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On the Variety of Linear Series on a Singular Curve.

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Sunto. – *Sia Y una curva proiettiva ridotta e irriducibile con $g := p_a(Y) \geq 2$. Per ogni d, r interi positivi sia $W_d^r(Y)(**)$ l'insieme di tutti gli $L \in \mathrm{Pic}^d(Y)$ con $h^0(Y, L) \geq r+1$ e L generato dalle sezioni globali. In questo lavoro si dimostra che se $d \leq g-2$, allora $\dim(W_d^r(Y)(**) \leq d-3r$ fatte salve rare eccezioni (essenzialmente il caso in cui Y sia un rivestimento doppio della retta proiettiva).*

Summary. – *Let Y be an integral projective curve with $g := p_a(Y) \geq 2$. For all positive integers d, r let $W_d^r(Y)(**)$ be the set of all $L \in \mathrm{Pic}^d(Y)$ with $h^0(Y, L) \geq r+1$ and L spanned. Here we prove that if $d \leq g-2$, then $\dim(W_d^r(Y)(**)) \leq d-3r$ except in a few cases (essentially if Y is a double covering).*

1. – Introduction.

Let Y be an integral projective curve defined over an algebraically closed field $\boldsymbol{K}$ with $\mathrm{char}(\boldsymbol{K}) = 0$. Set $g := p_a(Y)$. For all positive integers d, r let $W_d^r(Y)(**)$ be the set of all $L \in \mathrm{Pic}^d(Y)$ with $h^0(Y, L) \geq r+1$ and L spanned. We stress that in the definition of $W_d^r(Y)(**)$ we assume that L is both locally free and spanned. Indeed in general non-locally free rank 1 torsion free sheaves of degree d are not parametrized by an irreducible d-dimensional variety and if we start from a non-spanned line bundle, L, the subsheaf of L generated by $H^0(Y, L)$ may not be locally free. Furthermore, only spanned line bundles correspond to morphisms into projective spaces and this was the original reason to study line bundles on smooth curves. If Y is smooth and $d \leq g-1$ H. Martens proved that $\dim(W_d^r(Y)(**)) \leq d-2r$ ([9]). For any $L \in W_d^r(Y)(**)$, let $h_L: Y \to \boldsymbol{P}(H^0(Y, L)$ be the associated morphism. We believe that, even in the case of smooth curves, to get refined informations on $W_d^r(Y)(**)$ it is essential to distinguish between simple linear systems and linear systems composed with an involution. Hence we introduce the following notation. If $r \geq 2$, $n \geq 2$ and $q \geq 0$, set $A_d^r(Y)(**) := \{L \in W_d^r(Y)(**): h_L$ is birational$\}$, $B_d^r(Y)(**) := \{L \in W_d^r(Y)(**): h_L$ is not birational$\}$, $B_d^r(Y)(**)(n) := \{L \in B_d^r(Y)(**): \deg(h_L) = n\}$ and $B_d^r(Y)(**)(n, q) := \{L \in B_d^r(Y)(**)(n)$: the normalization $h_L(Y)$ has genus $q\}$. Set $B_d^1(Y)(**) := \{L \in W_d^1(Y)(**)$: there are morphisms $f: Y \to C$ and $h: C \to D$ with $\deg(f) \geq 2, \deg(h) \geq 2$ and

$h_L = h \circ f\}$ and $A^1_d(Y)(**) := W^1_d(Y)(**) \setminus B^1_d(Y)(**)$. Take $L \in B^r_d(Y)(**)(n)$; by the very definition of h_L the curve $h_L(Y)$ spans $\boldsymbol{P}(H^0(Y, L))$ and there is $R \in \mathrm{Pic}(h_L(Y))$ such that $R \in A^r_{d/n}(h_L(Y))(**)$, $h^0(h_L(Y), R) = h^0(Y, L)$ and $L \cong h_L^*(R)$: just take $R := \boldsymbol{O}_{h_L(Y)}(1)$.

In section 3 we will prove the following result.

THEOREM 1.1. – *Let Y be an integral projective curve and set $g := p_a(Y)$. Fix integers d and r with $1 \leqslant 2r \leqslant d \leqslant g-2$. Then:*

(a) $\dim(W^r_d(Y)(**)) \leqslant d - 2r$. *If Y is not hyperelliptic we have* $\dim(W^r_d(Y)(**)) \leqslant d - 2r - 1$.

