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Hausdorff Fréchet Closure Spaces with Maximum Topological Defect.

Riccardo Ghiloni

Sunto. – *È noto che il difetto topologico di ogni spazio di chiusura di Fréchet é minore o uguale al primo ordinale non numerabile ω_1. Nel caso di spazi di chiusura di Hausdorff Fréchet si ottengono alcune condizioni generali sufficienti affinché il difetto topologico sia pari a ω_1. Alcuni risultati classici e recenti sono dedotti dal nostro criterio.*

Summary. – *It is well-known that the topological defect of every Fréchet closure space is less than or equal to the first uncountable ordinal number ω_1. In the case of Hausdorff Fréchet closure spaces we obtain some general conditions sufficient so that the topological defect is exactly ω_1. Some classical and recent results are deduced from our criterion.*

Dedicated to Roberto Ghiloni and Giuseppina Gavazzi

Introduction.

One of the most important topological notions is undoubtedly the one of *convergence of sequences*. It not only satisfies Analysis's necessities, but has also been the main guide-idea for the foundation of General Topology (see [Fré]).

From this notion it seems natural to infer a closure operation in the following way. Let X be a topological space and let A be a fixed subset of X. One defines the sequential closure of A as the subset $\mathrm{cl}(A)$ of X formed by all limit points of sequences ranging in A. Such closure operation fulfils the properties $\mathrm{cl}(\emptyset)=\emptyset$, $A \subset \mathrm{cl}(A)$ for each $A \subset X$, $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$ for each $A, B \subset X$ but in general it's not idempotent as it's required for the closure operations derived from some topological structure for X.

To understand how this sequential closure operation is «near» the topology of X, one may consider the transfinite sequence of iterations of the sequential closure (this idea go back to Hausdorff [Ha]). It's well-known that it suffices to apply the sequential closure at the most ω_1 times (where ω_1 is the first uncountable ordinal) to have a topological closure.

In literature there exist examples for which really need ω_1 iterations. We recall some of these spaces: the set of all real-valued functions on the real line equipped with the topology of pointwise convergence (remember the existence of Baire functions of type α but not less than α for each fixed ordinal number α less than ω_1), the space S_ω of Arhangel'skiĭ and Franklin [AF] (this is the first example of countable homogeneous Hausdorff sequential topological space in which the topological closure is obtained by precisely ω_1 iterations of the sequential closure), the rational Féron cross topological plane (see [Gr]) (using this space G. H. Greco answered affirmatively to a question by Arhangel'skiĭ and Franklin [AF] about the existence of some countable homogeneous Hausdorff sequential topological space different from S_ω) and the radiolar topological plane (see [Fč]).

In this paper, using the language of closure spaces, we obtain a general criterion (see Theorem 6 and Theorem 9) which allows us to single out some type of such spaces; in particular we find again the space S_ω, the rational Féron cross topological plane and the radiolar topological plane above-mentioned.

We now mention a result (Theorem 11) obtained in a class of closure spaces which represents a natural ambient to apply our criterion that is the «limited topological spaces» (see subsection 3.3).

First we quickly recall some basic definitions about closure spaces.

A closure space is a pair (X, u) where X is a set and u is a closure operator for X that is a map from the family of all subsets of X into itself such that $u(\emptyset) = \emptyset$, $A \subset u(A)$ for each $A \subset X$, $u(A \cup B) = u(A) \cup u(B)$ for each $A, B \subset X$. For each $x \in X$ one defines the u-neighborhood filter $\mathcal{N}_u(x)$ of x in such way that $x \in \bigcap_{U \in \mathcal{N}_u(x)} U$ and, for each fixed $A \subset X$, $x \in u(A)$ iff (i.e. «if and only if») each $U \in \mathcal{N}_u(x)$ intersects A. A base system for u is a map that assigns to every $x \in X$ a base of the filter $\mathcal{N}_u(x)$. On the other hand, fixed a base system $\mathcal{B}$ on X (i.e. a map that assigns to every $x \in X$ a base of a filter in X) such that $x \in \bigcap_{U \in \mathcal{B}(x)} U$ for each $x \in X$, there exists only one closure operator v for X such that $\mathcal{B}$ is one of its base systems; one may verify that $v(A) = \{x \in X \mid U \cap A \neq \emptyset \ \forall U \in \mathcal{B}(x)\}$ for each $A \subset X$.

