Bollettino Unione Matematica Italiana

Jaime E. Muñoz Rivera, Vanilde Bisognin, Eleni Bisognin

Exponential decay to partially thermoelastic materials

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **5-B** (2002), n.3, p. 605–629.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2002_8_5B_3_605_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2002.

Bollettino U. M. I.
(8) 5-B (2002), 605-629

Exponential Decay to Partially Thermoelastic Materials (*).

JAIME E. MUÑOZ RIVERA - VANILDE BISOGNIN - ELENI BISOGNIN

Sunto. – *Studiamo il sistema termoelastico per materiali che siano parzialmente termoelastici. Consideriamo cioè un materiale diviso in due parti, una delle quali sia un buon conduttore di calore, in modo che ivi esistano fenomeni termoelastici. L'altra parte materiale è un cattivo conduttore di calore e quindi non esiste il flusso di calore. In questo lavoro dimostriamo che per tali modelli la soluzione decade esponenzialmente a zero quando il tempo tende all'infinito. Studiamo anche il caso non lineare.*

Summary. – *We study the thermoelastic system for material which are partially thermoelastic. That is, a material divided into two parts, one of them a good conductor of heat, so there exists a thermoelastic phenomenon. The other is a bad conductor of heat so there is not heat flux. We prove for such models that the solution decays exponentially as time goes to infinity. We also consider a nonlinear case.*

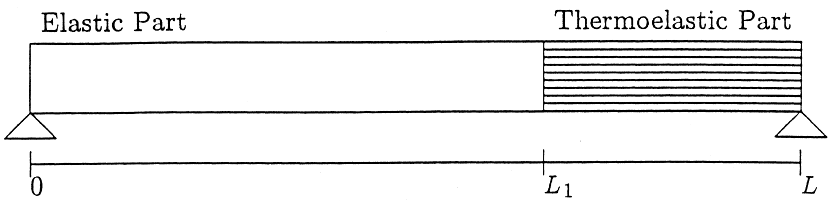
1. – Introduction.

Asymptotic stability for the n-dimensional thermoelastic system was study by C. Dafermos [1], who proved that the solution in general goes to zero when time goes to infinity, and depending on the domain operators and boundary conditions the solution may converge to a undamping function. For the one dimensional case, thanks to the work of [4], [8], [14] [15] among others, it is weel known by now that the solution allways decays to zero exponentially as time goes to infinity. This means that the dissipation given by the thermal difference is strong enough to produce uniform rate of decay, but not so strong to prevent blow up in a finite time as was proved by Hrusa and Messauodi [3]. They proved, for thermoelastic material which occupies the whole line, that

(*) Supported by a grant of CNPq.

there are smooth initial data for which the solution will develop singularities in finite time.

In this paper we consider the thermoelastic equation for mixed materials. That is, materials divided into two parts. One of them is a bad conductor of heat so there is not flux of heat along this part. The other part is a good heat conductor, therefore we have a thermoelastic phenomenon. Mathematically we can consider the above problem as a locally distribuited thermal dissipation.



Locally distribuited dissipation was study for several authors and the common point in all the works cited below, is that they consider such dissipations as an external source acting either in a part of the boundary (see for example [2], [5], [6], [7], [9], [12], [16], [20]), or in a part of the material (see [10], [11],[21]). The main difference between the above works and ours is that the local thermal mechanism appears not due to any external source of dissipative type, but due the structure of the material we are stuying.

Since we are reducing the effect of the thermal difference to only a small part of the material $[L_1, L]$, we may ask if such dissipation is strong enough to produces uniform rate of decay for the solution. The constitutive laws corresponding to mixed materias are given by

$$\sigma = \beta u_x - \alpha\theta$$

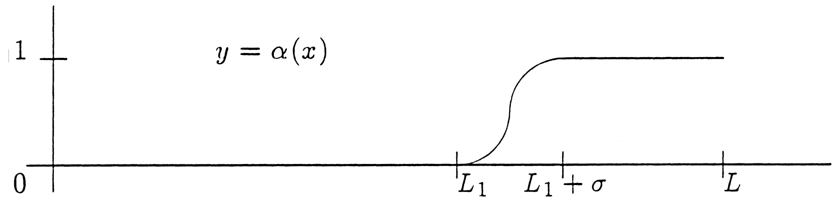
$$q = -\kappa\theta_x$$

$$e = \theta + \alpha u_x$$

where σ is the stress, q is the heat flux, and e is the internal energy. We are denoting by u the displacement, by $\theta = T_a - \tau_0$ the thermal difference, where T_a is the absolute temperature and τ_0 is the reference temperature which we will assume to be constant. Finally by α we are denoting a non decreasing C^2 function such that $\alpha(x) = 0$ for $x \in [0, L_1]$ and $\alpha(x) > 0$ for $x > L_1$. In that follows we will assume that exists $C > 0$ such that

$$|\alpha_x|^2 \leqslant C\alpha, \qquad |\alpha_{xx}|^2 \leqslant C\alpha \quad \text{for} \quad x \in [L_1, L_1 + \delta].$$

For $\delta > 0$ a small number. In this work α is a function that has the following behaviour,



The corresponding motion equations are given by

(1.1) $$u_{tt} - \beta u_{xx} + (\alpha\theta)_x = 0 \quad \text{in }]0, L[\times]0, \infty[,$$

(1.2) $$\theta_t - \kappa\theta_{xx} + \alpha u_{xt} = 0 \quad \text{in }]L_1, L[\times]0, \infty[,$$

(1.3) $$u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x), \qquad \theta(x, 0) = \theta_0 \quad \text{in }]0, L[.$$

Supporting the following boundary conditions.

(1.4) $$u(0, t) = u(L, t) = \theta(0, t) = \theta(L, t) = 0 \quad \text{for } t > 0.$$

The main result of this paper is to prove that this weak dissipation, given by the thermal difference, also produce exponential rate of decay of the solution as time goes to infinity. As an application of this result we also prove that there exist a global attractor for the quasi linear problem. Finally we show that the Kirchhoff's model for locally distribuited thermal dissipation, is well possed for small data.

To prove the exponential decay we explore the dissipative properties to construct a Liapunov functional whose derivative is negative proportional to itself. The main difficulty is that the dissipation only works in $[L_1, L]$ and we need estimates over the whole domain $[0, L]$. We overcome this problem introducing suitables multiplicators which allows us to control the energy only estimating u over $[L_1, L]$. See Lemmas 3.2-3.5 below.

2. – Existence for the linear system.

In this section we will use the semigroup approach to show the existence as well as the regularity of the solution to system (1.1)-(1.2). To do this we will introduce the following operator:

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 \\ \beta(\cdot)_{xx} & 0 & -[\alpha(x)(\cdot)]_x \\ 0 & -\alpha(x)(\cdot)_x & \kappa(\cdot)_{xx} \end{pmatrix}.$$

With domain

$$D(\mathcal{A}) = [H_0^1(0, L) \cap H^2(0, L)] \times H_0^1(0, L) \times [H_0^1(L_1, L) \cap H^2(L_1, L)].$$