(b) *If $r \geqslant 2$ we have* $\dim(A^r_d(Y)(**)) \leqslant d - 3r$.

(c) *If $r \geqslant 2$, $n \geqslant 2$, $q \geqslant 1$ and the normalization of Y has genus at least* 2 *then* $\dim(B^r_d(Y)(**)(n, q)) \leqslant d - 3r$.

(d) *If $r \geqslant 2$ and $n \geqslant 3$ we have* $\dim(B^r_d(Y)(**)(n)) \leqslant d - 3r$.

Theorem 1.1 is related to [3 Th. 3.2.1]. One of the two reasons for our inability to adapt almost verbatim the proof of [3 Th. 3.2.1] is the obvious failure of de Franchis' theorem for singular curves. The second reason is studied in 2.7. Assume $\dim(B^r_d(Y)(**)(2)) \geqslant d - 3r + 1$; we do not claim that for every irreducible component, T, of $(B^r_d(Y)(**)(2)$ with $\dim(T) \geqslant d - 3r + 1$ there is a degree 2 morphism $u : Y \to C$ such that for general $L \in T$ there is $M \in W^r_{d/2}(C)(**)$ with $L \cong u^*(M)$; we only claim that for a general $L \in T$ there is a degree 2 morphism $u_L : X \to C_L$ and $M_L \in A^r_{d/2}(C_L)(**)$ with $L \cong u^*(M_L)$; we do not know if a more general statement is true; our ignorance is related to our ignorance of any reasonably strong estension to singular curves of de Franchis' theorem. We discuss this topic in section 2. In section 2 we introduce the notion of saturated morphism between integral projective curves and of saturation of any morphism $f : Y \to C$. We believe that this is the right equivalent for multiple coverings $f : Y \to C$ between singular curves of the notion of passing to the normalization of Y and C when we do not want to change the domain, Y, of the morphism. Using this notion we prove a finiteness theorem (2.3 and 2.4) which is the exact analogous of de Franchis' theorem, but with two further assumptions: we restrict to saturated coverings and we assume that either the target has geometric genus 2 or the target has geometric genus 1 but we fix the degree of the morphism. This form of de Franchis' theorem is used to prove part (c) of 1.1. We show that the corresponding result is not true if the target has geometric genus 0, even if we work just with nodal curves (Example 2.6). Then we give a finiteness result which is essential for the proof of parts (c) and (d) of 1.1 (see Proposition 2.7).

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2. – Saturation and de Franchis' theorem.

The aim of this section is to give the «right» generalization of de Franchis' theorem to singular curves (see Corollary 2.3). Let Y and C be integral projective curves. We will say that the morphism $f : Y \to C$ is saturated (or that C is saturated for the pair (Y, f)) if for every integral projective curve C', every birational morphism $u : C' \to C$ and every morphism $f'' : Y \to C'$ with $u \circ f'' = f$ the morphism u is an isomorphism. Notice that if $\deg(f) = 1$ f is saturated if and only if f is an isomorphism. We will say that a pair (f'', u) with $f'' : Y \to C'$, $u : C' \to C$ with u birational and $u \circ f'' = f$ is a saturation of f if f'' is saturated.

LEMMA 2.1. – *Let $f : Y \to C$ be a morphism between integral projective curves. Then f has a saturation $f'' : Y \to C'$, $u : C' \to C$ which is unique, up to an automorphism of C'.*

PROOF. – Since the uniqueness part is obvious, we will check only the existence part. Let $\pi : X \to Y$ and $\pi' : D \to C$ the normalization. The morphism $f : Y \to C$ induces a morphism $f' : X \to D$. Since $\boldsymbol{O}_Y$ is in a natural way a coherent subsheaf of $\boldsymbol{K}$-algebras of $\pi_*(\boldsymbol{O}_X)$, we may see $f_*(\boldsymbol{O}_Y)$ as a subsheaf of $\boldsymbol{K}$-algebras of $f'_*(\boldsymbol{O}_X)$. The sheaf $f'_*(\boldsymbol{O}_X)$ is an $\boldsymbol{O}_D$-module. We see $\boldsymbol{O}_C$ as a coherent subsheaf of $\boldsymbol{O}_D$. The existence of $f : Y \to C$ shows that $\boldsymbol{O}_C$ sends $f_*(\boldsymbol{O}_Y)$ into itself. Let A be the coherent $\boldsymbol{K}$-subsheaf $\mathrm{Hom}(f_*(\boldsymbol{O}_Y), f_*(\boldsymbol{O}_Y))$ of $\boldsymbol{O}_D$. Thus $\boldsymbol{O}_C \subseteq A \subseteq \boldsymbol{O}_D$. Since there is an inclusion $A \subseteq \boldsymbol{O}_C$, it is easy to check that the sheaf A is a sheaf of local $\boldsymbol{K}$-algebras which defines a curve, C', birational to C and a morphism $u : C' \to C$. The inclusion $\boldsymbol{O}_C \subseteq A$ shows that f factors through u. For every pair (C'', h) with $u'' : C'' \to C$ birational map and $h : Y \to C''$ morphism with $u'' \circ h = f$ we have $\boldsymbol{O}_{C''} \subseteq A$ because f' is the same for f and f''. Thus (C', u, f'') is a saturation of f. ■