We introduce the limited topologies.

Let X be a set with topology τ, let $\mathcal{N}_\tau(x)$ be the τ-neighborhood filter of x for each $x \in X$ and let $\mathcal{D}$ be a map that assigns to each point $x \in X$ a subset $\mathcal{D}(x)$ of X in such way that $x \in \mathcal{D}(x)$ for each $x \in X$. $\mathcal{D}$ will be called set distribution in X. We define a base system $\mathcal{B}$ on X putting $\mathcal{B}(x) := \{U \cap \mathcal{D}(x)\}_{U \in \mathcal{N}_\tau(x)}$ for each $x \in X$. In this manner $\mathcal{B}$ is a base system for the closure operator $\tau_{\mathcal{D}}$ for X defined as follows

$$\tau_{\mathcal{D}}(A) := \{x \in X \mid U \cap \mathcal{D}(x) \cap A \neq \emptyset \ \ \forall U \in \mathcal{N}_\tau(x)\}$$

for all $A \subset X$. $\tau_{\mathcal{D}}$ will be called $\mathcal{D}$-limited topology τ. An example of this type of closure operators is the Féron cross closure operator (see [Gr] just quoted).

From our criterion follows the next result.

THEOREM 11. – Let (E, v) be a metrizable topological real (or complex) vector space and let $\mathcal{D}$ be a set distribution in E. Put $u := v_{\mathcal{D}}$. Suppose that $\mathcal{D}$ is invariant under translations and it assigns to the origin $\underline{0}$ of E a finite union of v-closed vectorial subspaces of E. Only one of the two following situations must occur: either $\mathcal{D}(\underline{0})$ is a vectorial subspace of E and hence u is a topological closure operator for E or $\mathcal{D}(\underline{0})$ is not a vectorial subspace of E and hence it needs ω_1 iterations of u to obtain a topological closure operator for E.

Really we prove some more. In fact we introduce the evolution function $\mathbf{ev}_u$ of u as the map that assigns to each $x \in X$ the smallest ordinal number α (which always exists) such that $\mathcal{N}_{u^\alpha}(x) = \mathcal{N}_{u^{\alpha+1}}(x)$ and we prove that, in the case «$\mathcal{D}(\underline{0})$ is not a vectorial subspace of E», $\mathbf{ev}_u$ is constantly equal to ω_1.

1. – Preliminaries.

In this section, using a more formal language, we introduce the basic ideas concerning closure spaces which include topological ones; in particular we review some classical definitions and results in terms of closure operators and we briefly investigate the relationship with the classical ones. The main references are [Če] (especially sections 14, 15, 16, 17 of Chapter III and sections 31, 33 of Chapter VI), [No1] (section 1) and [DG1]; moreover we point out [Fr1] and [Fr2] as fundamental works in the study of sequential topological spaces.

Let X be a set. A **closure operator** for X is a map u from the family $\mathcal{P}(X)$ of all subsets of X into itself such that: $u(\emptyset) = \emptyset$, $A \subset u(A)$ for each $A \subset X$, $u(A \cup B) = u(A) \cup u(B)$ for each $A, B \subset X$. The pair (X, u) will be called **closure space** (or **pretopological space**) and if $A \subset X$ then $u(A)$ will be called u-**closure** of A.

A closure u is said to be **finer** than a closure v, or v to be **coarser** than u, if $u(A) \subset v(A)$ for each $A \subset X$. We write $u > v$ or $v < u$. Evidently this is an order on the class of all closure operators for X.

Associated with any closure operator u for X there is the **interior operator** int_u from $\mathcal{P}(X)$ into itself defined by $\mathrm{int}_u := \complement \circ u \circ \complement$ where $\complement$ is the complement operator for X. If $A \subset X$ then $\mathrm{int}_u(A)$ will be called u-**interior** of A. A interior operator int for X is characterized by the following property: $\mathrm{int}(X) = X$, $\mathrm{int}(A) \subset A$ for each $A \subset X$, $\mathrm{int}(A \cap B) = \mathrm{int}(A) \cap \mathrm{int}(B)$ for each $A, B \subset X$. Obviously there is a bijective correspondence between the class of all interior operators for X defined by the former three conditions and the class of all closure operators for X; so one may define closure spaces by using of interior operators.

