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L^p-Improving Properties of Measures Supported on Curves on the Heisenberg Group. II.

Silvia Secco

Sunto. – *In questo lavoro riprendiamo la trattazione del cosiddetto fenomeno di L^p-improving per curve nel gruppo di Heisenberg iniziato nel precedente articolo* [7]. *Il problema riguarda lo studio delle proprietà di limitatezza L^p-L^q per operatori di convoluzione con misure finite a supporto su curve nel gruppo di Heisenberg. Sia Γ una curva C^∞ regolare nel gruppo di Heisenberg $\boldsymbol{H}_1$ definita da*

$$\Gamma : I \to \boldsymbol{H}_1 \qquad s \mapsto \Gamma(s) = (\psi_1(s), \psi_2(s), \psi_3(s))$$

dove I è un intervallo limitato di $\boldsymbol{R}$ e $\psi_1(s)$, $\psi_2(s)$, $\psi_3(s)$ sono funzioni C^∞ a valori reali. Definita la misura

$$\langle \mu, f \rangle = \int_I f(\Gamma(s))\, ds \qquad f \in C_c(\boldsymbol{H}_1),$$

consideriamo il corrispondente operatore di convoluzione a destra con μ

$$Tf(w) = f * \mu(w) = \int_I f(w \cdot (\Gamma(s))^{-1})\, ds \qquad w \in \boldsymbol{H}_1.$$

Nella prima parte di questo lavoro forniamo alcune limitazioni sull'insieme caratteristico

$$\mathcal{C} = \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) \in [0,1] \times [0,1] : T \text{ è limitato da } L^p(\boldsymbol{H}_1) \text{ a } L^q(\boldsymbol{H}_1) \right\}$$

dell'operatore T, precisamente proviamo che l'insieme $\mathcal{C}$ è contenuto nel trapezio chiuso di vertici

$$A = (0,0),\ B = (1,1),\ C = (2/3, 1/2),\ D = (1/2, 1/3).$$

Nella seconda parte di questo lavoro focalizziamo invece l'attenzione su curve nel gruppo di Heisenberg $\boldsymbol{H}_1$ aventi vettore tangente nell'origine parallelo al centro del gruppo. Più precisamente, consideriamo una curva $\gamma(s)$ data da

$$\gamma : I \to \boldsymbol{H}_1 \qquad s \mapsto \gamma(s) = (s^m, s^n, s) \tag{1}$$

dove $I = [0, R]$, $R > 0$, e m, n sono numeri reali distinti maggiori di uno. Proviamo che l'insieme caratteristico dell'operatore U definito dalla formula

$$Uf(w) = \int_I f(w \cdot (\gamma(s))^{-1})\, ds \qquad w \in \boldsymbol{H}_1$$

è contenuto nel trapezio chiuso di vertici $A=(0,0)$, $B=(1,1)$, $P_1=\left(\frac{m+n-1}{m+n+1}, \frac{m+n-2}{m+n+1}\right)$, $P_2=\left(\frac{3}{m+n+1}, \frac{2}{m+n+1}\right)$ *con la sola possibile eccezione del segmento chiuso congiungente i due punti* P_1 *e* P_2 *se* $m+n\geqslant 5$, *ed è l'intero trapezio chiuso ABCD se* $m+n<5$. *I risultati ottenuti per l'operatore U rimangono validi sostituendo la curva* (1) *con una più generale curva* $\Gamma(s)=(s^m+o(s^m), s^n+o(s^n), s)$, *per s in un intorno dell'origine.*

Summary. – L^p-L^q *estimates are obtained for convolution operators by finite measures supported on curves in the Heisenberg group whose tangent vector at the origin is parallel to the centre of the group.*

1. – Introduction.

In this paper we continue the study of the L^p-L^q boundedness properties of convolution operators by finite measures supported on curves in the Heisenberg group which we started in a previous paper [7].

As in [7], let $\boldsymbol{H}_1$ be the Heisenberg group, that is $\boldsymbol{R}^3$ with group law given by

$$(x, y, t)\cdot(x', y', t') = \left(x+x', y+y', t+t'+\frac{1}{2}(xy'-x'y)\right).$$

We consider a smooth regular curve Γ in the Heisenberg group $\boldsymbol{H}_1$ which is defined as

$$\Gamma : I \to \boldsymbol{H}_1 \qquad s \mapsto \Gamma(s) = (\psi_1(s), \psi_2(s), \psi_3(s)) \tag{1}$$

where I is a bounded interval of $\boldsymbol{R}$ and $\psi_1(s)$, $\psi_2(s)$, $\psi_3(s)$ are smooth real valued functions.

For any continuous compactly supported function f on $\boldsymbol{H}_1$ we define the measure

$$\langle \mu, f\rangle = \int_I f(\Gamma(s))\, ds \tag{2}$$

and we consider the corresponding right convolution operator by μ

$$Tf(w) = f*\mu(w) = \int_I f(w\cdot(\Gamma(s))^{-1})\, ds \qquad w\in \boldsymbol{H}_1. \tag{3}$$

We denote by

$$\mathcal{T}=\left\{\left(\frac{1}{p}, \frac{1}{q}\right)\in[0,1]\times[0,1]: T: L^p(\boldsymbol{H}_1)\to L^q(\boldsymbol{H}_1) \text{ boundedly}\right\}$$

the type set of the operator T in (3). The set $\mathcal{T}$ is a convex subset of the square $[0,1]\times[0,1]$, contained in the triangle below the diagonal $1/p=1/q$ and containing the diagonal itself, [1].

In the first part of this paper we obtain some limitations on the type set $\mathcal{T}$ of the operator T in (3). Analogously to the case of a curve in $\boldsymbol{R}^n$, the set $\mathcal{T}$ is subject to constraints which are due to dimensions and to a sort of local homogeneity of the curve Γ.

Suitably modifying some arguments which can be found in [4], we prove that the the type set $\mathcal{T}$ of the operator in (3) is contained in the closed trapezoid with vertices

$$A=(0,0),\quad B=(1,1),\quad C=(2/3,1/2),\quad D=(1/2,1/3) \tag{4}$$

within the closed triangle with vertices $A=(0,0)$, $B=(1,1)$, $P=(3/5,2/5)$.

