beta \in V^*$.

G defines a natural flat semi-Riemannian metric on the manifold $V \times V^*$. Let $\mathcal{B} = \{X_1, \ldots, X_m\}$ be a basis of V and let $\mathcal{B}^* = \{X_1^*, \ldots, X_m^*\}$ be the dual basis, let $\{x_1, \ldots, x_m, y_1, \ldots, y_m\}$ be coordinates on $V \times V^*$ with respect to the basis defined by $\mathcal{B}$ and $\mathcal{B}^*$, independently of the chosen basis $\mathcal{B}$, the metric G has the following expression:

$$G = \frac{1}{2}(dx_i \otimes dy_i + dy_i \otimes dx_i) \tag{4}$$

where here, as in the following, we use Einstein's convention on repeated indices.

3.2. *Symplectic structure.*

A natural structure of symplectic vector space is defined on $V \times V^*$ by:

$$\omega\left(\begin{pmatrix} v \\ \alpha \end{pmatrix}, \begin{pmatrix} w \\ \beta \end{pmatrix}\right) = \frac{1}{2}(\beta(v) - \alpha(w)). \tag{5}$$

ω defines a natural almost symplectic structure on the manifold $V \times V^*$; in coordinates, as before, ω has the following expression:

$$\omega = \frac{1}{2}(dx_i \otimes dy_i - dy_i \otimes dx_i) = dx_i \wedge dy_i \tag{6}$$

in particular $d\omega = 0$, then $(V \times V^*, \omega)$ is a symplectic manifold.

3.3. *Kähler structure.*

Let us suppose now that V is endowed with an euclidean scalar product, g, then $V \times V^*$ becomes an hermitian vector space; namely, denoted by $\flat : V \to V^*$ and $\natural = \flat^{-1} : V^* \to V$ the musical isomorphisms induced by g, we can define a complex structure J_g on $V \times V^*$ by:

$$J_g \begin{pmatrix} v \\ \alpha \end{pmatrix} = \begin{pmatrix} -\natural(\alpha) \\ \flat(v) \end{pmatrix}. \tag{7}$$

It is easily seen that:

$$\begin{aligned} &\omega(J_g \cdot, J_g \cdot) = \omega(\cdot, \cdot) \\ &\omega(\cdot, J_g \cdot) > 0 \\ &G(J_g \cdot, J_g \cdot) = -G(\cdot, \cdot). \end{aligned} \tag{8}$$

In particular let $g = g_{ij}\, dx_i \otimes dx_j$ be a Riemannian metric on the manifold V and let $T(V \times V^*)$ be the tangent bundle of $V \times V^*$; we define $J_g \in \operatorname{End}(T(V \times V^*))$ in the following way:

$$\begin{cases} J_g\left(\dfrac{\partial}{\partial x_i}\right) = g_{ij} \dfrac{\partial}{\partial y_j} \\[2ex] J_g\left(\dfrac{\partial}{\partial y_i}\right) = -g^{ij} \dfrac{\partial}{\partial x_j} \end{cases}$$

where $g^{ij} = (g^{-1})_{ij}$.

From (8) we have immediately that J_g is a *calibrated* almost complex structure on $(V \times V^*, \omega)$, in the sense of [1], then $\widetilde{G}_g(\cdot, \cdot) = \omega(\cdot, J_g \cdot)$ is a Riemannian metric on the manifold $V \times V^*$ and in coordinates we have the following expression:

$$\widetilde{G}_g = \frac{1}{2}(g_{ij}\, dx_i \otimes dx_j + g^{ij}\, dy_i \otimes dy_j) \tag{10}$$

$(V \times V^*, J_g, \widetilde{G}_g)$ is an almost Kähler manifold.