Let (X, u) be a closure space with associated interior operator int. A subset A of X will be called u**-closed** if $u(A) = A$ and u**-open** if its complement is u-closed (or equivalently if $A = \mathrm{int}(A)$). The family of all u-closed subsets of X has the usual properties of stability with respect to finite unions and arbitrary intersections, hence the family of all u-open subsets of X is closed under arbitrary unions and finite intersections. We point out that a closure operator for X is not uniquely determined by the corresponding collection of open subsets of X, namely there may exist different closure operators for X with the same collection of open sets.

A u**-neighborhood** of a subset A of X is any subset U of X such that $A \subset \mathrm{int}(U)$. The filter formed by the family of all u-neighborhoods of A will be indicated with $\mathcal{N}_u(A)$ and it will be called u**-neighborhood filter** of A; for convenience we put $\mathcal{N}_u(x) := \mathcal{N}_u(\{x\})$ for each $x \in X$. The **neighborhood system** of u (or «of (X, u)») is the map from X to the family $\mathcal{F}(X)$ of all filters in X that assigns to every x the filter $\mathcal{N}_u(x)$; we indicate this filter system with the symbol $\mathcal{N}_u$. If v is another closure operator for X such that $u > v$ (or $u < v$) then we have $\mathcal{N}_u > \mathcal{N}_v$ (resp. $\mathcal{N}_u < \mathcal{N}_v$) where this formula means that the filter $\mathcal{N}_u(x)$ is finer than the filter $\mathcal{N}_v(x)$ (resp. «is coarser») for each $x \in X$. If for each $x \in X$ we have a base $\mathcal{B}(x)$ of the filter $\mathcal{N}_u(x)$ then the correspondent map $\mathcal{B}$ is called **base system** for u (or «for (X, u)»); moreover we say that a map $\mathcal{C}: X \to \mathcal{P}(X)$ is a **section** of $\mathcal{N}_u$ (or «of u») if, for each $x \in X$, $\mathcal{C}(x) \in \mathcal{N}_u(x)$. It's very important to observe that u is completely determined by $\mathcal{N}_u$, in fact it's easy to see that a point $x \in X$ belongs to the closure of a subset A of X iff (i.e. «if and only if») each $U \in \mathcal{N}_u(x)$ intersects A; in this way $x \in \mathrm{int}(A)$ iff there exists $U \in \mathcal{N}_u(x)$ contained in A.

It is possible (and sometimes very convenient) to define a closure operator for X by specifying which filters (or filter bases) in X are neighborhood filters (resp. bases of the neighborhood filters) of points. For example if $\mathcal{B}$ is a base system on X (i.e. a map that assigns to each $x \in X$ a base $\mathcal{B}(x)$ of a filter in X) such that $x \in \bigcap_{U \in \mathcal{B}(x)} U$ for each $x \in X$ then there exactly exists one closure operator u for X such that $\mathcal{B}$ is a base system for u (obviously we have $u(A) = \{x \in X \mid U \cap A \neq \emptyset \ \forall U \in \mathcal{B}(x)\}$ for each $A \subset X$). In this manner we have another equivalent description of the closure spaces.

A **topological closure operator** τ for a set X is a closure operator for X satisfying the idempotent condition: $\tau \circ \tau = \tau$; in this case (X, τ) will be called **topological closure space** (or briefly **topological space**) and τ will also be called **topology** for X. It's easy to verify that a closure operator u for X is topological iff the u-closure of each subset of X is u-closed or iff, for each $x \in X$, the family of all u-open neighborhoods of x is a base of the filter $\mathcal{N}_u(x)$. These conditions ensure that the topological closure operators for X coincide with the usual closure operators associated with the classical topologies for X. In particular every topological closure operator is completely determined by the collection of its open sets (or closed sets).

Now we define the topological modification of a closure space.