It is well known that if in (3) we replace the Heisenberg group convolution by the ordinary convolution in $\boldsymbol{R}^3$, then the L^p-improving properties of the measure μ in (2) are closely related to the curvature and torsion properties of Γ, [2, 3, 6].

In our previous paper [7], we discussed the L^p-improving properties of a finite measure μ in the case in which its supporting manifold is a curve in $\boldsymbol{H}_1$ whose tangent vector at any point is not parallel to the centre of the group, i.e., without loss of generality, it has the form $\Gamma(s)=(s,\phi_1(s),\phi_2(s))$, $s\in I\subset\boldsymbol{R}$ where $\phi_1(s)$, $\phi_2(s)$ are smooth real valued functions.

We found the curvature condition that implies that the type set of the corresponding right convolution operator is the whole closed trapezoid $ABCD$ and we also established a notion of right curvature-torsion and a notion of left curvature-torsion which are not mutually equivalent.

In the second part of this paper we focus our attention on curves in the Heisenberg group having tangent vector at the origin which is parallel to the centre of the group. More precisely we consider a curve $\gamma(s)$ which is given by

$$\gamma: I\to\boldsymbol{H}_1 \qquad s\mapsto\gamma(s)=(s^m, s^n, s) \tag{5}$$

where $I=[0,R]$, $R>0$, and m, n are distinct real numbers greater than one.

We define the right convolution operator U

$$Uf(w)=\int_I f(w\cdot(\gamma(s))^{-1})\,ds \qquad w\in\boldsymbol{H}_1 \tag{6}$$

whose convolution kernel is a measure supported on the curve in (5) and we prove that its type set $\mathcal{U}$ is contained in the closed trapezoid with vertices $A = (0, 0)$, $B = (1, 1)$, $P_1 = \left(\frac{m+n-1}{m+n+1}, \frac{m+n-2}{m+n+1}\right)$, $P_2 = \left(\frac{3}{m+n+1}, \frac{2}{m+n+1}\right)$ with the only possible exception of the closed segment joining the two points P_1 and P_2 if $m+n \geq 5$, and it is the whole closed trapezoid $ABCD$ if $m+n<5$.

The fact that we have to distinguish between $m+n \geq 5$ and $m+n<5$ is due to the following remarks.

We know that the type set of the operator U in (6) is contained in the closed trapezoid $ABCD$, but we can also prove, by taking into account the local homogeneity near the origin of the curve in (5), that another necessary condition for the operator U to be bounded from $L^p(\boldsymbol{H}_1)$ to $L^q(\boldsymbol{H}_1)$ is that

$$\frac{1}{p} - \frac{1}{q} \leq \frac{1}{m+n+1}. \tag{7}$$

The inequality in (7) generates the closed trapezoid with vertices $A = (0, 0)$, $B = (1, 1)$, $P_1 = \left(\frac{m+n-1}{m+n+1}, \frac{m+n-2}{m+n+1}\right)$, $P_2 = \left(\frac{3}{m+n+1}, \frac{2}{m+n+1}\right)$ within the closed triangle with vertices A, B, $P = (3/5, 2/5)$. Depending on m and n, the optimal trapezoid $ABCD$ intersects the trapezoid ABP_1P_2 and since we are looking for boundedness in the intersection between the two previous closed trapezoids, we have to distinguish between the case $m+n \geq 5$ and the case $m+n<5$.

The results we have proved for the operator U also hold when we replace the curve (5) by the more general curve $\Gamma(s) = (s^m + o(s^m), s^n + o(s^n), s)$, for s in a neighborhood of the origin.

2. – Preliminary estimates.

THEOREM 2.1. – *Let $\Gamma(s)$, $s \in I$, be the smooth regular curve defined in* (1) *and let T be the operator defined in* (3), *then the type set $\mathcal{T}$ of T is contained in the closed trapezoid with vertices* $A = (0, 0)$, $B = (1, 1)$, $C = (2/3, 1/2)$, $D = (1/2, 1/3)$.

PROOF. – A result due to Ricci and Stein, [5], implies that if $\Gamma(s)$, $s \in I$, does not generate the Heisenberg group $\boldsymbol{H}_1$ (i.e. it is contained in some proper closed subgroup of $\boldsymbol{H}_1$) then the type set $\mathcal{T}$ of T is reduced to the diagonal $1/p = 1/q$.

Hence we can assume that the curve $\Gamma(s)$, $s \in I$, generates the full group

$\boldsymbol{H}_1$. Then there exists at least one point $s_0 \in I$ such that

$$
\begin{vmatrix} \psi_1'(s_0) & \psi_1''(s_0) & 0 \\ \psi_2'(s_0) & \psi_2''(s_0) & 0 \\ \psi_3'(s_0) & \psi_3''(s_0) & 1 \end{vmatrix} \neq 0 . \tag{8}
$$

Up to translations we can suppose that $s_0 = 0$ and $\Gamma(0) = (0, 0, 0)$.

Since inequality (8) holds for $s_0 = 0$, at least one value between $\psi_1'(0)$ and $\psi_2'(0)$ is not zero. Up to a rotation we can assume without loss of generality that $\psi_1'(0) \neq 0$.