LEMMA 2.2. – Let Y, C, Z be integral projective curves and $\pi : X \to Y$, $u : D \to C$ and $v : W \to Z$ their normalizations. Take morphisms $f : Y \to C$ and $h : Y \to Z$ and call $f' : X \to D$ and $h' : X \to W$ the induced morphisms. If $f' = h'$ (up to an isomorphism of D and W) and h is saturated, then f factors through h, $\deg(f) = \deg(h)$ and $p_a(C) \geq p_a(Z)$; we have $p_a(C) = p_a(Z)$ if and only if $C \cong Z$. If $f' = h'$ up to an isomorphism of D and W and both f and h are saturated, then $f = h$, up an isomorphism of C and Z.

PROOF. – We obviously have $\deg(f) = \deg(h)$. The thesis follows from the construction of the saturation, f'', of f given in the proof of Lemma 2.1 and the assumption that both f'' and h are saturated. ■

COROLLARY 2.3. – *Let Y be an integral projective curve such that its nor-*

malization has genus at least 2. Then there are only finitely many pairs (C, f), where C is an integral projective curve with normalization of genus at least 2 and $f : Y \to C$ is a saturated morphism.

PROOF. – Just use 2.2. and classical de Franchis' theorem for smooth curves.

REMARK 2.4. – Let Y be an integral projective curve such that its normalization has genus at least 2. Fix an integer $n \geq 2$. By Lemma 2.2 and a generalization of the classical de Franchis' theorem due to Tamme ([10]) there are only finitely many pairs (C, f) where C is an integral projective curve with normalization of genus 1 and $f : Y \to C$ is a degree n saturated morphism (up to isomorphisms of C if C is smooth).

REMARK 2.5. – Let $f : Y \to C$ be a non-constant morphism between integral projective curves, $\pi : X \to Y$ and $u : D \to C$ their normalizations. Let (f'', u') be the saturation of f with $f'': Y \to C'$ and $u' : C' \to C$. If $X = Y$, then (f', u) is the saturation of f. We claim that if Y is seminormal in the sense of [11] and [4], then C' is seminormal. The claim follows from the universal property of seminormalization. Alternatively, the claim is easily checked by applying the construction of Y from X (i.e. gluing together some points of X) given in [11] and [4]; indeed, if $Q \in \mathrm{Sing}(Y)$ is obtained by gluing together $P_1, \ldots, P_s \in X$, then $f''(Q)$ is obtained by gluing together $f'(P_1), \ldots, f'(P_s)$; in particular, $f''(Q) \in \mathrm{Sing}(C')$ if and only if $\mathrm{card}(\{f'(P_1), \ldots, f'(P_s)\}) < 1$. This description shows that if Y has only ordinary double points as singularities, then C' has only ordinary double points as singularities. A similar description holds if Y has some ordinary cusp; if Q is an ordinary cusp and $f'': Y \to C'$ is saturated, $f''(Q)$ is an ordinary cusp if f' is unramified at $\pi^{-1}(Q)$, while $f''(Q) \in C'_{\mathrm{reg}}$ if f' is ramified at $\pi^{-1}(Q)$.

In our opinion the following example shows that there is no natural generalization of Corollary 2.3 and Remark 2.4 to the case in which the normalization of C has genus 0, even if we assume Y nodal.