Let (X, u) be a closure space. The **topological modification** of u is the finest topological closure operator $\mathbf{t}u$ for X coarser than u and the topological modification of (X, u) is $(X, \mathbf{t}u)$. We point out that $\mathbf{t}u$ is the unique topology for X such that the collections of all $\mathbf{t}u$-open subsets (or $\mathbf{t}u$-closed subsets) and of all u-open subsets (resp. u-closed subsets) of X coincide; in particular it follows that the operation $\mathbf{t}$ preserves the order on the class of all closure operators for X that is if v is another closure operator for X such that $v > u$ (or $v < u$) then $\mathbf{t}v > \mathbf{t}u$ (resp. $\mathbf{t}v < \mathbf{t}u$).

There are further ways to characterize the topological modification of u; one of these is the following. Let $\{u^{\alpha}\}_{\alpha}$ be the transfinite sequence of closure operators for X defined as follows: for each $A \subset X$, $u^0(A) := A$, $u^{\alpha+1}(A) := u(u^{\alpha}(A))$ and $u^{\alpha}(A) := \bigcup_{\beta<\alpha} u^{\beta}(A)$ if α is a limit ordinal. u^{α} will be called α^{th} power of u. It's well-known that there exists an ordinal γ depending only on cardinality of X such that $u^{\gamma} = u^{\gamma+1}$; the latter equation is equivalent to $u^{\gamma} = u^{\gamma} \circ u^{\gamma}$ and we have $\mathbf{t}u = u^{\gamma}$.

Let $\mathbf{Ord}_{\gamma}$ be the segment of all nozero ordinal numbers β such that $\beta \leqslant \gamma$.

In the sequel we refer to the ordinal number

$$\mathbf{td}(u) := \min\{\beta \in \mathbf{Ord}_{\gamma} \mid u^{\beta} = u^{\beta+1}\}$$

as the **topological defect** of u (or «of (X, u)»).

We now explain the «neighborhood-version» of the precedent transfinite sequence and so we define the α^{th} power of the neighborhood system $\mathcal{N}_u$ which will must be equal to the neighborhood system of u^{α}.

First we examine the case $\alpha = 2$. Since $\mathrm{int}_{u^2} = \mathrm{int}_u \circ \mathrm{int}_u$ we observe that, for each $x \in X$, it holds

$$\mathcal{N}_{u^2}(x) = \{V \subset X \mid x \in \mathrm{int}_u(\mathrm{int}_u(V))\} = \{V \subset X \mid \mathrm{int}_u(V) \in \mathcal{N}_u(x)\}$$

that is $V \in \mathcal{N}_{u^2}(x)$ iff $\mathrm{int}_u(V)$ is a u-neighborhood of x. Fix $V \in \mathcal{N}_{u^2}(x)$ for some $x \in X$ and put $U := \mathrm{int}_u(V)$. By definitions and by the latter observation we have that $U \subset V$, $U \in \mathcal{N}_u(x)$ and $V \in \mathcal{N}_u(y)$ for each $y \in U$. If we assign to each $y \in U$ the set $V_y := V$ then we obtain the following expression of V: $V = \bigcup_{y \in U} V_y$ (we emphasize that $U \in \mathcal{N}_u(x)$ and $V_y \in \mathcal{N}_u(y)$ for every $y \in U$). Now let $U' \in \mathcal{N}_u(x)$ and, for each $y \in U'$, let $V'_y \in \mathcal{N}_u(y)$. Put $V' := \bigcup_{y \in U'} V'_y$. It's immediate to see that $U' \subset \mathrm{int}_u(V')$ and so $\mathrm{int}_u(V') \in \mathcal{N}_u(x)$ that is $V' \in \mathcal{N}_{u^2}(x)$. Summarizing we have obtained the following equation

$$\mathcal{N}_{u^2}(x) := \{\bigcup_{y\in U} V_y \mid U\in\mathcal{N}_u(x),\ V_y\in\mathcal{N}_u(y)\ \forall y\in U\}\,.$$

The latter equation suggests the correct meaning that we must assign to the following notion in order to define the α^{th} power of $\mathcal{N}_u$.