By the automorphism

$$
\begin{cases} x' = \dfrac{1}{\psi_1'(0)} x \\ y' = \dfrac{\psi_2'(0)}{\psi_1'(0)} x - y \\ t' = t - \dfrac{\psi_3''(0)\,\psi_2'(0) - \psi_3'(0)\,\psi_2''(0)}{\psi_2'(0)\,\psi_1''(0) - \psi_2''(0)\,\psi_1'(0)} x + \dfrac{\psi_3''(0)\,\psi_1'(0) - \psi_3'(0)\,\psi_1''(0)}{\psi_2'(0)\,\psi_1''(0) - \psi_2''(0)\,\psi_1'(0)} y , \end{cases}
$$

(which is well defined because $\psi_1'(0) \neq 0$ and (8) holds) and a change of parameter, the curve $\Gamma(s)$ can be written as

$$
\Gamma(s) = (s, as^2 + o(s^2), bs^3 + o(s^3)) \tag{9}
$$

in a neighborhood of the origin, where

$$
a = \frac{\psi_2'(0)\,\psi_1''(0) - \psi_2''(0)\,\psi_1'(0)}{2\psi_1'(0)}
$$

and

$$
b = -\frac{\begin{vmatrix} \psi_1'(0) & \psi_1''(0) & \psi_1'''(0) \\ \psi_2'(0) & \psi_2''(0) & \psi_2'''(0) \\ \psi_3'(0) & \psi_3''(0) & \psi_3'''(0) \end{vmatrix}}{6(\psi_2'(0)\,\psi_1''(0) - \psi_2''(0)\,\psi_1'(0))} .
$$

We notice that the coefficient b might be zero while the coefficient a is not zero because of the inequality in (8).

Suitably modifying the standard argument of testing the operator T on the characteristic function of a small euclidean ball, [2], we can prove that a necessary condition for the operator T to be bounded from $L^p(\boldsymbol{H}_1)$ to $L^q(\boldsymbol{H}_1)$

is that

$$\frac{3}{p} - \frac{2}{q} \leqslant 1 . \tag{10}$$

Recalling that the operator $T : f \mapsto f * \mu$ is bounded from $L^p(\boldsymbol{H}_1)$ to $L^q(\boldsymbol{H}_1)$ if and only if the operator $f \mapsto \mu * f$ is bounded from $L^{q'}(\boldsymbol{H}_1)$ to $L^{p'}(\boldsymbol{H}_1)$, the same test shows that also the inequality

$$\frac{3}{q} \geqslant \frac{2}{p} \tag{11}$$

must hold.

Conditions (10) and (11) together with the fact that it must be $0 \leqslant \frac{1}{q} \leqslant \frac{1}{p} \leqslant 1$, [1], imply that the type set $\mathcal{T}$ of T is contained in the closed triangle with vertices $A = (0, 0)$, $B = (1, 1)$, $P = (3/5, 2/5)$.

Finally, testing the operator T on the characteristic function f_ε of the set $\{(x, y, t) \in \boldsymbol{H}_1 : |x| < \varepsilon,\ |y| < \varepsilon^2,\ |t| < \varepsilon^3\}$ for a small positive ε, which takes into account the local homogeneity of the curve (9) in a neighborhood of the origin, yields the necessary condition

$$\frac{1}{p} - \frac{1}{q} \leqslant \frac{1}{6} \tag{12}$$

for the L^p-L^q boundedness of T to hold. The inequality in (12) generates the closed trapezoid with vertices $A = (0, 0)$, $B = (1, 1)$, $C = (2/3, 1/2)$, $D = (1/2, 1/3)$ within the closed triangle ABP and this concludes the proof. ∎

In order to study the type set of the operator U in (6), we need a preliminary lemma whose proof can be found in [7] and that we rewrite here for the sake of completeness.

LEMMA 2.2. – *Let I be a bounded interval of $\boldsymbol{R}$ and consider the curve in $\boldsymbol{R}^2$ given by $\Psi(s) = (\theta(s), \zeta(s))$ where $\theta(s)$ and $\zeta(s)$ are smooth real-valued functions such that*

$$|\theta'(s)\,\zeta''(s) - \theta''(s)\,\zeta'(s)| \geqslant C_1 \tag{13}$$

and

$$|\theta'(s)| \geqslant C_2 \tag{14}$$

for two positive constants C_1 and C_2.

Then the operator

$$Sf(x) = \int_I f(x - \Psi(s))\, ds, \qquad x \in \boldsymbol{R}^2$$

is bounded from $L^{3/2}(\boldsymbol{R}^2)$ *to* $L^3(\boldsymbol{R}^2)$ *and*

$$\|Sf\|_{L^3(\boldsymbol{R}^2)} \leq c \left(\frac{C_1 C_2}{M^3} \right)^{-1/3} \left(\frac{1}{C_2} + \frac{C_3}{C_2^3} |I| \right)^{2/3} \|f\|_{L^{3/2}(\boldsymbol{R}^2)}$$

where C_1, C_2 *are the constants in* (13) *and* (14) *respectively, and* $C_3 = \max_{s \in I} |\theta''(s)|$, $M = \max_{s \in I} |\theta'(s)|$.

3. – Type set for curves having tangent vector at the origin which is parallel to the centre of the Heisenberg group.

Let γ be the curve in the Heisenberg group $\boldsymbol{H}_1$ which is defined in (5). Since rotations in the first two coordinates of the Heisenberg group are group automorphisms, we can suppose without loss of generality $m > n > 1$. Moreover, since away from the origin the operator U is essentially a convolution operator by a curve which satisfies the hypotheses of Theorem 3.1 in [7], we assume $I = [0, \delta]$ for a sufficiently small positive parameter δ, (we will see later how small we must choose δ). We can state the following result.

THEOREM 3.1. – *Let* U *be the operator defined in* (6) *and let* $\mathcal{U}$ *be its type set, then*

(i) *if* $m + n \geq 5$, *the type set* $\mathcal{U}$ *is contained in the closed trapezoid with vertices* $A = (0, 0)$, $B = (1, 1)$, $P_1 = \left(\frac{m+n-1}{m+n+1}, \frac{m+n-2}{m+n+1} \right)$, $P_2 = \left(\frac{3}{m+n+1}, \frac{2}{m+n+1} \right)$ *with the only possible exception of the closed segment joining the two points* P_1 *and* P_2;

(ii) *if* $m + n < 5$, *the type set* $\mathcal{U}$ *is the whole closed trapezoid with vertices* $A = (0, 0)$, $B = (1, 1)$, $C = \left(\frac{2}{3}, \frac{1}{2} \right)$, $D = \left(\frac{1}{2}, \frac{1}{3} \right)$.

PROOF. – We split the argument into several steps.

STEP 1. – By Theorem 2.1 we know that the type set $\mathcal{U}$ of U is contained in the closed trapezoid with vertices $A = (0, 0)$, $B = (1, 1)$, $C = (2/3, 1/2)$, $D = (1/2, 1/3)$. We get some other limitations on $\mathcal{U}$ by applying U to some test functions.