Let $N(J_g)(\cdot, \cdot) = [J_g \cdot, J_g \cdot] - J_g[\cdot, J_g \cdot] - J_g[J_g \cdot, \cdot] - [\cdot, \cdot]$ be the Nijenhuis tensor of J_g, direct computation shows the following expression:

$$\begin{cases} N(J_g)\left(\dfrac{\partial}{\partial x_i}, \dfrac{\partial}{\partial x_j}\right) = \left(\dfrac{\partial g_{il}}{\partial x_j} - \dfrac{\partial g_{jl}}{\partial x_i}\right) J_g\left(\dfrac{\partial}{\partial y_l}\right) \\ N(J_g)\left(\dfrac{\partial}{\partial x_i}, \dfrac{\partial}{\partial y_j}\right) = g^{jr}\left(\dfrac{\partial g_{il}}{\partial x_r} - \dfrac{\partial g_{rl}}{\partial x_i}\right) \dfrac{\partial}{\partial y_l}\,. \\ N(J_g)\left(\dfrac{\partial}{\partial y_i}, \dfrac{\partial}{\partial y_j}\right) = \left(g^{ir}\dfrac{\partial g^{jl}}{\partial x_r} - g^{jr}\dfrac{\partial g^{il}}{\partial x_r}\right) \dfrac{\partial}{\partial x_l} \end{cases} \tag{11}$$

We get:

Proposition 3. – *The following facts are equivalent:*

(a) J_g *is an integrable almost complex structure on* $V \times V^*$;

(b) $\forall\, i, l, r = 1, \ldots, m$ *hold:*

$$\frac{\partial g_{il}}{\partial x_r} = \frac{\partial g_{rl}}{\partial x_i}\,;$$

(c) $\exists\, \phi \in C^\infty(V, \mathbb{R})$ *such that* $\forall\, i, j = 1, \ldots, m$ *is:*

$$g_{ij} = \frac{\partial^2 \phi}{\partial x_i\, \partial x_j}\,;$$

(d) $(X, Y, Z) \to (L_X g)(Y, Z)$ *is a symmetric* 3*-tensor on* V, *where* $L_X g$ *means Lie derivative of* g *along* X;

(e) $(V \times V^*, J_g, \widetilde{G}_g)$ *is a Kähler manifold.*

Proof. – (a) $\Leftrightarrow$ (b): From (11) it follows immediately that if $N(J_g) = 0$ then (b) is satisfied; conversely let us suppose that (b) holds, we need to prove that $g^{ir}\dfrac{\partial g^{jl}}{\partial x_r} = g^{jr}\dfrac{\partial g^{il}}{\partial x_r}$ $\forall\, i, j, l = 1, \ldots, m$, we have:

$$g^{ir}\frac{\partial g^{jl}}{\partial x_r} = g^{jr}\frac{\partial g^{il}}{\partial x_r} \Leftrightarrow g_{ih}\, g^{ir}\frac{\partial g^{jl}}{\partial x_r} = g_{ih}\, g^{jr}\frac{\partial g^{il}}{\partial x_r} \Leftrightarrow \frac{\partial g^{jl}}{\partial x_h} = -g^{jr}\frac{\partial g_{ir}}{\partial x_r}\, g^{il}$$

and, by (b):

$$\Leftrightarrow \frac{\partial g^{jl}}{\partial x_h} = -g^{jr}\frac{\partial g_{ir}}{\partial x_h} g^{il} \Leftrightarrow \frac{\partial g^{jl}}{\partial x_h} = g_{ir}\frac{\partial g^{jr}}{\partial x_h} g^{il};$$

$(b) \Leftrightarrow (c)$: let $\theta_j = g_{ij}\,dx_i$, θ_j is a closed 1-form on V if and only if (b) holds, but $d\theta_j = 0$ if and only if $\exists\, \varphi_j \in C^\infty(V, \mathbb{R})$ such that $\theta_j = d\varphi_j$, or $g_{ij} = \frac{\partial \varphi_j}{\partial x_i}$, $\forall i, j = 1, \ldots, m$, from $g_{ij} = g_{ji}$ it follows that the 1-form $\vartheta = \varphi_j\,dx_j$ is closed, then $\varphi_j = \frac{\partial \phi}{\partial x_j}$, for some $\phi \in C^\infty(V, \mathbb{R})$;

$(b) \Leftrightarrow (d)$ $(L_{\partial/\partial x_i} g)\left(\frac{\partial}{\partial x_r}, \frac{\partial}{\partial x_l}\right) = \frac{\partial g_{rl}}{\partial x_i}$;

$(a) \Leftrightarrow (e)$ as the Kähler form of $\widetilde{G}_g$ is ω and $d\omega = 0$, then $(V \times V^*, J_g, \widetilde{G}_g)$ is Kähler if and only if J_g is integrable. ■

Remark that metrics satisfying (c) are called metrics of *Hessian type*, [4].