Let $F\in\mathcal{F}(X)$ and $\mathcal{M}: X\to\mathcal{F}(X)$ a filter system in X (i.e. a map that assigns to each point of X a filter in X), we put

$$F\bullet\mathcal{M} := \left\{\bigcup_{y\in U} V_y \mid U\in F,\ V_y\in\mathcal{M}(y)\ \forall y\in U\right\};$$

it's easy to see that $F\bullet\mathcal{M}\in\mathcal{F}(X)$, $F > F\bullet\mathcal{M}$ and if $x\in\bigcap_{U\in F} U$ then $\mathcal{M}(x) > F\bullet\mathcal{M}$.

Let u be a closure operator for X, we define the filter system $\mathcal{N}_u^\alpha$ for each nozero ordinal α as follows: $\mathcal{N}_u^1 := \mathcal{N}_u$, $\mathcal{N}_u^{\alpha+1}(x) := \mathcal{N}_u(x)\bullet\mathcal{N}_u^\alpha$ for each $x\in X$, $\mathcal{N}_u^\alpha(x) := \bigcap_{\beta<\alpha}\mathcal{N}_u^\beta(x)$ for each $x\in X$ if α is a limit ordinal. One can verify that: if $\alpha\leqslant\beta$ then $\mathcal{N}_u^\alpha > \mathcal{N}_u^\beta$, $\mathcal{N}_u^\alpha = \mathcal{N}_{u^\alpha}$ and so $\mathcal{N}_{\mathbf{t}u} = \mathcal{N}_u^\gamma$ for some ordinal γ depending only on X (for more details see [DG1], section 3).

Now we furnish two pointwise versions of topological defect.

We define the **topological defect function** of u, $\mathbf{td}_u: X\to\mathbf{Ord}_\gamma$, and the **evolution function** of u, $\mathbf{ev}_u: X\to\mathbf{Ord}_\gamma$ (remember that $\mathbf{Ord}_\gamma$ is the segment of all nozero ordinal numbers β such that $\beta\leqslant\gamma$), putting respectively

$$\mathbf{td}_u(x) := \min\{\alpha\in\mathbf{Ord}_\gamma \mid \mathcal{N}_u^\alpha(x) = \mathcal{N}_{\mathbf{t}u}(x)\}\,,$$

$$\mathbf{ev}_u(x) := \min\{\alpha\in\mathbf{Ord}_\gamma \mid \mathcal{N}_u^\alpha(x) = \mathcal{N}_u^{\alpha+1}(x)\}\,.$$

We have $\mathbf{ev}_u\leqslant\mathbf{td}_u$ and $\mathbf{td}(u) = \sup_{x\in X}\{\mathbf{td}_u(x)\}$. Moreover u is topological iff $\mathbf{ev}_u\equiv 1$ (or equivalently $\mathbf{td}_u\equiv 1$ or $\mathbf{td}(u) = 1$).

We now restrict our attention to the sequential case.

Let X be a set and let u be a closure operator for X. The **sequential convergence class** Θ_u of u is the relation consisting of all pairs (S, x) such that S is a sequence in X (i.e. a map from the set $\mathbf{N}$ of all nozero natural numbers to X) converging to x with respect to u (i.e. S is eventually in each u-neighborhood of x). For convenience we write $S\xrightarrow{u}x$ instead of $(S, x)\in\Theta_u$. Clearly if S is the constant sequence $\{x\}_n$ then $S\xrightarrow{u}x$ and if $N\xrightarrow{u}x$ then every subsequences of N converges to x with respect to u. If v is another closure operator for X such that $v > u$ or $v < u$ then $\Theta_v\subset\Theta_u$, $\Theta_v\supset\Theta_u$ respectively (for more details see section 35 of [Če], [Do], [No2], [DG2] and the references of [DG2]). The **sequential modification** of u (or **sequential closure operator associated with** u) is the closure operator $\mathbf{s}u$ for X defined as follows

$$\mathbf{s}u(A) := \{x\in X \mid \exists S\xrightarrow{u}x,\ S(\mathbf{N})\subset A\}$$

for each $A \subset X$ and so we call $(X, \mathbf{s}u)$ sequential modification of (X, u). The sequential modification of u is the finest closure operator v for X such that $\Theta_v = \Theta_u$; in particular $\mathbf{s}u > u$ and if v is a closure operator for X finer (or coarser) than u then $\mathbf{s}v$ is finer (resp. coarser) than $\mathbf{s}u$. The $\mathbf{s}u$-closed (or $\mathbf{s}u$-open) subsets of X are traditionally called **u-sequentially closed** (resp. **u-sequentially open**) subsets of X.