Take, as a test function, the characteristic function f_ε of the set

$$\{(x, y, t) \in \boldsymbol{H}_1 \colon |x| < \varepsilon^m, |y| < \varepsilon^n, |t| < \varepsilon\}$$

where ε is a small positive parameter.

Let (x, y, t) be a point in the set

$$M_\varepsilon = \{(x, y, t) \in \boldsymbol{H}_1 \colon |x| < c_1 \varepsilon^m, |y| < c_1 \varepsilon^n, |t| < c_1 \varepsilon\}$$

where $c_1 > 0$ is a small constant and let $0 < s < c\varepsilon$. It is easy to verify that, if c is small enough, we get by term by term majorization

$$\begin{aligned} &|x - s^m| < \varepsilon^m \\ &|y - s^n| < \varepsilon^n \\ &\left| t - s - \frac{x}{2} s^n + \frac{y}{2} s^m \right| < \varepsilon . \end{aligned}$$

Therefore $Uf_\varepsilon(x, y, t) > c\varepsilon$ on M_ε. Since the Lebesgue measure of M_ε is a constant times ε^{m+n+1}, we have $\|Uf_\varepsilon\|_{L^q(\boldsymbol{H}_1)} > C\varepsilon^{1+(m+n+1)/q}$. If we impose the condition that U is bounded from $L^p(\boldsymbol{H}_1)$ to $L^q(\boldsymbol{H}_1)$, we must have

$$\|Uf_\varepsilon\|_q \leq C \|f_\varepsilon\|_p$$

that is

$$\varepsilon^{1+(m+n+1)/q} \leq C\varepsilon^{(m+n+1)/p}$$

for every $0 < \varepsilon < 1$. This gives

$$\frac{1}{p} - \frac{1}{q} \leq \frac{1}{m+n+1} .$$

This restriction implies that the type set $\mathcal{U}$ of U is contained in the closed trapezoid which lies above the line $\frac{1}{p} - \frac{1}{q} = \frac{1}{m+n+1}$ inside the optimal closed trapezoid $ABCD$, i.e. the closed trapezoid with vertices $A = (0, 0)$, $B = (1, 1)$, $P_1 = \left(\frac{m+n-1}{m+n+1}, \frac{m+n-2}{m+n+1} \right)$, $P_2 = \left(\frac{3}{m+n+1}, \frac{2}{m+n+1} \right)$.

STEP 2. – Let ε be a sufficiently small positive real value that will be determined later. We make a decomposition of $[0, \delta]$ into intervals $J_j = [(1+\varepsilon)^{-j-1}\delta, (1+\varepsilon)^{-j}\delta]$, $j = 0, 1, \ldots$, and we define the operators

$$U_j f(x, y, t) = \int_{(1+\varepsilon)^{-j-1}\delta}^{(1+\varepsilon)^{-j}\delta} f((x, y, t) \cdot (s^m, s^n, s)^{-1}) \, ds .$$

Then

$$(15)\qquad U = \sum_{j=0}^{+\infty} U_j .$$

We rescale the operators U_j in such a way to have a common range of integration. We hence define $U_{0,j}$ so that

$$(16)\qquad U_j f(x, y, t) =$$

$$(1+\varepsilon)^{-j} D_{((1+\varepsilon)^{mj}, (1+\varepsilon)^{nj}, (1+\varepsilon)^{(m+n)j})} (U_{0,j} D_{((1+\varepsilon)^{-mj}, (1+\varepsilon)^{-nj}, (1+\varepsilon)^{-(m+n)j})} f)(x, y, t)$$

where for a function f on $\boldsymbol{H}_1$ and $(\varepsilon_1, \varepsilon_2, \varepsilon_3) \in \boldsymbol{R}^3$, we set

$$D_{(\varepsilon_1, \varepsilon_2, \varepsilon_3)} f(x, y, t) = f(\varepsilon_1 x, \varepsilon_2 y, \varepsilon_3 t) .$$

Then

$$(17)\qquad U_{0,j} f(x, y, t) = \int_{\delta/(1+\varepsilon)}^{\delta} f((x, y, t) \cdot (s^m, s^n, (1+\varepsilon)^{(m+n-1)j} s)^{-1})\, ds .$$

Since $s \in [\delta/(1+\varepsilon), \delta]$, by a change of variable the operator in (17) becomes essentially the operator

$$(18)\qquad U_{0,j} f(x, y, t) = \int_{(\delta/(1+\varepsilon))^m}^{\delta^m} f((x, y, t) \cdot (s, s^{n/m}, (1+\varepsilon)^{(m+n-1)j} s^{1/m})^{-1})\, ds .$$

Hence, by (15) and (16)

$$(19)\qquad \|Uf\|_q \leqslant \sum_{j=0}^{+\infty} \|U_j f\|_q =$$

$$\sum_{j=0}^{+\infty} (1+\varepsilon)^{-j} \|D_{((1+\varepsilon)^{mj}, (1+\varepsilon)^{nj}, (1+\varepsilon)^{(m+n)j})} (U_{0,j} D_{((1+\varepsilon)^{-mj}, (1+\varepsilon)^{-nj}, (1+\varepsilon)^{-(m+n)j})} f)\|_q =$$

$$\sum_{j=0}^{+\infty} (1+\varepsilon)^{(-1-\frac{2m+2n}{q})j} \|U_{0,j} D_{((1+\varepsilon)^{-mj}, (1+\varepsilon)^{-nj}, (1+\varepsilon)^{-(m+n)j})} f\|_q \leqslant$$

$$\sum_{j=0}^{+\infty} (1+\varepsilon)^{(-1-\frac{2m+2n}{q}+\frac{2m+2n}{p})j} \|U_{0,j}\|_{p,q} \|f\|_p$$

where $U_{0,j}$ is the operator defined in (18).