EXAMPLE 2.6. – Fix integers q, u, v, n with $q \geq 0$, $n \geq 2$, and $u \geq v \geq 0$. Let X be a smooth curve of genus q and Y a nodal curve with $\pi : X \to Y$ as normalization and $\mathrm{card}(\mathrm{Sing}(y)) = u$. Set $\{P_1, \ldots, P_u\} := \mathrm{Sing}(Y)$ and $\pi^{-1}(P_i) := \{P_i', P_i''\}$, $1 \leq i \leq u$. Let $A(v)$ be the set of all integral nodal curves with $\boldsymbol{P}^1$ as normalization and exactly v nodes. $A(v)$ is in a natural way the quotient of a smooth variety of dimension $2v$ by an action of $\mathrm{Aut}(\boldsymbol{P}^1)$. For any $C \in A(v)$ the set of all degree n saturated morphisms $f : Y \to C$ may be described in the following way. Let $b: \boldsymbol{P}^1 \to C$ be the normalization, $\{Q_1, \ldots, Q_v\} := \mathrm{Sing}(C)$ and $\{Q_j', Q_j''\} := b^{-1}(Q_j)$, $1 \leq j \leq v$. Every degree n saturated morphism from Y onto C is uniquely determined by a degree n morphism $h' : X \to \boldsymbol{P}^1$. For any degree n morphism $f': X \to \boldsymbol{P}^1$, f' descends to a morphism $h: Y \to C$ if and only if for every integer i with $1 \leqslant i \leqslant v$ either $f'(P_i') = f'(P_i'')$ or there is an integer $j(i)$ such that $f'(\{P_i', P_i''\}) = \{Q_{j(i)}', Q_{j(i)}''\}$. Assume that h exists. The morphism h is saturated if and only if it does not factor through a partial normalization of C, i.e. if and only if for every integer j with $1 \leqslant j \leqslant v$ there exists an integer i such that $f'(\{P_i', P_i''\}) = \{Q_j', Q_j''\}$. This description shows that if there exists a degree n morphisms $f': X \to \boldsymbol{P}^1$, then for any u, v with $u \geqslant v > 0$ it is quite easy to construct Y such that for infinitely many $C \in A(v)$ there is a degree n saturated morphism $h: Y \to C$ with h' as associated morphism between the normalizations. Furthermore, for a general such Y we may even count the dimension of the set of all such curves $C \in A(v)$.

The following result gives a key property of saturated morphisms.

PROPOSITION 2.7. – *Let $f: X \to C$ be a saturated morphism between integral projective curves. Then the induced pull-back map $\alpha: \mathrm{Pic}(C) \to \mathrm{Pic}(X)$ has finite kernel.*

PROOF. – Let $u: U \to X$ and $v: V \to C$ be the normalization maps and $w: U \to V$ be the morphism induced by f. It is well known that the pul-back map $\beta: \mathrm{Pic}(V) \to \mathrm{Pic}(U)$ induced by w has finite kernel because w is a surjective morphism between smooth projective curves. The connected component $\mathrm{Pic}^0(C)$ (resp. $\mathrm{Pic}^0(X)$) of the locally algebraic group $\mathrm{Pic}(C)$ (resp. $\mathrm{Pic}(X)$) containing the trivial line bundle is obtained from the Abelian variety $\mathrm{Pic}(V)$ (resp. $\mathrm{Pic}(U)$) by making a finite number of extensions with the multiplicative group $\boldsymbol{K}^*$ and the additive group $\boldsymbol{K}$. Each of these extensions corresponds to a partial normalization of a partial normalization of C (resp. X). If α has infinite kernel, then there is a subgroup of $Ker(\alpha)$ isomorphic to the multiplicative group $\boldsymbol{K}^*$ or the additive group $\boldsymbol{K}$ and we may factorize f through the corresponding partial normalization of C, contradicting the saturatedness of f.

3. – Proof of 1.1.

The algebraic set $W_d^r(Y)(^{**})$ has a natural scheme structure and locally a determinantal description as in the case of smooth curves. We have the following lemma.