Let us denote by $\mathcal{H}$ the space

$$\mathcal{H} = H_0^1(0, L) \times L^2(0, L) \times L^2(L_1, L)$$

which is a Hilbert space with the inner product

$$(U, V)_{\mathcal{H}} = \beta \int_0^L u_x^1 v_x^1 \, dx + \int_0^L u_x^2 v_x^2 \, dx + \int_{L_1}^L u_x^3 v_x^3 \, dx$$

where $U = (u^1, u^2, u^3)$ and $V = (v^1, v^2, v^3)$. So, system (1.1)-(1.2) is equivalent to

$$U_t = \mathcal{A} U$$

$$U(0) = U_0$$

To show the existence of solutions we use the Lummer Phillips Theorem. It is not difficult to show that $\mathcal{A}$ is dissipative. In fact

$$(\mathcal{A} U, U)_{\mathcal{H}} = \beta \int_0^L u_x v_x \, dx + \beta \int_0^L u_{xx} v \, dx - \int_0^L (\alpha\theta)_x v \, dx - \int_0^L \alpha v_x \theta \, dx + \kappa \int_{L_1}^L \theta_{xx} \theta \, dx$$

$$= \kappa \int_{L_1}^L |\theta_x|^2 \, dx$$

Now we will show that $\mathcal{A}$ is maximal monotone, let us take $F = (f_1, f_2, f_3) \in \mathcal{H}$, and consider the equation,

$$U - \mathcal{A} U = F$$

which is equivalent to

$$u - v = f_1$$

$$v - \beta u_{xx} + (\alpha\theta)_x = f_2$$

$$\theta - \alpha v_x - \kappa \theta_{xx} = f_3.$$

Note that we can eliminate v in the above system, so we get

$$u - \beta u_{xx} + (\alpha\theta)_x = f_2 + f_1 := g_1$$

$$\theta - \alpha u_x - \kappa \theta_{xx} = f_3 + \alpha f_1 := g_2.$$

Now we introduce the spaces:

$$\mathcal{V} := H_0^1(0, L) \times H_0^1(L_1, L)$$

and the bilinear form:

$$a(V, W) = \int_0^L uw + \beta u_x w_x + (\alpha\theta)_x w \, dx + \int_{L_1}^L \theta\psi + \theta_x \psi_x + \alpha u_x \psi \, dx,$$

where $V = (u, \theta)$, $W = (w, \psi)$. It is easy to see that $a(\cdot, \cdot)$ is a continuous coercive bilinear form. Denoting by $G = (g_1, g_2)$ we conclude that there exists only one solution U to the equation

$$a(U, W) = \int_0^L g_1 w \, dx + \int_{L_1}^L g_2 \psi \, dx .$$

For any $W \in \mathcal{V}$. Using the elliptic regularity our conclusion follows.

3. – Exponential decay.

In this section we study the asymptotic behaviour of the linear equation (1.1)-(1.2). To do this, we define the following functionals

$$E_1(t; u; \theta) = E_1(t) = \frac{1}{2}\int_0^L |u_t|^2 + \beta |u_x|^2 dx + \int_{L_1}^L |\theta|^2 \, dx$$

$$E_2(t; u, \theta) = E_2(t) = \frac{1}{2}\int_0^L |u_{tt}|^2 + \beta |u_{xt}|^2 dx + \int_{L_1}^L |\theta_t|^2 dx$$

$$E_3(t; u, \theta) = E_3(t) = \frac{1}{2}\int_0^L |u_{xt}|^2 + \beta |u_{xx}|^2 dx + \int_{L_1}^L |\theta_x|^2 dx .$$

Let us multiply equation (1.1) by u_t and (1.2) by θ and summing up the product result we have

$$\frac{d}{dt} E_1(t; u, \theta) = -\kappa \int_{L_1}^L |\theta_x|^2 \, dx .$$

Assuming regular data, and since u_t and θ_t have the same boundary conditions, we get

$$\frac{d}{dt} E_2(t; u, \theta) = -\kappa \int_{L_1}^L |\theta_{xt}|^2 \, dx . \tag{3.1}$$

To get the above identity we use essentially the fact that u_t and θ_t have the same boundary condition than u and θ. But this is not the case for u_x and θ_x. This is the point where the tipical difficulty for boundary conditions of Dirichlet-Dirichlet type appears. Let us see in detail this fact. Multiplying equation (1.1) by $-u_{xxt}$ and (1.2) by $-\frac{\alpha}{\beta}\theta_{xx}$ we get

$$\frac{d}{dt}\left\{\int_0^L |u_{xt}|^2+\beta|u_{xx}|^2\,dx\right\} = \int_0^L (\alpha\theta)_x u_{xxt}\,dx$$

$$\frac{d}{dt}\left\{\int_{L_1}^L |\theta_x|^2\,dx\right\} = -\kappa\int_{L_1}^L |\theta_{xx}|^2\,dx + \int_{L_1}^L \alpha u_{xt}\theta_{xx}\,dx$$

Summing up we get

$$\frac{d}{dt}E_3(t;\,u,\,\theta) = -\kappa\int_{L_1}^L |\theta_{xx}|^2\,dx + \alpha(L)\,\theta_x(L,\,t)\,u_{xt}(L,\,t) - \int_0^L \{\alpha_{xx}\theta - 2\alpha_x\theta_x\}\,u_{xt}\,dx\,. \tag{3.2}$$

Note that

$$|\alpha(L)\,\theta_x(L,\,t)\,u_{xt}(L,\,t)| \leqslant \frac{\alpha(L)^2}{2\varepsilon}|\theta_x(L,\,t)|^2 + \frac{\varepsilon}{2}|u_{xt}(L,\,t)|^2. \tag{3.3}$$

From Gagliardo-Niremberg's inequality we get:

$$|\theta_x(x,\,t)|^2 \leqslant c\left\{\int_{L_1}^L |\theta_x|^2\,dx\right\}^{1/2}\left\{\int_{L_1}^L |\theta_x|^2 + |\theta_{xx}|^2\,dx\right\}^{1/2}$$

which implies

$$|\theta_x(x,\,t)|^2 \leqslant c_\varepsilon\int_{L_1}^L |\theta_x|^2\,dx + \frac{\varepsilon^2}{\alpha(L)^2}\int_{L_1}^L |\theta_{xx}|^2\,dx\,.$$

Inserting the above inequality into (3.3) we get

$$|\alpha(L)\,\theta_x(L,\,t)\,u_{xt}(L,\,t)| \leqslant C_\varepsilon\int_{L_1}^L |\theta_x|^2\,dx + \frac{\varepsilon}{2}\int_{L_1}^L |\theta_{xx}|^2\,dx + \frac{\varepsilon}{2}|u_{xt}(L,\,t)|^2.$$

So, identity (3.2) implies

$$\frac{d}{dt}E_3(t) \leqslant -\frac{\kappa}{2}\int_{L_1}^{L} |\theta_{xx}|^2 dx + c_\varepsilon \int_{L_1}^{L} |\theta_x|^2 dx + \frac{\varepsilon}{2}|u_{xt}(L,t)|^2 - \int_0^L \{\alpha_{xx}\theta - 2\alpha_x\theta_x\} u_{xt}\, dx \tag{3.4}$$

$$\leqslant -\frac{\kappa}{2}\int_{L_1}^{L} |\theta_{xx}|^2\, dx + c_\varepsilon \int_{L_1}^{L} |\theta_x|^2 dx + \frac{\varepsilon}{2}|u_{xt}(L,t)|^2 + \varepsilon\int_0^L \alpha |u_{xt}|^2\, dx\,.$$

The derivative of E_3 has a pointwise term involving second order derivatives, which is not possible to bound using the Sobolev's inequalities. To overcome this difficulty we will use the following Lemma.