STEP 3. – Let $U_{0,j}$ be the operator defined in (18). For an appropriate δ we prove that

(a) $U_{0,j}$ is bounded from $L^{3/2}(\boldsymbol{H}_1)$ to $L^2(\boldsymbol{H}_1)$ and

$$\|U_{0,j}\|_{3/2,\,2} \leqslant C(1+\varepsilon)^{-\frac{m+n-1}{6}j} \tag{20}$$

(b) $U_{0,j}$ is bounded from $L^2(\boldsymbol{H}_1)$ to $L^3(\boldsymbol{H}_1)$ and

$$\|U_{0,j}\|_{2,\,3} \leqslant C(1+\varepsilon)^{-\frac{m+n-1}{6}j} \tag{21}$$

where C is a positive constant which is independent of j.

We start by proving statement (a).

We notice that $U_{0,j}$ is a right convolution operator by a finite measure supported on the curve γ_j given by

$$\begin{aligned}\gamma_j(s) &= (s,\, s^{n/m},\, (1+\varepsilon)^{(m+n-1)j}\, s^{1/m}) \\ &= (s,\, \phi(s),\, \psi_j(s)), \qquad s \in [(\delta/(1+\varepsilon))^m,\, \delta^m]\end{aligned} \tag{22}$$

where we set $\phi(s) = s^{n/m}$ and $\psi_j(s) = (1+\varepsilon)^{(m+n-1)j}\, s^{1/m}$. We impose that γ_j satisfies the right curvature condition for every $s \in [(\delta/(1+\varepsilon))^m,\, \delta^m]$, as in Theorem 3.1 in [7]. Since

$$\left| \phi''(s)\, \psi_j'''(s) - \phi'''(s)\, \psi_j''(s) + \frac{(\phi''(s))^2}{2} \right| =$$

$$\frac{n(m-n)}{2m^5}\, s^{\frac{2n}{m}-4}\, \left|2(1+\varepsilon)^{(m+n-1)j}(m-1)(n-1)\, s^{\frac{1-m-n}{m}} - nm(m-n)\right| \geqslant$$

$$\frac{n(n-m)}{2m^5}\, \delta^{2n-4m} (1+\varepsilon)^{(m+n-1)j}\, (2(m-1)(n-1)\, \delta^{1-n-m} - nm(m-n))$$

for every $s \in [(\delta/(1+\varepsilon))^m,\, \delta^m]$, we choose δ sufficiently small so that

$$2(m-1)(n-1)\, \delta^{1-n-m} - nm(m-n) > 0\,. \tag{23}$$

Then

$$\left| \phi''(s)\, \psi_j'''(s) - \phi'''(s)\, \psi_j''(s) + \frac{(\phi''(s))^2}{2} \right| \geqslant C(1+\varepsilon)^{(m+n-1)j}.$$

Therefore, by applying Theorem 3.1 in [7], we know that $U_{0,j}$ is bounded from $L^{3/2}(\boldsymbol{H}_1)$ to $L^2(\boldsymbol{H}_1)$, but we need to check how the $L^{3/2}(\boldsymbol{H}_1) \to L^2(\boldsymbol{H}_1)$ norm of $U_{0,j}$ depends on j.

Proving that $U_{0,j}$ is bounded from $L^{3/2}(\boldsymbol{H}_1)$ to $L^2(\boldsymbol{H}_1)$ is equivalent to prove that the operator $U_{0,j}^* U_{0,j}$ is bounded from $L^{3/2}(\boldsymbol{H}_1)$ to $L^3(\boldsymbol{H}_1)$ and

$$\|U_{0,j}\|_{3/2,2} \leqslant \sqrt{\|U_{0,j}^* U_{0,j}\|_{3/2,3}}. \tag{24}$$

Following the proof of Theorem 3.1 in [7], for any $f \in C_c(\boldsymbol{H}_1)$ we write

$$U_{0,j}^* U_{0,j} f(x,y,t) = \int_{(\frac{\delta}{1+\varepsilon})^m}^{\delta^m} \int_{(\frac{\delta}{1+\varepsilon})^m}^{\delta^m} f((x,y,t)\cdot \tag{25}$$

$$(s, s^{n/m}, (1+\varepsilon)^{(m+n-1)j} s^{1/m}) \cdot (r, r^{n/m}, (1+\varepsilon)^{(m+n-1)j} r^{1/m})^{-1})\, dr ds$$

$$= \int_{(\frac{\delta}{1+\varepsilon})^m}^{\delta^m} \int_{(\frac{\delta}{1+\varepsilon})^m}^{\delta^m} f\Big((x,y,t)\cdot \Big(r-s,\, r^{n/m}-s^{n/m}, (1+\varepsilon)^{(m+n-1)j}(r^{1/m}-s^{1/m}) +$$

$$\frac{1}{2}(sr^{n/m} - s^{n/m} r)\Big)^{-1}\Big)\, dr ds\,.$$

By changing variable

$$\begin{cases} r - s = u \\ \quad\; s = v \end{cases}$$

the operator in (25) becomes

$$U_{0,j}^* U_{0,j} f(x,y,t) = \int_{(\delta/(1+\varepsilon))^m - \delta^m}^{\delta^m - (\delta/(1+\varepsilon))^m} \int_{I(u)} f\Big((x,y,t)\cdot \Big(u, (v+u)^{n/m} - v^{n/m}, \tag{26}$$

$$(1+\varepsilon)^{(m+n-1)j}((v+u)^{1/m} - v^{1/m}) + \frac{v(v+u)^{n/m} - (v+u)v^{n/m}}{2}\Big)^{-1}\Big)\, dv du =$$

$$\int_{(\delta/(1+\varepsilon))^m - \delta^m}^{\delta^m - (\delta/(1+\varepsilon))^m} (f_{x-u} *_{\boldsymbol{R}^2} \gamma_{x,u,j})\Big(y, t + \frac{1}{2} uy\Big)\, du$$

where we set

$$f_{x-u}(y,t) = f(x-u, y, t)$$

$$I(u) = \begin{cases} [(\delta/(1+\varepsilon))^m, \delta^m - u], & 0 \leqslant u \leqslant \delta^m - (\delta/(1+\varepsilon))^m \\ [(\delta/(1+\varepsilon))^m - u, \delta^m], & \delta/(1+\varepsilon))^m - \delta^m \leqslant u < 0\,. \end{cases}$$

and for a fixed u with $|u| \leqslant \delta^m - (\delta/(1+\varepsilon))^m$ and a fixed $x \in \boldsymbol{R}$, $\gamma_{x,u,j}$ is the curve given by

$$\gamma_{x,u,j}(v) = \Big((v+u)^{n/m} - v^{n/m}, (1+\varepsilon)^{(m+n-1)j}((v+u)^{1/m} - v^{1/m}) + \tag{27}$$

$$\frac{v(v+u)^{n/m} - (v+u)v^{n/m}}{2} + \frac{x((v+u)^{n/m} - v^{n/m})}{2}\Big), \qquad v \in I(u).$$