LEMMA 3.1. – *Fix $L \in W_d^r(Y)(^{**})$. Let $\mu_L: H^0(Y, L) \otimes H^0(Y, \omega_Y \otimes L^*) \to H^0(Y, \omega_Y)$ be the cup product. Then* $\mathrm{Coker}(\mu_L)$ *is isomorphic to the Zariski tangent space of $W_d^r(Y)(^{**})$ at L.*

PROOF. – Notice that the cup product is well-defined because L is locally free and hence $L \otimes (\omega_Y \otimes L^*) \cong \omega_Y$. Let $(Y_{\mathrm{reg}})^{(d)}$ be the symmetric product and $u_d\colon (Y_{\text{reg}})^{(d)} \to \text{Pic}^d(Y)$ the natural morphism. Fix $D \in (Y_{\text{reg}})^{(d)}$. Use D to identify $\text{Pic}^d(Y)$ and $\text{Pic}^0(Y)$. As in [1. 1.2], with this identification the differential of u_d at D is the coboundary linear map $H^0(Y, \boldsymbol{O}_D(D)) \to H^1(Y, \boldsymbol{O}_Y)$ induced by the exact sequence

$$0 \to \boldsymbol{O}_Y \to \boldsymbol{O}_Y(D) \to \boldsymbol{O}_D(D) \to 0 \tag{1}$$

Now we take as D a general effective divisor with $\boldsymbol{O}_Y(D) \cong L$; this is admissible because by the spannedness of L at each point of $\text{Sing}(Y)$ we may take $D \subset Y_{\text{reg}}$. Look at the determinantal description of the variety of special divisors on a smooth curve ([2, p. 154]). Since L is spanned, we may assume D reduced; here we use $\text{char}(\boldsymbol{K}) = 0$. Since $D \subset Y_{\text{reg}}$, ω_Y is locally free at each point of D. Hence for any choice of a basis $\omega_1, \dots, \omega_g$ of $H^0(Y, \omega_Y)$ we may form the Brill-Noether $g \times d$ matrix obtained by evaluating each ω_i at each point of D; the rank of this matrix does not depend on the choice of local coordinates at each point of D needed to evaluate each ω_i. Now just use the relevant local calculations at each point of D made in [2, Ch. IV, §1, Lemma 1.1 and Lemma 1.5].

Using Lemma 3.1 the proof of [2, Th. 4.5.1] works verbatim. Now we will show that this observation gives the following extension of Martens' theorem which is the second assertion of part (a) of Theorem 1.1.

PROPOSITION 3.2. – *Let Y be an integral projective curve with $g := p_a(Y) \geq 3$. Assume Y not hyperelliptic. Then for all positive integers d, r with either $0 < 2r \leq d \leq g-2$ or $d = g-1 > 2r > 0$ we have* $\dim(W^r_d(Y)(^{**})) \leq d - 2r - 1$.

PROOF. – Fix an irreducible component T of $W^r_d(Y)(^{**})$ with $\dim(T) \geq d - 2r$ and a general $L \in T$. By Lemma 3.1 we may copy the proof of [2, Th. 5.1], which uses only the Petri map, the base point free pencil trick and obtain $h^0(Y, L^{\otimes 2}) \geq d+1$. Now in [2] one uses Clifford's theorem; we may use the form of Clifford's theorem proved in [5, Th. 1 of the Appendix with J. Harris]. Since Y is assumed to be not hyperelliptic (in the usual sense of being a double cover of $\boldsymbol{P}^1$) we have to exclude just a curve, $T(g)$, with rational normalization and with a unique singular point; such a curve has no spanned line bundle of degree d (or even of degree g) by the last Remark in [7]. Hence, as in [2, top of p. 193], it remains only the case $d = g - 1$ and $L^{\otimes 2} \cong \omega_Y$. In particular we may assume Y Gorenstein. Since even in the singular case Y has only finitely many theta-characteristics and $d > 2r$, we obtain a contradiction.

PROOF OF 1.1. – Since for hyperelliptic curves much more is known by [6], we may assume $d \geq 3$ and Y not hyperelliptic. Hence part (a) of 1.1 follows from 3.2. We assume the existence of an irreducible subvariety, T, of $W^r_d(Y)(^{**})$ with $\dim(T) \geq d - 3r + 1$ and take a general $L \in T$.

(i) Here we assume $L \in A^r_d(Y)(**)$ and prove part (b) of 1.1. In particular we have $r \geq 2$. We want to find a contradiction. We need to solve (in our set-up) a few exercises from [2]. We will do that in detail to correct a few misprints contained in the relevant part of [2].