LEMMA 3.1. – *Let us suppose that v belongs to $W^{2,\infty}(a, b; H^v 2)$ and satisfies the equation:*

$$v_{tt} - \beta v_{xx} = f\,.$$

Then for any $q \in C^1(a, b)$ we have,

$$-\frac{d}{dt}\int_a^b q(x)\, v_t v_x\, dx = -\frac{q(x)}{2}\left[|v_t(x,t)|^2 + \beta|v_x(x,t)|^2\right]_{x=a}^{x=b} + \frac{1}{2}\int_a^b q'(x)\left\{|v_t|^2 + \beta|v_x|^2\right\} dx - \int_a^b q(x)\, v_x f\, dx\,. \tag{3.5}$$

PROOF. – Note that

$$-\frac{d}{dt}\int_a^b q(x)\, v_t v_x\, dx = -\int_a^b q(x)\, v_{tt} v_x\, dx - \int_a^b q(x)\, v_t v_{xt}\, dx \tag{3.6}$$

$$= \underbrace{-\int_a^b q(x)\, v_{tt} v_x\, dx}_{:= I_1} - \left[\frac{q(x)}{2}|v_t(x,t)|^2\right]_{x=b}^{x=b} + \frac{1}{2}\int_a^b q'(x)|v_t(x,t)|^2 dx\,.$$

On the other hand

$$I_1 = -\beta \int_a^b q(x)\, v_{xx} v_x \, dx - \int_a^b q(x)\, f(x,\, t)\, v_x \, dx$$

$$= -\frac{\beta}{2}\left[q(x)\,|v_x|^2\right]_{x=\alpha}^{x=\beta} + \frac{\beta}{2}\int_a^b q\,'(x)\,|v_x|^2\, dx - \int_a^b q(x)\, f(x,\, t)\, v_x \, dx\,.$$

Going back to identity (3.6) formula (3.5) follows. The proof is now complete. ∎

LEMMA 3.2. – *There exist a positive constant C such that*

$$\frac{d}{dt}\int_0^L \theta u_{xt}\, dx \leqslant -\frac{1}{2}\int_0^L \alpha\,|u_{xt}|^2\, dx + C_\delta \int_{L_1}^L |\theta_{xx}|^2 + |\theta_x|^2\, dx + \delta E_2(t)\,.$$

PROOF. – Multiplying equation (1.2) by u_{xt} we get

$$\frac{d}{dt}\int_0^L \theta u_{xt}\, dx = \int_0^L \theta_t\, u_{xt}\, dx + \int_0^L \theta u_{xtt}\, dx$$

$$= \int_0^L \theta_{xx}\, u_{xt}\, dx - \int_0^L \alpha\,|u_{xt}|^2\, dx - \int_0^L (\alpha\theta)_x\, u_{xx}\, dx + \int_0^L (\alpha\theta)_x\, \theta_x\, dx$$

$$\leqslant C_\delta \int_{L_1}^L |\theta_{xx}|^2 + |\theta_x|^2\, dx - \int_0^L \alpha\,|u_{xt}|^2\, dx + \frac{\delta}{2}\int_0^L |u_{xx}|^2 + |u_{xt}|^2\, dx\,.$$

From where our conclusion follows. ∎

LEMMA 3.3. – *Let us denote by α_2 the C^2-function given by*

$$\alpha_2(x) = \begin{cases} 0 & \textit{for } 0 < x < L_1 \\ 1 & L - \delta_0 < x < L \end{cases}$$

where δ_0 is such that $L_1 < L - \delta_0$. In this conditions we have

$$-\frac{d}{dt}\int_0^L \alpha_2\, u_t\, u_{xx}\, dx \leqslant c_0 \int_{L-\delta_0}^L |u_{xt}|^2\, dx - \frac{\beta}{2}\int_0^L \alpha_2\,|u_{xx}|^2\, dx + C\int_{L_1}^L |\theta_x|^2\, dx\,.$$

PROOF. – Differentiating the expression $\alpha_2 u_t u_{xx}$ and using the equation (1.1) we get

$$\frac{d}{dt}\int_0^L \alpha_2 u_t u_{xx}\,dx = \int_0^L \alpha_2 u_{tt} u_{xx}\,dx + \int_0^L \alpha_2 u_t u_{xxt}\,dx$$
$$= \int_0^L \alpha_2 |u_{xx}|^2\,dx - \int_0^L (\alpha_2)_x u_t u_{xt}\,dx - \int_0^L \alpha_2 |u_{xt}|^2\,dx$$
$$- \int_0^L \alpha_2 (\alpha\theta)_x u_{xx}\,dx\,.$$

From where it follows that

$$-\frac{d}{dt}\int_0^L \alpha_2 u_t u_{xx}\,dx \leqslant c_0 \int_{L-\delta_0}^L |u_{xt}|^2\,dx - \frac{\beta}{2}\int_0^L \alpha_2 |u_{xx}|^2\,dx + C\int_{L_1}^L |\theta_x|^2\,dx\,.$$

The proof is now complete. ■

LEMMA 3.4. – *Let us take* $\delta_0 < L - L_1 - \sigma$ *and let us denote by* α_3 *a* C^2 *function such that* $\operatorname{supp}(\alpha_3) \subset]L-\delta_0, L[$ *and* $\alpha_3(L) > 0$. *In this conditions we have,*

$$-\frac{d}{dt}\int_0^L \alpha_3 u_{xt} u_{tt}\,dx \leqslant -\frac{\alpha_3(L)}{2}|u_x(L,t)|^2 +$$
$$c\int_{L-\delta_0}^L \left(|u_{xx}|^2 + \beta |u_{xt}|^2\right) dx + \int_{L_1}^L |\theta_{xt}|^2\,dx\,.$$

PROOF. – Using Lemma 3.1 for $q = \alpha_3$ and $v = u_t$, we have

$$-\frac{d}{dt}\int_0^L \alpha_3 u_{xt} u_{tt}\,dx = -\frac{\alpha_3(L)}{2}|u_x(L,t)|^2 +$$
$$\int_0^L \alpha_3'\left(|u_{xx}|^2 + \beta |u_{xt}|^2\right) dx + \int_0^L \alpha_3 u_{xt} (\alpha\theta_t)_x\,dx\,.$$

From where our conclusion follows. ■

Using Lemma 3.2 and Lemma 3.3

$$\frac{d}{dt}\left\{\int_{L_1}^{L}\theta u_{xt}\,dx-\frac{1}{2c}\int_{L-\delta}^{L}\alpha_2 u_t u_{xx}\,dx\right\}\leqslant-\frac{\beta}{4C}\int_{L-\delta}^{L}|u_{xx}|^2\,dx-\frac{1}{4}\int_{L_1}^{L}\alpha|u_{xt}|^2\,dx+$$

$$C_\delta\int_{L_1}^{L}|\theta_{xx}|^2+|\theta_x|^2\,dx+\delta E_2(t).$$

From Lemma 3.4 we arrive at

$$\frac{d}{dt}\underbrace{\left\{\int_{L_1}^{L}\theta u_{xt}\,dx-\frac{1}{8c}\int_{L-\delta}^{L}\alpha_2 u_t u_{xx}\,dx+\frac{\gamma}{c_1}\int_0^{L}\alpha_3 u_{xt}u_{tt}\,dx\right\}}_{:=\mathcal{F}(t)}\leqslant$$

$$-\frac{\beta}{4}\int_{L-\delta}^{L}|u_{xx}|^2\,dx-\frac{1}{4}\int_0^{L}\alpha|u_{xt}|^2\,dx-\frac{\alpha_3(L)}{2}|u_x(L,t)|^2+$$

$$C_\delta\int_{L_1}^{L}|\theta_{xx}|^2+|\theta_x|^2+|\theta_t|^2\,dx+\delta E_2(t)$$

where $\gamma=\frac{1}{8}\min\{1,\beta\}$. Denoting by $\mathcal{L}$ the functional

$$\mathcal{L}(t)=N_1E_1(t)+N_1E_2(t)+NE_3(t)+\mathcal{F}(t)$$

we conclude that

$$\frac{d}{dt}\mathcal{L}(t)\leqslant-\frac{\beta}{4}\int_{L-\delta}^{L}|u_{xx}|^2\,dx-\frac{1}{4}\int_0^{L}\alpha|u_{xt}|^2\,dx-\frac{\alpha_3(L)}{2}|u_x(L,t)|^2 \tag{3.7}$$

$$-\left(\frac{\kappa N}{2}-C_\delta\right)\int_{L_1}^{L}|\theta_{xx}|^2\,dx-\left(\frac{\kappa N_1}{2}-C_\delta\right)\int_{L_1}^{L}|\theta_x|^2+$$

$$|\theta_{xt}|^2\,dx+\delta E_2(t).$$

To prove the exponential decay we will use the following Lemma.