Hence, if $\gamma_{x,u,j}$ is the curve in (27), we are reduced to prove the following estimate

$$\|g *_{\boldsymbol{R}^2} \gamma_{x,u,j}\|_{L^3(\boldsymbol{R}^2)} \leqslant C(j)\|g\|_{L^{3/2}(\boldsymbol{R}^2)}\,|u|^{-2/3} \tag{28}$$

for any $g \in C_c(\boldsymbol{R}^2)$, where $C(j) = C(1+\varepsilon)^{-(m+n-1)j/3}$.

In fact, assuming that (28) holds, we have by (25)

$$\|U_{0,j}^* U_{0,j}\|_3 = \tag{29}$$

$$\Bigg\| \Bigg\| \int_{(\delta/(1+\varepsilon))^m - \delta^m}^{\delta^m - (\delta/(1+\varepsilon))^m} (f_{x-u} *_{\boldsymbol{R}^2} \gamma_{x,u,j})\Big(y, t + \frac{1}{2}uy\Big)\, du \Bigg\|_{L^3(\boldsymbol{R}^2)} \Bigg\|_{L^3(\boldsymbol{R})} \leqslant$$

$$\Bigg\| \int_{(\delta/(1+\varepsilon))^m - \delta^m}^{\delta^m - (\delta/(1+\varepsilon))^m} \Big\|(f_{x-u} *_{\boldsymbol{R}^2} \gamma_{x,u,j})\Big(y, t + \frac{1}{2}uy\Big)\Big\|_{L^3(\boldsymbol{R}^2)}\, du \Bigg\|_{L^3(\boldsymbol{R})} \leqslant$$

$$C(j) \Bigg\| \int_{(\delta/(1+\varepsilon))^m - \delta^m}^{\delta^m - (\delta/(1+\varepsilon))^m} \|f_{x-u}\|_{L^{3/2}(\boldsymbol{R}^2)}\,|u|^{-2/3}\, du \Bigg\|_{L^3(\boldsymbol{R})} \leqslant C(j)\|f\|_{3/2}$$

where the last inequality follows from the boundedness of the Riesz potential of order $1/3$ as a mapping from $L^{3/2}(\boldsymbol{R})$ to $L^3(\boldsymbol{R})$.

To prove inequality (28) we write by a scaling argument

$$(g *_{\boldsymbol{R}^2} \gamma_{x,u,j})(\xi, \eta) = D_{(u^{-1}, u^{-1})}(D_{(u,u)} g *_{\boldsymbol{R}^2} \tilde{\gamma}_{x,u,j})(\xi, \eta) \tag{30}$$

for any $g \in C_c(\boldsymbol{R}^2)$, where

$$\tilde{\gamma}_{x,u,j}(v) = \Big(\frac{(v+u)^{n/m} - v^{n/m}}{u}, (1+\varepsilon)^{(m+n-1)j}\,\frac{(v+u)^{1/m} - v^{1/m}}{u} + \tag{31}$$

$$\frac{v(v+u)^{n/m} - (v+u)v^{n/m}}{2u} + \frac{x((v+u)^{n/m} - v^{n/m})}{2u}\Big), \qquad v \in I(u).$$

Moreover, by conjugating the convolution operator $g \mapsto g *_{\boldsymbol{R}^2} \tilde{\gamma}_{x,u,j}$ by the linear change of coordinates in $\boldsymbol{R}^2$

$$(\xi, \eta) \mapsto \left(\xi, \eta - \frac{x}{2}\xi\right),$$

we can replace the curve in (31) by the curve

$$\overline{\gamma}_{u,j}(v) = \left(\frac{(v+u)^{n/m} - v^{n/m}}{u}, (1+\varepsilon)^{(m+n-1)j} \frac{(v+u)^{1/m} - v^{1/m}}{u} + \frac{v(v+u)^{n/m} - (v+u)\, v^{n/m}}{2u} \right), \quad v \in I(u) \tag{32}$$

which is independent of x, and

$$\|g \mapsto g *_{\boldsymbol{R}^2} \tilde{\gamma}_{x,u,j}\|_{L^{3/2}(\boldsymbol{R}^2), L^3(\boldsymbol{R}^2)} = \|g \mapsto g *_{\boldsymbol{R}^2} \overline{\gamma}_{u,j}\|_{L^{3/2}(\boldsymbol{R}^2), L^3(\boldsymbol{R}^2)}. \tag{33}$$

Therefore inequality (28) will then follow from the estimate

$$\|g *_{\boldsymbol{R}^2} \overline{\gamma}_{u,j}\|_{L^3(\boldsymbol{R}^2)} \leqslant C(j) \|g\|_{L^{3/2}(\boldsymbol{R}^2)} \qquad \text{for any } g \in C_c(\boldsymbol{R}^2) \tag{34}$$

where $\overline{\gamma}_{u,j}$ is the curve in (32).

In fact if we assume that (34) holds and we take into account the equality in (30), we get

$$\begin{aligned} \|g *_{\boldsymbol{R}^2} \gamma_{x,u,j}\|_{L^3(\boldsymbol{R}^2)} &= \|D_{(u^{-1}, u^{-1})}(D_{(u,u)} g *_{\boldsymbol{R}^2} \tilde{\gamma}_{x,u,j})\|_{L^3(\boldsymbol{R}^2)} \\ &= \|D_{(u^{-1}, u^{-1})}(D_{(u,u)} g *_{\boldsymbol{R}^2} \overline{\gamma}_{u,j})\|_{L^3(\boldsymbol{R}^2)} \\ &= |u|^{2/3} \|D_{(u,u)} g *_{\boldsymbol{R}^2} \overline{\gamma}_{u,j}\|_{L^3(\boldsymbol{R}^2)} \\ &\leqslant C(j)\, |u|^{-2/3} \|g\|_{L^{3/2}(\boldsymbol{R}^2)} \end{aligned} \tag{35}$$

which gives the inequality in (28).