3.4. *Curvature computation.*

From now on we suppose that g is a Riemannian metric on V of Hessian type. First of all we need some formulas:

LEMMA 4. – $\forall i, k = 1, \ldots, m$ *the following formula holds:*

$$g^{hr}\frac{\partial^2 g^{ik}}{\partial x_h\,\partial x_r} - \frac{\partial g^{hk}}{\partial x_l}\frac{\partial g^{il}}{\partial x_h} = g^{li}\frac{\partial^2 g^{hk}}{\partial x_h\,\partial x_l} - \frac{\partial g^{hr}}{\partial x_h}\frac{\partial g^{ik}}{\partial x_r}. \tag{12}$$

PROOF. – From Proposition 3 we get: $g^{hr}\frac{\partial g^{ik}}{\partial x_r} = g^{li}\frac{\partial g^{hk}}{\partial x_l}$, then:

$$\frac{\partial}{\partial x_h}\left(g^{hr}\frac{\partial g^{ik}}{\partial x_r}\right) = \frac{\partial}{\partial x_h}\left(g^{li}\frac{\partial g^{hk}}{\partial x_l}\right),$$

or:

$$\frac{\partial g^{hr}}{\partial x_h}\frac{\partial g^{ik}}{\partial x_r} + g^{hr}\frac{\partial^2 g^{ik}}{\partial x_h\,\partial x_r} = g^{li}\frac{\partial g^{hk}}{\partial x_h\,\partial x_l} + \frac{\partial g^{li}}{\partial x_h}\frac{\partial g^{hk}}{\partial x_r},$$

thus the proof is complete. ■

The following Lemma is an exercise:

LEMMA 5. – *The derivative of the determinant of a square matrix* g, *of order* m *is the sum of* m *terms,* $d_1, \ldots, d_m$, *where* d_i *is the determinant of the matrix obtained by* g *substituting the i-eme column with its derivative.*

COROLLARY 6. – $\forall k = 1, \ldots, m$ *the following formulas hold:*

$$g^{ij} \frac{\partial g_{ij}}{\partial x_k} = \frac{\partial \ln \det g}{\partial x_k} \tag{13}$$

$$\frac{\partial g^{kj}}{\partial x_j} = -g^{rk} \frac{\partial \ln \det g}{\partial x_r} . \tag{14}$$

PROOF. – Let us denote by $g_{(1)}, \ldots, g_{(m)}$ the columns of the matrix g and by $g_{[ij]}$ the algebraic complement of g_{ij}, we have:

$$\frac{\partial \det g}{\partial x_k} = \sum_{i=1}^{m} \det \left(g_{(1)}, \ldots, \frac{\partial g_{(i)}}{\partial x_k}, \ldots, g_{(m)} \right) = \sum_{i,j=1}^{m} (-1)^{i+j} \frac{\partial g_{ij}}{\partial x_k} \det g_{[ij]} = g^{ij} \frac{\partial g_{ij}}{\partial x_k} \det g ,$$

thus (13) is proved.

(14) follows from:

$$g^{ij} \frac{\partial g_{ij}}{\partial x_k} = \frac{\partial \ln \det g}{\partial x_k}, \quad \frac{\partial g_{ij}}{\partial x_k} = \frac{\partial g_{ik}}{\partial x_j}, \quad \text{and } g^{ij} \frac{\partial g_{ik}}{\partial x_j} = -g_{ik} \frac{\partial g^{ij}}{\partial x_j} . \qquad \blacksquare$$

In the following we denote $y_i = x_{i+m}$, $\widetilde{G}_g = \widetilde{G}$, $\widetilde{\nabla}$ the Levi-Civita connection of $\widetilde{G}$, $\widetilde{\Gamma}^{\alpha}_{\beta\gamma}$ Cristoffel's symbols of $\widetilde{G}$ and $\widetilde{R}(X, Y) = [\widetilde{\nabla}_X, \widetilde{\nabla}_Y] - \widetilde{\nabla}_{[X, Y]}$ the Riemann curvature tensor of $\widetilde{G}$, moreover we assume that italic letters $i, j, k, \ldots$ run from 1 to m, and greek letters $\alpha, \beta, \gamma, \ldots$ run from 1 to $2m$.