LEMMA 3.5. – *There exists a positive constant* C *such that*

$$\left(1-\frac{2L}{T\sqrt{\beta}}\right)\int_0^T E_2(t)\,dt \leqslant C\int_0^T\int_0^L \alpha|u_{xt}|^2\,dx\,dt + C\int_0^T\int_{L_1}^L |\theta_{xt}|^2\,dx\,dt + \frac{L}{2}\int_0^T |u_{xt}(L,t)|^2\,dt\,,$$

for $T>\dfrac{2L}{\sqrt{\beta}}$.

PROOF. – Using Lemma 3.1 for $q=x$ and $v=u_t$ we arrive at

$$\frac{1}{2}\int_0^L |u_{tt}|^2+\beta|u_{xt}|^2\,dx = \frac{\beta L}{2}|u_{xt}(L,t)|^2 - \frac{d}{dt}\int_0^L xu_{tt}u_{xt}\,dx - \int_0^L xu_{xt}(\alpha\theta_t)_x\,dx\,.$$

Integrating over $[0,T]$ and summing up the term $\int_{L_1}^L |\theta_t|^2\,dx$ we get that

$$\int_0^T E_2(t)\,dt = \frac{L}{2}\beta\int_0^T |u_{xt}(L,t)|^2\,dt - \left(\int_0^L xu_{tt}u_{xt}\,dx\right)\Bigg|_{t=0}^{t=T} - \int_0^T\int_0^L xu_{xt}(\alpha\theta_t)_x\,dx\,dt + \frac{1}{2}\int_0^T\int_{L_1}^L |\theta_t|^2\,dx\,dt\,.$$

Since

$$E_2(t)=E_2(0)-\kappa\int_0^T\int_{L_1}^L |\theta_{xt}|^2\,dx\,dt,\qquad \int_0^T E_2(t)\,dt \geqslant TE_2(T),$$

$$\left|\int_0^L xu_{tt}u_{xt}\,dx\right| \leqslant \frac{L}{\sqrt{\beta}}E_2(0)\,.$$

It follows that

$$\begin{aligned}\int_0^T E_2(t)\,dt &\leqslant \frac{L\beta}{2}\int_0^T |u_{xt}(L,t)|^2\,dt + \frac{2L}{\sqrt{\beta}}E_2(0) - \int_0^T\int_0^L xu_{xt}(\alpha\theta_t)_x\,dx\,dt\\ &\leqslant \frac{L\beta}{2}\int_0^T |u_{xt}(L,t)|^2\,dt + \frac{2L}{\sqrt{\beta}}E_2(T) + \frac{2\kappa L}{\sqrt{\beta}}\int_0^T\int_{L_1}^L |\theta_{xt}|^2\,dx\,dt\end{aligned}$$

$$-\int_0^T\int_0^L xu_{xt}(\alpha\theta_t)_x\,dx\,dt$$

$$\leqslant \frac{L\beta}{2}\int_0^T |u_{xt}(L,t)|^2\,dt+\frac{2L}{T\sqrt{\beta}}\int_0^T E_2(t)\,dt+\frac{2\kappa L}{\sqrt{\beta}}\int_0^T\int_{L_1}^L |\theta_{xt}|^2\,dx\,dt$$

$$-\int_0^T\int_0^L xu_{xt}(\alpha\theta_t)_x\,dx\,dt\,.$$

Finally using the inequality

$$\int_0^L xu_{xt}(\alpha\theta_t)_x\,dx=\int_{L_1}^L xu_{xt}(\alpha\theta_t)_x\,dx$$

$$\leqslant c\int_0^L \alpha(x)\,|u_{xt}|^2\,dx + c\int_{L_1}^L |\theta_{xt}|^2\,dx$$

our conclusion follows. The proof is now complete ■

Let us introduce the following functionals

$$\mathcal{N}(t) = \int_0^L \alpha\,|u_{xt}|^2\,dx + \int_{L-\delta_0}^L |u_{xx}|^2\,dx + \int_{L_1}^L |\theta_x|^2 + |\theta_{xx}|^2\,dx + |u_{xt}(L,\,t)|^2.$$

We are now able to show the main result of this section.

THEOREM 3.1. – *Under the above notations, the energy associated to the thermoelastic system* (1.1)-(1.2) *decays exponentially. That is, there exist positive constants* C, γ *such that*

$$E_2(t) \leqslant CE_2(0)\,e^{-\gamma t}.$$

PROOF. – It is not difficult to see that there exists positive constants such that

$$C_0E_2(t) \leqslant \mathcal{L}(t) \leqslant C_1E_2(t)\,, \tag{3.8}$$

for N large enough. Recalling the definition of $\mathcal{N}$ and using (3.7) we get

$$\frac{d}{dt}\,\mathcal{L}(t) \leqslant -c\mathcal{N}(t) + \delta E_2(t)\,.$$

Lemma 3.5 implies that

$$\int_0^T E_2(t)\,dt \leqslant c\int_0^T \mathcal{N}(t)\,dt$$

and taking δ small enough after an integration we have

$$\mathcal{L}(T) \leqslant \mathcal{L}(0) - \frac{c}{2}\int_0^T \mathcal{N}(t)\,dt\,.$$

Using Lemma 3.5 once more, we conclude that

$$\begin{aligned}\mathcal{L}(T) &\leqslant \mathcal{L}(0) - c_3\int_0^T E_2(t)\,dt\\ &\leqslant \mathcal{L}(0) - c_3 T E_2(T)\,dt\\ &\leqslant \mathcal{L}(0) - c_4 T\mathcal{L}(T)\,dt\end{aligned}$$

which implies

$$(1 + c_4 T)\,\mathcal{L}(T)\,dt \leqslant \mathcal{L}(0)\,.$$

Finally from the semigroup property our conclusion follows. The proof is now complete ■

COROLLARY 3.1. – *Under conditions of Theorem* 3.1 *if*

$$(u_0, u_1, \theta_0) \in H_0^1(0, L) \times L^2(0, L) \times L^2(L_1, L)\,,$$

then, there exist positive constants C and γ *such that the first order energy decays exponentially*

$$E_1(t) \leqslant CE_1(0)\,e^{-\gamma t}.$$

PROOF. – Let us denote by

$$v(\cdot, \tau) = \int_0^t u(\cdot, \tau)\,d\tau + \chi_1, \qquad \psi(\cdot, \tau) = \int_0^t \theta(\cdot, \tau)\,d\tau + \chi_2.$$

In this condition the couple (v, ψ) satisfies

$$v_{tt} - u_1 - \beta v_{xx} - \beta\chi_{1,\,xx} + (\alpha\theta)_x + (\alpha\chi_{2,\,x}) = 0 \quad \text{in} \quad]0, L[\times]0, \infty[,$$

$$\psi_t - \theta_0 - \kappa\psi_{xx} - \kappa\chi_{2,\,xx} + \alpha v_{xt} + \alpha u_{0,\,x} = 0 \quad \text{in} \quad]L_1, L[\times]0, \infty[,$$

$$v(x, 0) = \chi_1(x), \quad u_t(x, 0) = u_0(x), \quad \psi(x, 0) = \chi_2 \quad \text{in} \quad]0, L[$$

$$v(0, t) = v(L, t) = \psi(L_1, t) = \psi(L, t) = 0 \quad \text{for } t > 0.$$

Choosing χ_1 and χ_2 such that

$$-\beta\chi_{1,xx} + \alpha\chi_{2,x} = u_1$$
$$= \kappa\chi_{2,xx} = -\alpha u_{0,\,x} + \theta_0$$

$$\chi_1(0) = \chi_1(L) = \chi_2(L_1) = \chi_2(L) = 0\,.$$

The couple $(v,\,\psi)$ satisfies system (1.1)-(1.2) for the initial data

$$v(x,\,0) = \chi_1, \qquad v_t(x,\,0) = u_0, \qquad \psi(x,\,0) = \chi_2.$$

From Theorem 3.1 we conclude that E_2 decays for v and ψ instead of u and θ. Since

$$C_1 E_1(t,\,u,\,\theta) \leqslant E_2(t;\,v,\,\psi) \leqslant C_0 E_1(t,\,u,\,\theta)$$

then our conclusion follows. The proof is now complete ■

4. – Global attractor.