Two very important classes of closure operators are the **Fréchet closure operators** cl defined by $\mathbf{s}(\mathrm{cl}) = \mathrm{cl}$ and the **sequential topologies** v (or **elementary topologies**) which are topological closure operators v with the property $\mathbf{t}(\mathbf{s}v) = v$; we say that (X, cl) is a **Fréchet space** and (X, v) is a **sequential topological space**. Moreover, fixed $x \in X$, we say that a closure operator u for X is **Fréchet at** x iff for every subset A of X such that $x \in u(A)$ there exists a sequence in A which converges to x with respect to u that is $x \in \mathbf{s}u(A)$; in this way u is a Fréchet closure operator iff it's Fréchet at each point of X. Clearly the sequential modification of any closure operator for X is a Fréchet closure operator and any **Fréchet topology** for X (i.e. Fréchet topological closure operator for X) is sequential, while the vice versa is not true (see for example [DG1], section 7 or Theorem 3, section 2 and section 3 of this paper). We point out that a topological closure space is sequential iff every sequentially open subset of X is open (or equivalently «iff every sequentially closed subset of X is closed»). For any sequential topology v for X we define the **sequential order** of v by $\mathbf{td}(\mathbf{s}v)$ (in this manner v is Fréchet iff its sequential order is 1). We recall that it's known that the topological defect of every Fréchet closure operator and in particular the sequential order of any sequential topology is less than or equal to the first uncountable ordinal number ω_1 (see [Ku], [Do], [No1], [No2]).

2. – The theorems.

We start this section with some definitions and lemmas. For short in the sequel of the paper we'll often use the term «space» to indicate «closure space».

DEFINITION 1. – *Let (X, τ) be a topological space. (X, τ) is a T_3^1-space if it's a T_3-space and it satisfies the first axiom of countability (in the usual way); in this case we also say that τ is a T_3^1-topology for X.*

LEMMA 2. – *Let (X, cl) be a Fréchet space and let v be a T_3^1-topology for X such that $\mathrm{cl} > v$. Also let Ω be a v-open subset of X, let x be a point not in Ω and let $\{y_n\}_n$ be a sequence in Ω converging to x with respect to* cl.

There exists a subsequence $\{x_k\}_k$ of $\{y_n\}_n$ and, for each k, there are three v-open subsets $\Omega_k^{(1)}$, $\Omega_k^{(2)}$ and D_k such that

1. *for each* k, $x_k \in \Omega_k^{(1)}$, $v(\Omega_k^{(1)}) \subset \Omega_k^{(2)}$, $\Omega_k^{(1)}$ *and* $\Omega_k^{(2)}$ *are contained in* Ω (*remember that, in our notation, if* $A \subset X$ *then* $v(A)$ *is the closure of* A *in the space* (X, v));

2. $\{D_k\}_k$ *is a non-increasing sequence of subsets of* X *which forms a base for the* v*-neighborhood filter of* x;

3. *for each* k, $\Omega_k^{(2)} \cap D_{k+1} = \emptyset$ *and* $\bigcup_{n \geq k} \Omega_n^{(2)} \subset D_k$ (*in particular* $\Omega_k^{(2)} \cap \Omega_h^{(2)} = \emptyset$ *for each* $k \neq h$);

4. $\left(\bigcup_k v(\Omega_k^{(1)})\right) \cup \{x\}$ *is* v*-closed* (*and hence* cl*-closed*) *and it's the* v*-closure* (*and so the* cl*-closure*) *of* $\bigcup_k v(\Omega_k^{(1)})$.

PROOF. – Since v is T_3^1 and $\mathrm{cl} > v$ we always have that v is a Hausdorff Fréchet topology (remember that T_3 implies T_2 and the first axiom of countability implies that v is Fréchet), cl is Hausdorff (use the usual condition on neighborhoods to define Hausdorff closure operators and hence Hausdorff closure spaces) and $\{y_n\}_n \xrightarrow{v} x$ (remember that $\Theta_{\mathrm{cl}} \subset \Theta_v$).