Now we prove the estimate (34) by applying Lemma 2.2.

For $v \in I(u)$, we rewrite the curve in (32) as

$$\overline{\gamma}_{u,j}(v) = (\theta_u(v), \zeta_u(v))$$

where

$$\theta_u(v) = \frac{(v+u)^{n/m} - v^{n/m}}{u}$$

$$\zeta_u(v) = (1+\varepsilon)^{(m+n-1)j} \frac{(v+u)^{1/m} - v^{1/m}}{u} + v \frac{(v+u)^{n/m} - v^{n/m}}{2u} - \frac{v^{n/m}}{2}.$$

By the mean-value Theorem there exist τ_i, $i = 1, \ldots, 5$, between v and $v+u$

such that

$$\theta'_u(v) = \frac{n}{m}\left(\frac{n}{m} - 1\right)\tau_1^{(n/m)-2}$$

$$\theta''_u(v) = \frac{n}{m}\left(\frac{n}{m} - 1\right)\left(\frac{n}{m} - 2\right)\tau_2^{(n/m)-3}$$

$$\zeta'_u(v) = \frac{(1+\varepsilon)^{(m+n-1)j}}{m}\left(\frac{1}{m} - 1\right)\tau_3^{(1/m)-2} +$$

$$\frac{n}{2m}\tau_4^{(n/m)-1} + \frac{n}{2m}\left(\frac{n}{m} - 1\right) v\tau_1^{(n/m)-2} - \frac{n}{2m} v^{(n/m)-1}$$

$$\zeta''_u(v) = \frac{(1+\varepsilon)^{(m+n-1)j}}{m}\left(\frac{1}{m} - 1\right)\left(\frac{1}{m} - 2\right)\tau_5^{(1/m)-3} + \frac{n}{m}\left(\frac{n}{m} - 1\right)\tau_1^{(n/m)-2} +$$

$$\frac{n}{2m}\left(\frac{n}{m} - 1\right)\left(\frac{n}{m} - 2\right) v\tau_2^{(n/m)-3} - \frac{n}{2m}\left(\frac{n}{m} - 1\right) v^{(n/m)-2}.$$

Since v and $v+u$ belong to $[(\delta/(1+\varepsilon))^m, \delta^m]$ and τ_i, $i = 1, \ldots, 5$ are between v and $v+u$, there exist three positive constants C_2, C_3, M such that

$$C_2 \leqslant |\theta'_u(v)| \leqslant M, \qquad \text{for every } v \in I(u) \tag{36}$$

$$|\theta''_u(v)| \leqslant C_3, \qquad \text{for every } v \in I(u). \tag{37}$$

Moreover we can prove that there exists a positive constant C_1 which is independent of j such that

$$|\theta'_u(v)\,\zeta''_u(v) - \theta''_u(v)\,\zeta'_u(v)| \geqslant C_1(1+\varepsilon)^{(m+n-1)j} \tag{38}$$

for every $j \geqslant 0$.

Since $m+n-1 > 0$ then $(1+\varepsilon)^{(m+n-1)j} \geqslant 1$ when $j \geqslant 0$. Moreover, since $v, \tau_1, \ldots, \tau_5 \in [(\delta/(1+\varepsilon))^m, \delta^m]$, and $(1+\varepsilon)^{4m-2n} \leqslant (1+\varepsilon)^{5m-n-1}$, we have

$$|\theta'_u(v)\,\zeta''_u(v) - \theta''_u(v)\,\zeta'_u(v)| \geqslant \frac{n(m-n)\,\delta^{2n-4m}}{2m^5}(1+\varepsilon)^{(m+n-1)j}\cdot \tag{39}$$

$$(2(m-1)(2m-1)\,\delta^{-m-n+1} + (3m-2n)\,mn -$$

$$(2(m-1)(2m-n)\,\delta^{-m-n+1} + mn(4m-3n))(1+\varepsilon)^{5m-n-1}).$$

Now, suppose we have fixed a δ such that (23) holds; then we can choose an arbitrarily small positive ε such that

$$|\theta_u'(v)\,\zeta_u''(v) - \theta_u''(v)\,\zeta_u'(v)| \geq (1+\varepsilon)^{(m+n-1)j} C_1$$

which is the inequality in (38).

Therefore, by applying Lemma 2.2, taking into account the estimates in (36), (37), (38) and the fact that $|I(u)| \leq \delta^m - (\delta/(1+\varepsilon))^m$, we have

$$\|g *_{\boldsymbol{R}^2} \overline{\gamma}_{u,j}\|_{L^3(\boldsymbol{R}^2)} \leq C\left(\frac{C_1(1+\varepsilon)^{(m+n-1)j} C_2}{M^3}\right)^{-1/3}\left(\frac{1}{C_2} + \frac{C_3}{C_2^3}\,|I(u)|\right)^{2/3} \|g\|_{L^{3/2}(\boldsymbol{R}^2)} \tag{40}$$

$$\leq C(1+\varepsilon)^{-\frac{m+n-1}{3}j}\|g\|_{L^{3/2}(\boldsymbol{R}^2)}.$$

By combining (28), (29) and (24), we have

$$\|U_{0,j}\|_{3/2,\,2} \leq C(1+\varepsilon)^{-\frac{m+n-1}{6}j}. \tag{41}$$

which is the inequality in (20).

For what regards statement (b), we notice that if we have chosen a sufficiently small $\delta > 0$, then the curve in (22) also satisfies the left curvature condition for every $s \in [(\delta/(1+\varepsilon))^m, \delta^m]$, therefore, by Theorem 3.1 in [7] we know that the operator $U_{0,j}$ is bounded from $L^2(\boldsymbol{H}_1)$ to $L^3(\boldsymbol{H}_1)$ and we can repeat the previous proof by replacing the operator $U_{0,j}^* U_{0,j}$ by the operator $U_{0,j} U_{0,j}^*$, to get inequality (21).