In this section we will show, as a consequence of the exponential decay, the existence of a global attractor to the non linear system

$$u_{tt} - \beta u_{xx} + (\alpha\theta)_x + g(u) = f_1 \quad \text{in} \quad]0,\,L[\times]0,\,\infty[, \tag{4.1}$$

$$\theta_t - \kappa\theta_{xx} + \alpha u_{xt} = f_2 \quad \text{in} \quad]L_1,\,L[\times]0,\,\infty[, \tag{4.2}$$

$$u(x,\,0) = u_0(x), \quad u_t(x,\,0) = u_1(x), \quad \theta(x,\,0) = \theta_0 \quad \text{in }]0,\,L[$$

$$u(0,\,t) = u(L,\,t) = \theta(0,\,t) = \theta(L,\,t) = 0 \quad \text{for } t > 0\,.$$

To do this we will assume that

$$g \in C^1(\mathbb{R}), \qquad g(s)\,s \geqslant 0. \tag{4.3}$$

In this conditions it is not difficult to show that there exists only one solution to the system (4.1)-(4.2). This will be summarized in the following theorem:

THEOREM 4.1. – *Under the above notations, if g satisfies condition* (4.3) *then for any initial data*

$$(u_0, u_1, \theta_0) \ \textit{in}\ H_0^1(0,L)\times L^2(0,L)\times L^2(L_1,L), \quad f_1 \in L^2(0,L), \quad f_2 \in L^2(L_1,L)$$

there exist only one solution satisfying

$$u \in C^0([0, T]; H_0^1(0, L)) \cap C^1([0, T]; L^2(0, L)),$$

$$\theta \in L^2([0, T]; H_0^1(L_1, L)) \cap C^0([0, T]; L^2(L_1, L)).$$

Moreover if

$$(u_0, u_1, \theta_0) \quad in \quad H_0^1(0, L) \cap H^2(0, L) \times H_0^1(0, L) \times H_0^1(L_1, L),$$

then the solution satisfy:

$$u \in C^0([0,T]; H_0^1(0,L) \cap H^2(0,L)) \cap C^1([0,T]; H_0^1(0,L)) \cap C^2([0,T]; L^2(0,L)),$$

$$\theta \in L^2([0, T]; H_0^1(L_1, L) \cap H^2(L_1, L)) \; C^0([0, T]; H_0^1(L_1, L)).$$

In this conditions we are able to show the existence of a global attractor to system (4.1)-(4.2).

THEOREM 4.2. – *Under the above conditions the dynamical system defined by the system* (4.1)-(4.2) *supplemented by the Dirichlet boundary condition possesses a global attractor* $\mathcal{C}$ *which is compact, connected, and maximal in* $H_0^1(0, L) \times L^2(0, L) \times L^2(L_1, L)$. *Moreover* $\mathcal{C}$ *is included in* $H_0^1(0, L) \cap H^2(0, L) \times H_0^1(0, L) \times H_0^1(L_1, L)$.

PROOF. – Let us denote by $S(t)$ the semigroup associated with the dynamical sistem (4.1)-(4.2) and let us decompose it into two parts:

$$S(t) = S_1(t) + S_2(t),$$

where by S_1 we are denoting the semigroup associated with the linear homogeneous part. By S_2 we are denoting the semigroup associated by the dynamical system $S_2(t)\{u_0, u_1, \theta_0\} = \{\widehat{u}, \widehat{u}_t, \widehat{\theta}\}$ where $\widehat{u}, \widehat{u}_t, \widehat{\theta}$ is the solution of

$$\widehat{u}_{tt} - \beta \widehat{u}_{xx} + (\alpha \widehat{\theta})_x = f_1 - g(u) \text{ in }]0, L[\times]0, \infty[,$$

$$\widehat{\theta}_t - \kappa \widehat{\theta}_{xx} + \alpha \widehat{u}_{xt} = f_2 \text{ in }]L_1, L[\times]0, \infty[,$$

$$\widehat{u}(x, 0) = \widehat{u}_t(x, 0) = \widehat{\theta}(x, 0) = 0$$

$$\widehat{u}(0, t) = \widehat{u}(L, t) = \widehat{\theta}(0, t) = \widehat{\theta}(L, t) = 0 .$$

Thanks to Theorem 4.1, it is not difficult to show that S_2 is uniformly compact in $H_0^1(0, L) \times L^2(0, L) \times L^2(L_1, L)$. On the other hand, since

$$\|S_1(t)\|_{\mathcal{L}(\mathcal{H})} \leq c_0 e^{-\gamma t}.$$

Using Theorem 1.1 of Chapter 1 of [18] our conclusion follows. The proof is now complete. ■

5. – Small solutions.

In this section we will study the existence of solutions for the locally distribuided thermoelastic system of Kirchhoff type

$$u_{tt} - M\left(\int_0^L |u_x|^2 dx\right) u_{xx} + (\alpha\theta)_x = 0 \text{ in }]0, L[\times]0, \infty[, \tag{5.1}$$

$$\theta_t - \kappa\theta_{xx} + \alpha u_{xt} = 0 \text{ in }]L_1, L[\times]0, \infty[, \tag{5.2}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \theta(x, 0) = \theta_0 \quad \text{in }]0, L[$$

$$u(0, t) = u(L, t) = \theta(0, t) = \theta(L, t) = 0 \quad \text{for } t > 0,$$

where

$$M \in C^2(\mathbb{R}_+), \qquad M(s) \geqslant m_0 > 0\,. \tag{5.3}$$

The main result of this section is the global existence of solutions to system (5.1)-(5.2) provided the initial data is small. As a consequence of the prove we also conclude that the solution of the nonlinear system decay exponentially as time goes to infinity. The proof is based on the following local existence result, which is proved by standard fixed point argument.

THEOREM 5.1. – *Let us suppose that the initial data satisfies*

$$(u_0, u_1, \theta_0) \quad \text{in} \quad [H_0^1(0,L)\cap H^2(0,L)]\times H_0^1(0,L)\times[H^2(L_1,L)\cap H_0^1(L_1,L)]\,.$$

Then there exist $T > 0$ *and a solution* u, θ *of system* (5.1)-(5.2) *satisfying:*

$$u \in C^0([0, T]; H_0^1(0, L)\cap H^2(0, L))\cap C^1([0, T]; H_0^1(0, L))\cap C^2([0, T]; L^2(0, L)), \tag{5.4}$$

$$\theta \in L^2([0, T]; H_0^1(L_1, L)\cap H^2(L_1, L))\; C^0([0, T]; H_0^1(L_1, L)). \tag{5.5}$$

Moreover given $T > 0$ *there exist* $\varepsilon > 0$ *such that for any initial data* (u_0, u_1, θ_0) *satisfying*

$$\|u_{0,\,xx}\|^2 + \|u_{1,\,x}\|^2 + \|\theta_{0,\,xx}\|^2 < \varepsilon\,,$$

there exist only one solution (u, θ) *satisfying condition* (5.4) *and* (5.5).