Direct computation gives:

$$\widetilde{\Gamma}^i_{jk} = \frac{1}{2} g^{il} \frac{\partial g_{jl}}{\partial x_k} ,$$

$$\widetilde{\Gamma}^i_{j+m\, k+m} = -\frac{1}{2} g^{il} \frac{\partial g^{jk}}{\partial x_l} ,$$

$$\widetilde{\Gamma}^{i+m}_{j+m\, k} = \frac{1}{2} g_{il} \frac{\partial g^{jl}}{\partial x_k} ,$$

$$\widetilde{\Gamma}^{i+m}_{jk} = \widetilde{\Gamma}^{i+m}_{j+m\, k+m} = \widetilde{\Gamma}^i_{j+m\, k} = 0 ;$$

and, posed:

$$\widetilde{R}(X_\alpha, X_\beta)\, X_\gamma = \widetilde{R}^{\delta}_{\alpha\beta\gamma} X_\delta ,$$

we get:

$$\widetilde{R}^j_{ijk} = \frac{1}{4}\left(-\frac{\partial g^{jl}}{\partial x_j}\frac{\partial g_{il}}{\partial x_k} + \frac{\partial g^{js}}{\partial x_i}\frac{\partial g_{js}}{\partial x_k}\right);$$

$$\widetilde{R}^{j+m}_{i\ j+m\ k} = \frac{1}{2}\frac{\partial}{\partial x_i}\left(g_{jl}\frac{\partial g^{jl}}{\partial x_k}\right) - \frac{1}{4}\frac{\partial g^{js}}{\partial x_s}\frac{\partial g_{ij}}{\partial x_k} - \frac{1}{4}\frac{\partial g_{jr}}{\partial x_k}\frac{\partial g^{jr}}{\partial x_i};$$

$$\widetilde{R}^j_{i+m\ j\ k+m} = \frac{1}{2}g^{jl}\frac{\partial^2 g^{ik}}{\partial x_j \partial x_l} + \frac{1}{4}\frac{\partial g^{ik}}{\partial x_r}\frac{\partial g^{rs}}{\partial x_s} - \frac{1}{4}\frac{\partial g^{il}}{\partial x_r}\frac{\partial g^{kr}}{\partial x_l};$$

$$\widetilde{R}^{j+m}_{i+m\ j+m\ k+m} = \frac{1}{4}\left(-\frac{\partial g^{ir}}{\partial x_j}\frac{\partial g^{jk}}{\partial x_r} + \frac{\partial g^{jr}}{\partial x_j}\frac{\partial g^{ik}}{\partial x_r}\right);$$

$$\widetilde{R}^j_{ij\ k+m} = \widetilde{R}^{j+m}_{i\ j+m\ k+m} = 0\,.$$

In particular the Ricci tensor is given by:

$$(15)\qquad \begin{cases} \widetilde{R}_{ik} & = \dfrac{1}{2}\left(\dfrac{\partial^2 \ln \det g}{\partial x_i \partial x_k} - g^{ls}\dfrac{\partial g_{ik}}{\partial x_l}\dfrac{\partial \ln \det g}{\partial x_s}\right) \\ \widetilde{R}_{i\ k+m} & = 0 \\ \widetilde{R}_{i+m\ k+m} & \dfrac{1}{2}g^{li}\left(g^{kr}\dfrac{\partial^2 \ln \det g}{\partial x_l \partial x_r} + \dfrac{\partial g^{kr}}{\partial x_l}\dfrac{\partial \ln \det g}{\partial x_r}\right) = g^{li}g^{kr}\widetilde{R}_{rl} \end{cases};$$

and the scalar curvature of $\widetilde{G}$ is:

$$(16)\qquad \widetilde{S} = g^{ij}\left(\frac{\partial^2 \ln \det g}{\partial x_i \partial x_j} - \frac{\partial \ln \det g}{\partial x_i}\frac{\partial \ln \det g}{\partial x_j}\right).$$

From previous computation we get immediately the following:

PROPOSITION 7. – *If g is a Riemannian metric on V of Hessian type then $(V \times V^*, J_g, \widetilde{G}_g)$ is a Ricci flat Kähler manifold if and only if $\forall i, k = 1, \ldots, m$ holds:*

$$\frac{\partial^2 \ln \det g}{\partial x_i \partial x_k} - g^{ls}\frac{\partial g_{ik}}{\partial x_l}\frac{\partial \ln \det g}{\partial x_s} = 0\,.$$

4. – Moduli space of special Lagrangian submanifolds.