STEP 4. – Let $\overline{BC}$ be the segment with endpoints $C = (2/3, 1/2)$ and $B = (1, 1)$ and let $\overline{AD}$ be the segment with endpoints $A = (0, 0)$ and $D = (1/2, 1/3)$. We prove that the operator U in (6) is bounded on the points of $\overline{BC}$ of coordinates $\left(\frac{1}{p}, \frac{3}{2p} - \frac{1}{2}\right)$ for $\frac{1}{p} > \frac{m+n-1}{m+n+1}$ and it is bounded on the points of $\overline{AD}$ of coordinates $\left(\frac{1}{p}, \frac{2}{3p}\right)$ for $\frac{1}{p} < \frac{3}{m+n+1}$.

Let $\left(\frac{1}{p}, \frac{3}{2p} - \frac{1}{2}\right)$, $\frac{1}{p} \in \left(\frac{2}{3}, 1\right)$, be a point on the segment $\overline{BC}$, we estimate the $L^p(\boldsymbol{H}_1) \to L^{\frac{2p}{3-p}}(\boldsymbol{H}_1)$ norm of the operator $U_{0,j}$ in (18) by interpolating between the estimates $L^{3/2}(\boldsymbol{H}_1) \to L^2(\boldsymbol{H}_1)$, $L^1(\boldsymbol{H}_1) \to L^1(\boldsymbol{H}_1)$. Let $r \in (0, 1)$ be a value such that

$$\frac{1}{p} = \frac{1-r}{3/2} + \frac{r}{1}$$

that is

$$r = \frac{3-2p}{p}.$$

Since the operator $U_{0,j}$ is uniformly bounded on the points of the diagonal $\overline{AB}$ and since (20) holds, by applying the Riesz-Thorin interpolation theorem we get

$$\|U_{0,j}\|_{p,\frac{2p}{3-p}} \leqslant \|U_{0,j}\|_{3/2,2}^{1-r} \|U_{0,j}\|_{1,1}^{r} \leqslant C(1+\varepsilon)^{-\frac{(m+n-1)(p-1)}{2p}j}. \tag{42}$$

Therefore by (19) and (42) we have

$$\begin{aligned} \|Uf\|_{\frac{2p}{3-p}} &\leqslant C\sum_{j=0}^{+\infty}(1+\varepsilon)^{(1-\frac{(m+n)(3-p)}{p}+\frac{2m+2n}{p}-\frac{(m+n-1)(p-1)}{2p})j}\|f\|_p \\ &= C\sum_{j=0}^{+\infty}(1+\varepsilon)^{\frac{(m+n-1)p-m-n-1}{2p}j}\|f\|_p \end{aligned} \tag{43}$$

and the series in (43) converges if

$$\frac{(m+n-1)p-m-n-1}{2p} < 0$$

that is

$$\frac{1}{p} > \frac{m+n-1}{m+n+1}.$$

Now, let $\left(\frac{1}{p}, \frac{2}{3p}\right)$, $\frac{1}{p} \in \left(0, \frac{1}{2}\right)$, be a point on the segment $\overline{AD}$, analogously to the previous case, by taking into account the estimate (21) and the fact that $U_{0,j}$ is uniformly bounded from $L^\infty(\boldsymbol{H}_1)$ to $L^\infty(\boldsymbol{H}_1)$, we get by the Riesz-Thorin interpolation theorem

$$\|U_{0,j}\|_{p,\frac{3p}{2}} \leqslant \|U_{0,j}\|_{\infty,\infty}^{(p-2)/p} \|U_{0,j}\|_{2,3}^{2/p} \leqslant C(1+\varepsilon)^{-\frac{m+n-1}{3p}j}. \tag{44}$$

Therefore by (19) and (44) we have

$$\begin{aligned} \|Uf\|_{3p/2} &\leqslant C\sum_{j=0}^{+\infty}(1+\varepsilon)^{(-1-\frac{4m+4n}{3p}+\frac{2m+2n}{p}-\frac{m+n-1}{3p})j}\|f\|_p \\ &= C\sum_{j=0}^{+\infty}(1+\varepsilon)^{\frac{m+n+1-3p}{3p}j}\|f\|_p \end{aligned} \tag{45}$$

and the series in (45) converges if

$$\frac{1}{p} < \frac{3}{m+n+1}.$$

STEP 5. – We now conclude the proof of Theorem 3.1.

We notice that if $m+n \geqslant 5$ then, by what we have proved in Step 1 and in step 4 it follows that the type set $\mathcal{U}$ of the operator U defined in (6) is contained in the closed trapezoid with vertices $A=(0,0)$, $B=(1,1)$, $P_1=\left(\frac{m+n-1}{m+n+1}, \frac{m+n-2}{m+n+1}\right)$, $P_2=\left(\frac{3}{m+n+1}, \frac{2}{m+n+1}\right)$ with the only possible exception of the closed segment $\overline{P_1P_2}$, and this gives statement (i).

If $m+n<5$ we know by what we have proved in step 1 that the type set $\mathcal{U}$ of U is contained in the closed trapezoid with vertices $A=(0,0)$, $B=(1,1)$, $C=(2/3, 1/2)$, $D=(1/2, 1/3)$. Moreover, since in this case we have $\frac{2}{3}>\frac{m+n-1}{m+n+1}$ and $\frac{1}{2}<\frac{3}{m+n+1}$, we know by what we have seen in step 4, that U is bounded on the points $C=(2/3, 1/2)$ and $D=(1/2, 1/3)$. Therefore, by interpolating with the trivial estimates $L^\infty(\boldsymbol{H}_1)\to L^\infty(\boldsymbol{H}_1)$, and $L^1(\boldsymbol{H}_1)\to L^1(\boldsymbol{H}_1)$, we get that the operator U is bounded on the whole closed trapezoid $ABCD$, and this proves statement (ii). ∎

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