Note that the last part of the above Theorem, $T = T(\varepsilon) < \infty$. Here we will show that for ε small enough, T does not depent on ε, that is $T = \infty$, which means that the solution is global in time. Which is equivalent to say that the second order derivatives are uniformly bounded for any $t > 0$. Let us denote by

$$M_1 = \sup \left\{ M(s);\ s \in \left[0,\ \frac{E_1(0)}{m_0} \right] \right\}.$$

From the local existence Theorem we have that for $T > \dfrac{2L}{\sqrt{M_1}}$ there exists $\varepsilon > 0$, such that for any initial data satisfying

$$E_2(0) + E_3(0) \leqslant \varepsilon$$

there exist only one solution $(u,\ \theta)$ solution of (4.1)-(4.2), defined on $[0,\ T]$. Let us take $\varepsilon_0 < \varepsilon$ and let us take initial data such that

$$E_2(0) + E_3(0) \leqslant \varepsilon_0. \tag{5.6}$$

By the continuity of the solutions there exists a positive $T_2 > T$ such that

$$E_2(t) + E_3(t) \leqslant d\varepsilon_0, \qquad \forall t \in [0,\ T_2], \tag{5.7}$$

where $d > 1$ is a positive constant to be fixed later. Let us denote by

$$T^* = \sup \{ t > 0;\ E_2(t) + E_3(t) \leqslant d\varepsilon_0 \}.$$

We will show that $T^* = \infty$, which will prove that there exists a global in time solution for sufficient small initial data. To do this we will define the following functionals:

$$E_1(t;\ u;\ \theta) = E_1(t) = \frac{1}{2} \int_0^L |u_t|^2 \, dx + \widehat{M} \left(\int_0^L |u_x|^2 \, dx \right) + \int_{L_1}^L |\theta|^2 \, dx$$

$$E_2(t;\ u,\ \theta) = E_2(t) = \frac{1}{2} \int_0^L |u_{tt}|^2 + M \left(\int_0^L |u_x|^2 \, dx \right) |u_{xt}|^2 \, dx + \int_{L_1}^L |\theta_t|^2 \, dx$$

$$E_3(t;\ u,\ \theta) = E_3(t) = \frac{1}{2} \int_0^L |u_{xt}|^2 + M \left(\int_0^L |u_x|^2 \, dx \right) |u_{xx}|^2 \, dx + \int_{L_1}^L |\theta_x|^2 \, dx .$$

Where $\widehat{M}(\sigma) = \int_0^\sigma M(s) \, ds$. Let us multiply equation (5.1) by u_t and (5.2) by θ and summing the product result we have

$$\frac{d}{dt} E_1(t;\ u,\ \theta) = -\kappa \int_{L_1}^L |\theta_x|^2 \, dx . \tag{5.8}$$

Similarly, differentiating in time equations (5.1) and (5.2) multiplying by u_{tt}

and θ_t respectively and summing up the product result we get

$$\frac{d}{dt}E_2(t;\,u,\,\theta) = -\kappa\int_{L_1}^{L} |\theta_{xt}|^2\,dx + R_2 \tag{5.9}$$

where

$$R_2 = M'\left(\int_0^L |u_x|^2\,dx\right)\int_0^L u_x u_{xt}\,dx\int_0^L u_{xx}u_{tt}\,dx + \frac{1}{2}M'\left(\int_0^L |u_x|^2\,dx\right)\int_0^L u_x u_{xt}\,dx\int_0^L |u_{xt}|^2\,dx\,.$$

Note that

$$|R_2| \leqslant c\varepsilon_0 E_2(t)\,.$$

From (5.7) we can rewrite identity (5.9) as

$$\frac{d}{dt}E_2(t;\,u,\,\theta) \leqslant -\kappa\int_{L_1}^{L} |\theta_{xt}|^2\,dx + C\varepsilon_0 E_2(t)\,. \tag{5.10}$$

Using similar arguments as in section 2 we can show that

$$\frac{d}{dt}E_3(t) = -\kappa\int_{L_1}^{L} |\theta_{xx}|^2\,dx + \alpha(L)\,\theta_x(L,\,t)\,u_{xt}(L,\,t) - \int_0^L \{\alpha_{xx}\theta - 2\alpha_x\theta_x\}\,u_{xt}\,dx + R_3 \tag{5.11}$$

where

$$R_3 = \frac{1}{2}M'\left(\int_0^L |u_x|^2\,dx\right)\int_0^L u_x u_{xt}\,dx\int_0^L |u_{xx}|^2\,dx\,.$$

We also have that

$$|R_3| \leqslant c\varepsilon_0\left\{E_2(t) + \int_{L_1}^{L} |\theta_x|^2\,dx\right\}.$$

As in the proof of inequality (3.4) and using (5.9) we get

$$\frac{d}{dt}E_3(t) \leqslant -\frac{\kappa}{2}\int_{L_1}^{L} |\theta_{xx}|^2\,dx + c_\varepsilon \int_{L_1}^{L} |\theta_x|^2\,dx + \frac{\varepsilon}{2}|u_{xt}(L,t)|^2$$

$$+\varepsilon \int_0^L \alpha |u_{xt}|^2\,dx + C\varepsilon_0 E_2(t).$$

In the following Lemma we will summarize the nonlinear version of Lemmas 3.2, 3.3.

LEMMA 5.1. – *Under the above notations the following inequalities holds.*

$$\left(1 - \frac{2L}{\sqrt{M_1}T} - C\varepsilon_0\right)\int_0^T E_2(t)\,dt \leqslant$$

$$\frac{L}{2}M\left(\int_0^L |u_x|^2\,dx\right)|u_{xt}(L,t)|^2 + c_2\int_0^T \alpha|u_{xt}|^2 + |\theta_{xt}|^2\,dx$$

for a positive constant C.

PROOF. – Using the same technique as in section 3 we get,

$$\frac{d}{dt}\int_0^L x u_{tt} u_{xt}\,dx =$$

$$-\frac{L}{2}M\left(\int_0^L |u_x|^2\,dx\right)|u_{xt}(L,t)|^2 + \underbrace{\frac{1}{2}\int_0^L |u_{tt}|^2 + M\left(\int_0^L |u_x|^2\,dx\right)|u_{xt}|^2\,dx}_{=E_2(t)}$$

$$-\int_0^L x(\alpha\theta_t)_x u_{xt} - 2M'\left(\int_0^L |u_x|^2\,dx\right)\int_0^L u_{xt}u_x\,dx\int_0^L x u_{xt}u_{xx}\,dx\,.$$

From where it follows that

$$E_2(t) = \frac{d}{dt}\int_0^L x u_{tt}u_{xt}\,dx + \frac{L}{2}M\left(\int_0^L |u_x|^2\,dx\right)|u_{xt}(L,t)|^2 +$$

$$\int_0^L x(\alpha\theta_t)_x u_{xt} + 2M'\left(\int_0^L |u_x|^2\,dx\right)\int_0^L u_{xt}u_x\,dx\int_0^L x u_{xt}u_{xx}\,dx\,.$$

Integrating from 0 to T we have that

$$\int_0^T E_2(t)\,dt = \left(\int_0^L x u_{tt} u_{xt}\,dx\right)_{t=0}^{t=T} + \frac{L}{2}\int_0^T M\left(\int_0^L |u_x|^2\,dx\right)|u_{xt}(L,t)|^2\,dt$$

$$\int_0^T\int_0^L x(\alpha\theta_t)_x u_{xt}\,dx\,dt + 2\int_0^T M'\left(\int_0^L |u_x|^2\,dx\right)\int_0^L u_{xt}u_x\,dx\int_0^L x u_{xt}u_{xx}\,dx\,dt$$

$$\leqslant \frac{L}{\sqrt{M_1}}(E_2(T)+E_2(0)) + \frac{L}{2}\int_0^T M\left(\int_0^L |u_x|^2\,dx\right)|u_{xt}(L,t)|^2\,dt$$

$$+\int_0^T\int_0^L \alpha|u_{xt}|^2\,dx\,dt + \int_0^T\int_{L_1}^L |\theta_{xt}|^2 + |\theta_x|^2\,dx\,dt + c\varepsilon_0 E_2(t).$$