Let us go back to the case in which we are interested: using notations of section 2, let $\iota : L \to X$ be a compact special Lagrangian submanifold of a complex Calabi-Yau manifold X, let $\mathcal{M}(L)$ be the local moduli space, let $F : \mathcal{M}(L) \to H^1(L, \mathbb{R}) \times H^1(L, \mathbb{R})^*$ and let $\phi : H^1(L, \mathbb{R}) \to \mathbb{R}$ as in theorem

2. Let $m = \dim_{\mathbb{R}} H^1(L, \mathbb{R})$, posing $V = H^1(L, \mathbb{R})$ we can apply the results of section 3, in particular we have that $(H^1(L, \mathbb{R}) \times H^1(L, \mathbb{R})^*, G)$ is a flat semi-Riemannian manifold and $(H^1(L, \mathbb{R}) \times H^1(L, \mathbb{R})^*, \omega)$ is a symplectic manifold, moreover in coordinates $\mathcal{M}(L)$ is defined by:

$$y_i = \frac{\partial \phi}{\partial x_i} \qquad i = 1, \dots, m \tag{17}$$

the tangent space $T(\mathcal{M}(L))$ is spanned by:

$$X_i = \frac{\partial}{\partial x_i} + \frac{\partial^2 \phi}{\partial x_i \partial x_j} \frac{\partial}{\partial y_j} \qquad i, j = 1, \dots, m \tag{18}$$

and:

$$(G_{|\mathcal{M}(L)})_{ij} = g_{ij} = \frac{\partial^2 \phi}{\partial x_i \partial x_j}; \tag{19}$$

thus $(H^1(L, \mathbb{R}) \times H^1(L, \mathbb{R})^*, J_g, \widetilde{G}_g)$ is a Kähler manifold.

Let us denote $J_g X_i = N_i$, then $N_i = -\frac{\partial}{\partial x_i} + g_{ij} \frac{\partial}{\partial y_j}$, and $J_g N_i = -X_i$; we get immediately:

LEMMA 8. – $\forall\, i, j = 1, \dots, m$, *the following facts hold:*

(1) $\widetilde{G}_g(X_i, N_j) = 0$,

(2) $\widetilde{G}_g(N_i, N_j) = g_{ij}$,

(3) $G(X_i, N_j) = 0$,

(4) $G(N_i, N_j) = -g_{ij}$,

(5) $\omega(N_i, N_j) = 0$.

In particular $\{N_1, \dots, N_m\}$ span the normal bundle of $\mathcal{M}(L)$, $(T(\mathcal{M}(L))^\perp$, with respect to $\widetilde{G}_g$ and with respect to G.

4.1. *Extrinsic geometry.*

In this section we will study the extrinsic geometry of the Riemannian manifold $(\mathcal{M}(L), g)$ embedded isometrically in the Kähler manifold $(H^1(L, \mathbb{R}) \times H^1(L, \mathbb{R})^*, J_g, \widetilde{G}_g)$ first, and in the flat semi-Riemannian manifold $(H^1(L, \mathbb{R}) \times H^1(L, \mathbb{R})^*, G)$ after.