(a) ([2, Ex. IV.E-1]). Fix $M \in W^r_d(Y)(**)$ with $h^0(Y, M) = r + 1$ and $h^1(Y, M) \neq 0$. We want to check that

$$d \geq g - d + 2r - h^0(Y, \omega_Y \otimes M^{*\otimes 2}). \tag{2}$$

We take a linear subspace $V \subseteq H^0(Y, M)$ with $\dim(V) = 2$ and V spanning L. As in the base point free pencil trick V induces an exact sequence

$$0 \to \boldsymbol{O}_Y \to M \oplus M \to M^{\otimes 2} \to 0 \tag{3}$$

From (3) we obtain $2(h^0(Y, M)) \leq 1 + h^0(Y, M^{\otimes 2}) = 1 + 2d + 1 - g + h^0(Y, \omega_Y \otimes M^{*\otimes 2})$.

(b) ([2, Ex. III.B-6]). Fix $M \in A^r_d(Y)(**)$ and $N \in W^x_y(Y)(**)$. Let $\mu : H^0(Y, M) \otimes H^0(Y, N) \to H^0(Y, M \otimes N)$ be the multiplication map. We want to check that if $d \geq h^0(Y, M) + h^0(Y, N) - h^0(Y, N \otimes M^*) - 1$, then

$$\dim(\operatorname{Im}(\mu) \geq h^0(Y, M) + 2h^0(Y, N) - h^0(Y, N \otimes M^*) - 2 \tag{4}$$

The misprint in the corresponding formula in [2] was remarked in [8, lines between Lemmas 5 and 6]. Notice that in any tensor product appearing in (4) one of the factors is locally free and we take the dual of a line bundle. We follow the hint given in [2]. We may assume $h^0(Y, N) \geq 2$. Take a general effective divisor D with $\boldsymbol{O}_Y(D) \cong M$. Hence $\deg(D) = d$ and D is the sum of d distinct regular points of Y; identify D with the corresponding subset of Y_{reg}. Take $B \subset D$ with $\operatorname{card}(B) = r$ and $h^0(Y, M(-B)) = 1$ and set $A := D \backslash B$. Since h_M is birational, we may apply the uniform position principle and obtain $h^0(Y, N(-A) = \max\{h^0(Y, N \otimes M^*), h^0(Y, N) - d + r\}$. Since $d \geq h^0(Y, M) + h^0(Y, N) - h^0(Y, N \otimes M^*) - 1$, we have $h^0(Y, N(-A)) = h^0(Y, N \otimes M^*)$. We claim that $\dim(\operatorname{Im}(\mu)) - \dim(\operatorname{Im}(\mu)(-D)) \geq (h^0(Y, M) - h^0(Y, M(-B))) + (h^0(Y, N) - h^0(Y, N(-A))) - 1$, i.e. we claim that the number of conditions imposed by D on $\operatorname{Im}(\mu)$ is at least the number, p, of conditions imposed by B on $H^0(Y, M)$ plus the number, q, of conditions imposed by A on $H^0(Y, N)$ minus 1. To check the claim, let $B' \subseteq B$ with $\operatorname{card}(B') = p$ and $A' \subseteq A$ with $\operatorname{card}(A') = q$ imposing independent conditions respectively on $H^0(Y, M)$ and $H^0(Y, N)$. We may chose $P \in B'$, $Q \in A'$ and $\alpha \in H^0(Y, M)$, $\beta \in H^0(Y, N)$, such that $\alpha(P) \neq 0$, $\alpha(V) = 0$ for every $V \in (B' \backslash \{P\})$, $\beta(P) \neq 0$, $\beta(Q) \neq 0$ and $\beta(W) = 0$ for every $W \in (A' \backslash \{Q\})$. Hence $\alpha \otimes \beta(Z) \neq 0$ for every $Z \in (A' \cup B' \backslash \{P, Q\})$ and $\alpha \otimes \beta(P) \neq 0$. Hence the number of conditions imposed by D on $\operatorname{Im}(\mu)$ is at least $(p-1) + (q-1) + 1$, as claimed. The obvious inequality $\dim(\operatorname{Im}(\mu)(-D)) \geq h^0(Y, N)$ allows us obtain (4).