From (5.9) it follows

$$E_2(0) = E_2(T) + \kappa\int_0^T\int_{L_1}^L |\theta_{xt}|^2\,dx - \int_0^T R_2\,dt$$

$$\leqslant E_2(T) + \kappa\int_0^T\int_{L_1}^L |\theta_{xt}|^2\,dx + C\varepsilon_0\int_0^T E_2(t)\,dt$$

from where we have

$$\int_0^T E_2(t)\,dt \leqslant \frac{2L}{\sqrt{M_1}}E_2(T) + \frac{L}{2}\int_0^T M\left(\int_0^L |u_x|^2\,dx\right)|u_{xt}(L,t)|^2\,dt \tag{5.13}$$

$$+\int_0^T\int_0^L \alpha|u_{xt}|^2\,dx\,dt + c\int_0^T\int_{L_1}^L |\theta_{xt}|^2 + |\theta_x|^2\,dx\,dt + c\varepsilon_0\int_0^T E_2(t)\,dt \tag{5.14}$$

Using relation (5.9) once more we have

$$\frac{d}{dt}\left\{E_2(t) - \int_0^t R_2(\tau)\,d\tau\right\} \leqslant -\kappa\int_0^t\int_0^L |\theta_{xt}|^2\,dx\,dt \leqslant 0\,.$$

So we have that

$$\int_0^T E_2(t)\,dt - \int_0^T\int_0^t R_2(\tau)\,d\tau\,dt \geqslant TE_2(T) - T\int_0^T R_2(\tau)\,d\tau\,.$$

From where it follows that

$$E_2(T) \leqslant \frac{1}{T}\int_0^T E_2(t)\,dt + \int_0^T R_2(t)\,dt - \frac{1}{T}\int_0^T \int_0^t R_2(\tau)\,d\tau\,dt$$

$$\leqslant \frac{1}{T}\int_0^T E_2(t)\,dt + C\varepsilon_0\int_0^T E_2(t)\,dt + \int_0^T |R_2(\tau)|\,dt\,.$$

Inserting the above inequality into (5.14) our conclusion follows.

LEMMA 5.2. – *Under the above conditions we have:*

$$\frac{d}{dt}\int_0^L \theta u_{xt}\,dx \leqslant C_\delta\int_{L_1}^L |\theta_{xx}|^2 + |\theta_x|^2\,dx - \int_0^L \alpha |u_{xt}|^2\,dx + C\delta E_2(t)$$

$$-\frac{d}{dt}\int_0^L \alpha_2 u_t u_{xx}\,dx \leqslant c_0 \int_{L-\delta_0}^L \alpha_2 |u_{xt}|^2\,dx - \frac{m_0}{2}\int_0^L \alpha_2 |u_{xx}|^2\,dx + C\int_{L_1}^L |\theta_x|^2\,dx$$

$$\frac{d}{dt}\int_0^L \alpha_3 u_{tt} u_{xt}\,dx \leqslant -\frac{\alpha_3(L)\,m_0}{2}|u_{xt}(L,t)|^2 + c\int_{L-\delta_0}^L \alpha_3(|u_{xx}|^2 + |u_{xt}|^2)\,dx$$

$$+\int_{L_1}^L |\theta_{xt}|^2\,dx + C\varepsilon_0 E_2(t)\,.$$

PROOF. – We only prove the third inequality, the others can be proved with the same arguments as in Lemma 3.2 and Lemma 3.3. Differentiating equation (5.1) and using Lemma 3.1 we arrive at

$$\frac{d}{dt}\int_0^L \alpha_3 u_{tt} u_{xt}\,dx =$$

$$-\frac{\alpha_3(L)}{2}M\left(\int_0^L |u_x|^2\,dx\right)|u_{xt}(L,t)|^2 + \int_0^L \alpha_3'(|u_{tt}|^2 + |u_{xt}|^2)\,dx$$

$$-\int_0^L \alpha_3(\alpha\theta_t)_x u_{xt} - M'\left(\int_0^L |u_x|^2\,dx\right)\int_0^L u_{xt}u_x\,dx\int_0^L \alpha_3 u_{xt}u_{xx}\,dx\,.$$

Using the inequality

$$2|(\alpha\theta)_x|^2 - 2C|u_{xx}|^2 \leqslant |u_{tt}|^2 \leqslant 2|(\alpha\theta)_x|^2 + 2C|u_{xx}|^2$$

together with relation (5.7) our conclusion follows. The proof is now complete. ■

From the above Lemma we conclude that

$$\frac{d}{dt}\left\{\int_{L_1}^{L}\theta u_{xt}\,dx-\frac{1}{2c_0}\int_{L-\delta}^{L}\alpha_2 u_t u_{xx}\,dx\right\}\leqslant$$

$$-\frac{m_0}{4c_0}\int_{L-\delta}^{L}|u_{xx}|^2\,dx-\frac{1}{2}\int_{L_1}^{L}\alpha|u_{xt}|^2\,dx+C_\delta\int_{L_1}^{L}|\theta_{xx}|^2+|\theta_x|^2\,dx+C\delta E_2(t).$$

From the third inequality of Lemma 5.2 we conclude that

$$\frac{d}{dt}\underbrace{\left\{\int_{L_1}^{L}\theta u_{xt}\,dx-\frac{1}{8c}\int_{L-\delta}^{L}\alpha_2 u_t u_{xx}\,dx+\frac{\gamma}{c_1}\int_{0}^{L}\alpha_3 u_{xt}u_{tt}\,dx\right\}}_{:=\mathcal{F}(t)}\leqslant$$

$$-\frac{m_0}{8c_0}\int_{L-\delta}^{L}|u_{xx}|^2\,dx-\frac{1}{4}\int_{0}^{L}\alpha|u_{xt}|^2\,dx-\frac{\alpha_3(L)\,m_0}{2c_1}|u_x(L,t)|^2$$

$$+C_\delta\int_{L_1}^{L}|\theta_{xx}|^2+|\theta_x|^2\,dx+C\int_{L_1}^{L}|\theta_{xt}|^2\,dx+(c_1\delta+c_2\varepsilon_0)\,\delta E_2(t)$$

where $\gamma=\frac{1}{8}\min\{1,m_0\}$. Let us denote by $\mathcal{L}$ the functional

$$\mathcal{L}(t)=N_1E_1(t)+N_1E_2(t)+NE_3(t)+\mathcal{F}(t).$$

It is not difficult to see that there exist positive constants σ_1 and σ_2 for which we have,

$$\sigma_1\{E_2(t)+E_3(t)\}\leqslant\mathcal{L}(t)\leqslant\sigma_2\{E_2(t)+E_3(t)\}.$$

Now let us take $d=\frac{\sigma_2}{\sigma_1}$. In this conditions we have

THEOREM 5.2. – *Let us suppose that the initial data satisfies condition* (5.6) *then there exists only one solution* (u,θ) *of system* (5.1)-(5.2) *satisfying*

$$u\in C([0,\infty[,H^2(0,L)\cap H_0^1(0,L))\cap C^1([0,\infty[,H_0^1(0,L)),$$

$$\theta_0(0,L)\in C([0,\infty[,H^2(0,L)\cap H_0^1(0,L))\cap C^1([0,\infty[,L^2(\Omega)),$$