We observe at once that there exists a subsequence $\{y'_n\}_n$ of $\{y_n\}_n$ with distinct values; in fact the range of the sequence is necessarily infinite otherwise there exists a subsequence of $\{y_n\}_n$ constantly equal to some y_m which is different from x because $y_m \in \Omega$ while $x \notin \Omega$. This constant subsequence v-converges both to x and to y_m but this contradicts the Hausdorff's condition on v. In the sequel of the proof we indicate again $\{y'_n\}_n$ with the symbol $\{y_n\}_n$.

Let $\{V_n\}_n$ be a base for the v-neighborhood system of x such that V_n is v-open and $V_{n+1} \subset V_n$ for each n and put $n_h := \min\{n \mid y_m \in V_h \ \forall m \geq n\}$ for each non-negative integer h. Of course the sequence $\{n_h\}_h$ is non-decreasing and it's unbounded, otherwise there would exist a bound M and so $y_M \in \bigcap_h V_h = \{x\}$ (remember that T_3 implies T_1) which is impossible. In this manner there exists a subsequence $\{x_k\}_k$ of the subsequence $\{y_{n_h}\}_h$ with distinct values such that $x_k \in V_n$ for each $k \geq n$.

We now construct by induction two sequences $\{B_k\}_k$ and $\{C_k\}_k$ of v-open subsets of X in such way that, put $S_k := \{x_n \mid n \geq k\} \cup \{x\}$, it holds: $x_k \in B_k \subset \Omega$, $S_{k+1} \subset C_k \subset V_{k+1}$, $B_k \cap C_k = \emptyset$ and $B_{k+1} \cup C_{k+1} \subset C_k$ for each k.

First we observe that S_k is v-closed for each fixed k; in fact if N is a sequence in S_k that v-converges to some y in X then one of the two following situations must occur. Either the range of N is infinite, in which case N and $\{x_k\}_k$ have in common a subsequence and hence $y = x$, or the range of N is finite and so $y = x_n$ for some $n \geq k$. In any case $y \in S_k$, therefore S_k is v-sequentially closed and so it's v-closed (remember that v is Fréchet and hence it's a sequential topology).

We now start with the induction.

Let $k = 1$. Since v is T_3 and $x_1 \notin S_2$ we can choose two disjoint v-open subsets B_1' and C_1' such that $x_1 \in B_1'$ and $S_2 \subset C_1'$. We put $B_1 := B_1' \cap \Omega$ and $C_1 := C_1' \cap V_2$.

Let $k = n+1$. According to the inductive assumption, there are v-open subsets B_k and C_k of X with $k = 1, 2, \dots, n$ such that $x_k \in B_k \subset \Omega$, $S_{k+1} \subset C_k \subset V_{k+1}$, $B_k \cap C_k = \emptyset$ for each k and $B_{k+1} \cup C_{k+1} \subset C_k$ for each $k < n$. Since $x_{n+1} \notin S_{n+2}$ we can choose again two disjoint v-open subsets B_{n+1}' and C_{n+1}' such that $x_{n+1} \in B_{n+1}'$ and $S_{n+2} \subset C_{n+1}'$. We define $B_{n+1} := B_{n+1}' \cap \Omega \cap C_n$ and $C_{n+1} := C_{n+1}' \cap C_n \cap V_{n+2}$. In this manner the required properties (which are $x_{n+1} \in B_{n+1} \subset \Omega$, $S_{n+2} \subset C_{n+1} \subset V_{n+2}$, $B_{n+1} \cap C_{n+1} = \emptyset$ and $B_{n+1} \cup C_{n+1} \subset C_n$) follow evidently by construction.

Now we put $\Omega_k^{(2)} := B_k \cap V_k$ for every k, $D_1 := X$ and $D_k := C_{k-1}$ if $k > 1$. Using the T_3-separation property of v and the relation $x_k \in \Omega_k^{(2)}$, we can fix, for each k, a v-open neighborhood $\Omega_k^{(1)}$ of x_k such that $v(\Omega_k^{(1)}) \subset \Omega_k^{(2)}$ (observe that $\Omega_k^{(1)}$ may be chosen arbitrarily small about x_k).