(c) ([2, Ex. IV.E-2]. By (2) and Rieman-Roch we have $d \geq h^0(Y, L) + h^0(Y, \omega_Y \otimes L^*) - h^0(Y, \omega_Y \otimes L^{*\otimes 2}) - 1$. Hence we may apply (b) to the case $M := L$ and $N := \omega_Y \otimes L^*$. By Riemann-Roch we obtain $\mathrm{rank}(\mu) \geq 2g - 2d + 3r - 1 - h^0(Y, \omega_Y \otimes L^{*\otimes 2})$. By Lemma 2.1 we obtain in our situation as in [2, Ex. IV.E-2] $\dim(T) \leq -g + 2d - 3r + 1 + h^0(Y, \omega_Y \otimes L^{\otimes 2}) = h^0(Y, L^{\otimes 2}) - 3r \leq d - 3r$ (Clifford and assumption $d \leq g-2$). Thus the case h_L birational is over.

(ii) Now we assume h_L not birational. Set $n := \deg(h_L)$. We may assume that parts (c) and (d) of 1.1 are true for all pairs (r', d') with $2 \leq r' < r$ and $d' < d$; if $r = 2$ we will use part (a) of 1.1 for the integer $r' := 1$. If $h^0(Y, L) > r + 1$ the generality of L implies the existence of an irreducible subvariety, A, of $B_d^r(Y)(**)$ with $\dim(A) > \dim(T)$ and such that a general $M \in A$ has $h^0(Y, M) = r + 1$. Hence it is sufficient to find a contradiction under the assumption $h^0(Y, L) = r + 1$. Set $C := h_L(X) \subset \boldsymbol{P}^r$ and $R := \boldsymbol{O}_C(1)$, so that $h_L^*(R) \cong L$, $h^0(C, R) = r + 1$ and $n = d/\deg(C)$. Take a general $Q \in Y_{\mathrm{reg}}$ and set $P := h_L(Q)$. We have $P \in C_{\mathrm{reg}}$ and P is general in C. We have $h^0(C, R(-P)) = r$ and $h_L^{-1}(P) \subset Y_{\mathrm{reg}}$. Thus $L' := L(-h_L^{-1}(P))$ is locally free, $h^0(Y, L') = r$ and $\deg(L') = d - n$. Since $\mathrm{char}(\boldsymbol{K}) = 0$ and P is general, $R(-P)$ is spanned by the trisecant lemma. Thus L' is spanned, i.e. $L' \in W_{d-n}^{r-1}(Y)(**)$. We claim that the rational map from T into $W_{d-n}^{r-1}(Y)(**)$ sending L to L' is finite. Since $h_L(Q)$ is smooth , the saturations of the morphisms h_L and h_L' are the same. Let $f: Y \to Z$ be the common saturation. Every element in T comes from at least one element of $W^r d/n(Z)(**)$ through the morphism α: $W_{d/n}^r(Z)(**) \to W_d^r(Y)(**)$ induced by f. Since $h_L(Q)$ is a smooth point of C and hence of Z and $R(-P)$ is spanned by the trisecant lemma, we have a natural injective map from $W_{d/n}^r(Z)(**)$ into $W_{d/n-1}^{r-1}(Z)(**)$ induced by the projection from P. We have also a morphism β: $W_{d/n-1}^{r-1}(Z)(**) \to W_{d-n}^{r-1}(Y)(**)$ induced by f. From Proposition 2.7 it follows that β is finite. Hence the claim is proved. By the claim there is an irreducible subvariety, G, of $W_{d-n}^{r-1}(Y)(**)$ with $\dim(T) \leq \dim(G)$ and such that a general element of G corresponds to a degree n covering. By the inductive hypothesis (or, if $r = 2$, by Proposition 3.2) we have $\dim(G) \leq d - n - 3(r-1)$, which is absurd if $n \geq 3$. If $T \subseteq B_d^r(Y)(**)(n, q)$ we use 2.3 (case $q \geq 2$) and 2.4 (case $q = 1$) and apply this forms of de Franchis' theorem to the saturation of the morphisms $h_L: Y \to h_L(Y)$, $L \in B_d^r(Y)(**)(n, q)$. Hence we may repeat the proof of [3. Th. 3.2.1], case not simple at p. 254, taking as C' the target of the saturation $u: Y \to C'$ of the morphism h_L. As in [3, part (ii) at p. 254] we need to apply the weaker form of part (a) corresponding to classical H. Martens' theorem $\dim(A_{d/n}^r(C')(**)) \leq d/n - 2r$ to the curve C' if $p_a(C') \geq 2$ and, taking $R \in \mathrm{Pic}(C')$ with $u^*(R) \cong L$ and $h^0(C', R) = h^0(Y, L)$, we have $h^1(C', R) > 0$.

Hence we have proved parts (c) and (d) of 1.1.

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