Let us denote by ∇ the Levi-Civita connection of g on $\mathcal{M}(L)$, then by using the notations of previous sections we get:

$$\widetilde{\nabla}_{X_i} X_j = g^{kl} \frac{\partial g_{ij}}{\partial x_l} \frac{\partial}{\partial x_k} = \nabla_{X_i} X_j + N_{ij}$$

where:

$$N_{ij} = -\frac{1}{2} g^{rl} \frac{\partial g_{ij}}{\partial x_l} N_r,$$

the second fundamental form of g with respect to $\widetilde{G}_g$ at the point $p \in \mathcal{M}(L)$ is then given by:

$$\widetilde{B}: T_p\mathcal{M}(L) \times T_p\mathcal{M}(L) \to (T_p\mathcal{M}(L))^\perp$$

$$\widetilde{B}(X_i, X_j) = N_{ij}$$

and the corresponding shape operator with respect to a normal vector N is $\mathcal{S}_N: T_p\mathcal{M}(L) \to T_p\mathcal{M}(L)$ defined by:

$$\widetilde{G}_g(\mathcal{S}_N(X), Y) = \widetilde{G}_g(\widetilde{B}(X, Y), N);$$

in particular the *mean curvature vector*, $H_{\widetilde{G}_g}$, is given by

$$H_{\widetilde{G}_g} = -\frac{1}{2} \frac{\partial g^{rk}}{\partial x_k} N_r,$$

or:

$$H_{\widetilde{G}_g} = -\frac{1}{2} g^{rk} \frac{\partial \ln \det g}{\partial x_k} N_r. \tag{20}$$

From (20) we get immediately the following:

PROPOSITION 9. – $(\mathcal{M}(L), g) \hookrightarrow (H^1(L, \mathbb{R}) \times H^1(L, \mathbb{R})^*, \widetilde{G}_g)$ *is minimal if and only if* $\det g$ *is constant; moreover if* $(\mathcal{M}(L), g)$ *is minimal then the Kähler manifold* $(H^1(L, \mathbb{R}) \times H^1(L, \mathbb{R})^*, J_g, \widetilde{G}_g)$ *is Ricci flat.*

Let $\flat_{\widetilde{G}_g}: T(H^1(L, \mathbb{R}) \times H^1(L, \mathbb{R})^*) \to T^*(H^1(L, \mathbb{R}) \times H^1(L, \mathbb{R})^*)$ be the musical isomorphism induced by $\widetilde{G}_g$, starting from $H_{\widetilde{G}_g}$ we can define the *mean curvature form*, η_g, in the following way:

$$\eta_g = \flat_{\widetilde{G}_g}(J_g H_{\widetilde{G}_g}), \tag{21}$$

we have:

$$\begin{cases} \eta_g(X_i) = \dfrac{1}{2} \dfrac{\partial \ln \det g}{\partial x_i} \\ \eta_g(N_i) = 0\,. \end{cases} \tag{22}$$

We get the following:

PROPOSITION 10. – *$(H^1(L, \mathbb{R}) \times H^1(L, \mathbb{R})^*, \widetilde{G}_g)$ is Ricci flat if and only if η_g is d-closed.*

PROOF. – Let us compute $d\eta_g$:

$$d\eta_g(X_j, X_k) = X_j \eta_g(X_k) - X_k \eta_g(X_j) = 0\ ;$$

$$d\eta_g(X_j, N_k) = -N_k \eta_g(X_j) - \eta_g([X_j, N_k]) = \frac{1}{2} \frac{\partial^2 \ln \det g}{\partial x_k \partial x_j} - \eta_g\left(2 \frac{\partial g_{jk}}{\partial x_l} \frac{\partial}{\partial y_l}\right) =$$

$$= \frac{1}{2}\left(\frac{\partial^2 \ln \det g}{\partial x_k \partial x_j} - \frac{\partial g_{jk}}{\partial x_l} \frac{\partial \ln \det g}{\partial x_r} g^{rl}\right) = \widetilde{R}_{jk};$$

$$d\eta_g(N_j, N_k) = 0\ ;$$

thus we get the statement. ■

We remark that in [5] is proved that the mean curvature form of a Lagrangian submanifold is d-closed if the ambient space is Ricci flat.