PROOF. – To show the global existence of solutions it is enough to show that $T^* = \infty$. In fact let us suppose that $T^* < \infty$. Using relations (5.8), (5.10) and (5.12) we conclude that the functional $\mathcal{L}$ satisfies

$$\frac{d}{dt}\mathcal{L}(t) \leqslant -\kappa_0 \int_{L-\delta}^{L} |u_{xx}|^2 dx - \kappa_0 \int_0^L \alpha |u_{xt}|^2 dx - \kappa_0 |u_x(L,t)|^2 \tag{5.15}$$

$$-\left(\frac{\kappa N}{2} - C_\delta\right)\int_{L_1}^{L} |\theta_{xx}|^2 dx - \left(\frac{\kappa N_1}{2} - C_\delta\right)\int_{L_1}^{L} |\theta_x|^2 + |\theta_{xt}|^2 dx$$

$$+(NC\varepsilon_0 + N_2 C\varepsilon_0 + c\varepsilon_0 + c_1\delta)\, E_2(t). \tag{5.16}$$

Let us take δ such that $\delta C_1 \leqslant \frac{\kappa_0}{8}$ then take N_2 and N such that $\kappa N_2 - C_\delta > \kappa_0$ and $\kappa N - C_\delta > \kappa_0$ in this conditions we have that

$$\frac{d}{dt}\mathcal{L}(t) \leqslant -\kappa_0 \mathcal{N}(t) + (NC + N_2 C + C)\,\varepsilon_0 + c\delta)\, E_2(t).$$

Using Lemma 5.1 we arrive at

$$\mathcal{L}(t) - \mathcal{L}(0) \leqslant -\kappa_0 c_0 \int_0^t E_2(\tau)\, d\tau + [(NC + N_2 C + C)\,\varepsilon_0 + c\delta] \int_0^t E_2(\tau)\, d\tau\,.$$

Taking ε_0 and δ small we conclude that

$$\mathcal{L}(t) - \mathcal{L}(0) \leqslant -\frac{\kappa_0 c_0}{2}\int_0^t E_2(\tau)\, d\tau\,.$$

From where it follows that

$$\{E_2(t) + E_3(t)\} \leqslant \frac{\sigma_2}{\sigma_1}\{E_2(0) + E_3(0)\} - \frac{1}{\sigma_1}\frac{\kappa_0 c_0}{2}\int_0^t E_2(\tau)\, d\tau$$

$$\leqslant \frac{\sigma_2}{\sigma_1}\varepsilon_0 - \frac{\kappa_0 c_0}{2\sigma_1}\int_0^t E_2(\tau)\, d\tau < d\varepsilon_0.$$

Letting $t \to T^*$ we conclude that

$$E_2(T^*) + E_3(T^*) \leqslant \frac{\sigma_2}{\sigma_1}\varepsilon_0 - \frac{\kappa_0 c_0}{2\sigma_1}\int_0^{T^*} E_2(\tau)\, d\tau < d\varepsilon_0.$$

But this is contratictory with the maximality of T^* because by the continuity of the solution, there exists $\eta > 0$ such that $E_2(T^* + \eta) + E_3(T^* + \eta) < d\varepsilon_0$. Therefore $T^* = \infty$. The proof is now complete. ■

REMARK 5.1. – *The exponential decay to the partial thermoelastic model, means that we can stabilize the moviment of an elastic string intruducing another thermoelastic part, no matter how small it is. That is, to stabilize the moviment is not necessary to introduce neither an external sources nor external controls, but to compose the elastic material with another thermoelastic one.*

REFERENCES

[1] C. M. DAFERMOS, *On the existence and the asymptotic stability of solution to the equation of linear thermoelasticity*, Arch. Rat. Mech Anal., **29** (1968), 241-271.

[2] J. M. GREENBERG - LI TATSIEN, *The effect of the boundary damping for the quasilinear wave equation*, Journal of Differential Equations, **52** (1) (1984), 66-75.

[3] W. J. HRUSA - S. A. MESSAOUDI, *On formation of singularities in one dimensional nonlinear thermoelasticity*, Arch. Rational Mech. Anal., **111** (1990), 135-151.

[4] J. U. KIM, *On the energy decay of a linear thermoelastic bar and plate*, SIAM J. Math. Anal., **23** (1992), 889-899.

[5] V. KOMORNIK, *Rapid boundary stabilization of the wave equation*, SIAM J. Control and Optimization, **29** (1991), 197-208.

[6] V. KOMORNIK - ENRIKE ZUAZUA, *A direct method for the boundary stabilization of the wave equation*, J. Math. pures et appl., **69** (1990), 33-54.

[7] I. LASIECKA, *Global uniform decay rates for the solution to the wave equation with nonlinear boundary conditions*, Applicable Analysis **47** (1992), 191-212.

[8] J. E. MUÑOZ RIVERA, *Energy decay rates in linear thermoelasticity*, Funkcialaj Ekvacioj, **35** (1992), 19-30.

[9] J. E. MUÑOZ RIVERA - M. L. OLIVERA, *Stability in inhomogeneous and anisotropic thermoelasticity*, Bollettino della Unione Matematica Italiana, (7) **11-A** (1997), 115-127.

[10] M. NAKAO, *Decay of solutions of the wave equation with a local degenerate dissipation*, Israel J. Math., **95** (1996), 25-42.

[11] M. NAKAO, *Decay of solutions of the wave equation with a local nonlinear dissipation*, Math. Ann., **305**, No. 3 (1996), 403-417.

[12] K. ONO, *A stretched string equation with a boundary dissipation*, Kyushu J. of Math., **28**, No. 2 (1994), 265-281.

[13] R. RACKE, *On the time asymptotic behaviour of solutions in thermoelasticity*, Proceeding of the Royal Society of Edinburgh, **107A** (1987), 289-298.

[14] R. RACKE - Y. SHIBATA, *Global smooth solution and asymptotic stability in one dimensional nonlinear thermoelasticity*, Arch. Rat. Mech. Anal., **116** (1991), 1-34.

[15] R. RACKE - Y. SHIBATA - S. ZHENG, *Global solvability and exponential stability in one dimensional nonlinear thermoelasticity*, Quarterly Appl. Math., **51**, No. 4 (1993), 751-763.

[16] SHEN WEIXI - ZHENG SONGMU, *Global smooth solution to the system of one dimensional Thermoelasticity with dissipation boundary condition*, Chin. Ann. of Math., **7B** (3) (1986), 303-317.
[17] M. SLEMROD, *Global existence, uniqueness and asymptotic stability of classical smooth solution in one dimensional non linear thermoelasticity*, Arch. Rat Mech. Ana., **76** (1981), 97-133.
[18] R. TEMAN, *Infinite Dimensional Dynamical systems in Mechanics and Physics*, Springer-Verlag New York Inc. (1988).
[19] S. ZHENG, *Global solution and application to a class of quasi linear hyperbolic - parabolic coupled system*, Sci. Sínica, Ser. A, **27** (1984), 1274-1286.
[20] E. ZUAZUA, *Uniform stabilization of the wave equation by nonlinear wave equation boundary feedback*, SIAM J. control and optimization, **28** (1990), 466-477.
[21] E. ZUAZUA, *Exponential decay for the semilinear wave equation with locally distribuited damping*, Comm. P.D.E., **15** (1990), 205-235.

Jaime E. Muñoz Rivera: Laboratório Nacional de Computação Científica
Rua Getulio Vargas 333, Quitandinha 25651-070, Petrópolis, Rio de Janeiro, Brasil

Vanilde Bisognin: Department of Mathematics of Santa Maria University
Rio Grande do Sul, Brasil

Eleni Bisognin: Department of Mathematics of Santa Maria University
Rio Grande do Sul, Brasil

Pervenuta in Redazione
l'11 luglio 2001