The extrinsic geometry of $(\mathcal{M}(L), g) \hookrightarrow (H^1(L, \mathbb{R}) \times H^1(L, \mathbb{R})^*, G)$ is quite similar: let $\overline{\nabla}$ be the Levi-Civita connection of the semi-Riemannian metric G, then we have:

$$\overline{\nabla}_{X_i} X_j = \nabla_{X_i} X_j + \overline{N}_{ij}$$

where

$$\overline{N}_{ij} = \frac{1}{2} \frac{\partial g_{ij}}{\partial x_k}\left(\frac{\partial}{\partial y_k} - g^{kl} \frac{\partial}{\partial x_l}\right) = -N_{ij};$$

denote $\overline{N}_k = -g^{kl} \dfrac{\partial}{\partial x_l} + \dfrac{\partial}{\partial y_k}$, then the mean curvature vector H_G is given by:

$$H_G = -H_{\widetilde{G}_g},$$

hence we get:

PROPOSITION 11. – *The following facts are equivalent:*

(1) $\det g$ *is constant*;

(2) $(\mathcal{M}(L), g) \hookrightarrow (H^1(L, \mathbb{R}) \times H^1(L, \mathbb{R})^*, \widetilde{G}_g)$ *is minimal*;

(3) $(\mathcal{M}(L), g) \hookrightarrow (H^1(L, \mathbb{R}) \times H^1(L, \mathbb{R})^*, G)$ *is minimal.*

Before closing this section, we introduce a tensor that will be useful for curvature: let $p \in \mathcal{M}(L)$, define $h : T_p\mathcal{M}(L) \times T_p\mathcal{M}(L) \times T_p\mathcal{M}(L) \to \mathbb{R}$

by:

$$h(X_i, X_j, X_k) = -\,\omega(X_i, \tilde{\nabla}_{X_j} X_k),$$

the following hold:

LEMMA 12. – $h(X_i, X_j, X_k) = \frac{1}{2}(L_{X_i} g)(X_j, X_k)$.

PROOF. – From the definition of h it follows that

$$h(X_i, X_j, X_k) = -\,\widetilde{G}_g(J_g(X_i), \tilde{\nabla}_{X_j} X_k) = -\,\widetilde{G}_g(N_i, N_{jk}) =$$

$$= \frac{1}{2} g^{rl} \frac{\partial g_{jk}}{\partial x_l} g_{ir} = \frac{1}{2} \frac{\partial g_{jk}}{\partial x_i} = \frac{1}{2}(L_{X_i} g)(X_j, X_k). \quad \blacksquare$$

COROLLARY. – 13. – *h is a symmetric 3-tensor on $\mathcal{M}(L)$.*

4.2. *Curvature.*

Let us denote by R the Riemann curvature tensor of the metric g on $\mathcal{M}(L)$, using the isometric embedding $(\mathcal{M}(L), g) \hookrightarrow (H^1(L, \mathbb{R}) \times H^1(L, \mathbb{R})^*, G)$, as G is flat, by Gauss equation, we get:

$$G(R(X, Y)Z, T) = -G(\overline{B}(Y, T), \overline{B}(X, Z)) + G(\overline{B}(X, T), \overline{B}(Y, Z)),$$

$\forall\ X, Y, Z, T$ vector fields tangent to $\mathcal{M}(L)$; in particular direct computation shows that:

$$R_{ijkl} = G(N_{jl}, N_{ik}) - G(N_{il}, N_{jk}) = \frac{1}{4} g^{rs} \left(\frac{\partial g_{jl}}{\partial x_s} \frac{\partial g_{ik}}{\partial x_r} - \frac{\partial g_{il}}{\partial x_s} \frac{\partial g_{jk}}{\partial x_r} \right),$$

then the Ricci tensor of g has the following expression:

$$R_{ik} = R^j_{ijk} = \frac{1}{4} \left(\frac{\partial g^{lj}}{\partial x_j} \frac{\partial g_{ik}}{\partial x_l} - \frac{\partial g^{lj}}{\partial x_i} \frac{\partial g_{jk}}{\partial x_l} \right).$$

The following Lemma provides us useful formulas for the Ricci tensor and the scalar curvature, S, of the metric g:

LEMMA 14. – *The following formulas hold*:

$$R_{ik} = -\,\widetilde{G}_g(H_{\widetilde{G}_g}, N_r)\, h^r_{ik} + h^l_{si} h^s_{lk}$$

$$S = -\,\widetilde{G}_g(H_{\widetilde{G}_g}, H_{\widetilde{G}_g}) + h_{klj} h^{jlk}.$$

PROOF. – From previous expression of the Ricci tensor, using (13) and (14), we get:

$$R_{ik} = -\frac{1}{4}\left(g^{lr}\frac{\partial g_{ik}}{\partial x_l}\frac{\partial \ln \det g}{\partial x_r} - g^{rj}\frac{\partial g_{rs}}{\partial x_i}g^{sl}\frac{\partial g_{jl}}{\partial x_k}\right) = -\widetilde{G}_g(H_{\widetilde{G}_g}, N_r)\, h^r_{ik} + h^l_{si}h^s_{lk};$$

moreover:

$$S = g^{ik}R_{ik} = -\frac{1}{4}g^{ik}\frac{\partial \ln \det g}{\partial x_i}\frac{\partial \ln \det g}{\partial x_k} - \frac{1}{4}g^{ik}\frac{\partial g^{lj}}{\partial x_i}\frac{\partial g_{jk}}{\partial x_i} =$$

$$= -\widetilde{G}_g(H_{\widetilde{G}_g}, H_{\widetilde{G}_g}) + \frac{1}{4}g^{rj}g^{sl}\frac{\partial g_{rs}}{\partial x_i}g^{ik}\frac{\partial g_{jk}}{\partial x_i} = -\widetilde{G}_g(H_{\widetilde{G}_g}, H_{\widetilde{G}_g}) + h_{klj}h^{jlk}. \qquad \blacksquare$$

From previous Lemma we get the main result:

PROPOSITION 15. – *If* $\det g$ *is constant then the Ricci tensor of* g *is non negative, moreover the scalar curvature is zero if and only if* $(\mathcal{M}(L), g)$ *is totally geodesic with respect to* $\widetilde{G}_g$ *or, equivalently, with respect to* G.

PROOF. – If $\det g$ is constant then $H_{\widetilde{G}_g} = 0$, and from previous Lemma we get $R_{ik} = h^l_{si}h^s_{lk}$, where $h^l_{si} = g^{lk}h_{ksi}$, this provides the first statement; moreover is $S = h_{klj}h^{jlk} = \|h\|^2$, where $\|h\|$ means norm of the tensor h with respect to g, then $S \geqslant 0$, and $S = 0$ if and only if $h = 0$, or $\widetilde{B} = 0$. $\blacksquare$

We remark that, via Proposition 3 of [2], the minimality of $(\mathcal{M}(L), g)$ is equivalent to be special Lagrangian in Hitchin's sense.

Before closing, we also remark that Propositions 3 and 7 describe the differential geometry of the local moduli space of special Lagrangian submanifolds with flat unitary line bundles, introduced by Strominger, Yau and Zaslov, [6]; more precisely, J_g is the natural Kähler structure and, under the condition $\det g$ constant, $\widetilde{G}_g$ is the natural Calabi-Yau metric, as obtained in [2]. Moreover Proposition 10 can be restate as in the following:

PROPOSITION 16. – *The natural Kähler structure on the local moduli space of special Lagrangian submanifolds with unitary line bundles is Calabi-Yau if and only if the mean curvature form of Hitchin's embedding is d-closed.*

Note Added. Recently D. Matessi (Math. DG/0011061) proved that Hitchin's conjecture about the constancy of the determinant of g is not true if the dimension of the special lagrangian submanifold L is 3.

REFERENCES

[1] M. GROMOV, *Pseudo holomorphic curves in symplectic manifolds*, Inventiones Mathematicae, **82** (1985), 307-347.
[2] N. J. HITCHIN, *The moduli space of special Lagrangian submanifolds*, Annali Scuola Normale Superiore di Pisa (to appear), dg-ga/9711002.
[3] R. C. MCLEAN, *Deformations of calibrated submanifolds*, Duke University preprint, January (1996).
[4] V. RUUSKA, *Riemannian polarizations*, Ann. Acad. Sci. Fenn. Math. Diss. No. 106 (1996).
[5] K. SMOCZYK, *A canonical way to deform a Lagrangian submanifold*, dg-ga/9605005.
[6] A. STROMINGER - S.-T. YAU - E. ZASLOV, *Mirror Symmetry is T-duality*, Nuclear Phys. B, **479**, no. 1-2 (1996), 243-259.

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