Conditions 1, 2 and 3 in the terms of this lemma hold by construction of $\{B_k\}_k$ and $\{C_k\}_k$.

Remain to be shown condition 4. Let $A := \bigcup_k v(\Omega_k^{(1)})$, $x' \notin A \cup \{x\}$ and let U be a v-neighborhood of x such that $x' \notin U$. Since v is T_3 we can suppose $U := v(V_n)$ for some n; it holds that

$$\bigcup_{k \geqslant n} v(\Omega_k^{(1)}) \subset \bigcup_{k \geqslant n} \Omega_k^{(2)} \subset \bigcup_{k \geqslant n} V_k = V_n \subset U\,,$$

$A \cup U = U \cup \bigcup_{k < n} v(\Omega_k^{(1)})$ and so $A \cup U$ is v-closed. Since $v(A) \subset v(A \cup U) = A \cup U$ and $x' \notin A \cup U$ we have $x' \notin v(A)$; therefore, for each $x' \notin A \cup \{x\}$, $x' \notin v(A)$ and so $v(A) \subset A \cup \{x\}$. On the other hand $\{x_k\}_k$ is a sequence in A which converges to x with respect to v and hence $v(A) = A \cup \{x\}$. Furthermore we have $A \subset \mathrm{cl}(A) \subset v(A) = A \cup \{x\}$ and $\{x_k\}_k \xrightarrow{\mathrm{cl}} x$, therefore $\mathrm{cl}(A) = A \cup \{x\}$. This completes the proof. ∎

Before presenting next lemma we recall some general notions.

Let (X, u) be a space and let A be a subset of X. We define the **u-derived** of A as the subset $\mathrm{Der}_u(A)$ of X formed by the points x such that $x \in u(A \backslash \{x\})$. We now fix another subset B of X and we observe that the map u_B from $\mathcal{P}(B)$ into itself that assigns to each $A \subset B$ the set $u(A) \cap B$ is a closure operator for B. u_B is called **relativization** of u to B and (B, u_B) is a **subspace** of (X, u). By a straightforward transfinite induction argument (see [Gr], Lemma 1 and Lemma 2) one can prove the following

LEMMA 3. – *Let (X, u) be a space, let α be an ordinal number and let A be a subset of X.*

It holds

1. $u^{\alpha+1}(A) = A \cup \mathrm{Der}_u(u^{\alpha}(A))$;

2. *if* B *is a* u*-open subset of* X *then* $B \cap u^{\alpha}(A) = (u_B)^{\alpha}(B \cap A)$ *and in particular* $B \cap u^{\alpha}(A) \subset u^{\alpha}(B \cap A)$.

DEFINITION 4. – *Let* (X, u) *be a space and let* τ *be a topology for* X. *A section* $\mathcal{C}$ *of* $\mathcal{N}_u$ *is* τ-locally closed (*or* τ-loc. closed *for short*) *if, for each* $x \in X$, $\mathcal{C}(x)$ *is locally closed with respect to* τ (*in the usual way*).

DEFINITION 5. – *Let* (X, u) *be a space and let* A *be a subset of* X. *We put*

$$\partial_u(A) := A \cap u(\mathsf{C}A)$$

where C *is the complement operator for* X *and we call* $\partial_u(A)$ proper frontier *of* A *with respect to* u.

Elaborating the transfinite induction idea of G. H. Greco in [Gr] and using Lemma 2 and Lemma 3 we obtain the main theorem of this paper which we state and we prove below.

THEOREM 6. – *Let* (X, cl) *be a Fréchet space and let* v *be a* T_3^1*-topology for* X *such that* $\mathrm{cl} > v$. *Assume that there exists a section* $\mathcal{C}$ v*-loc. closed of* $\mathcal{N}_{\mathrm{cl}}$ *such that, for each fixed* $x \in X$, *putting* $\mathcal{C}_x := \mathcal{C}(x)$ *it holds*

1. $x \in \mathrm{cl}(\partial_{\mathrm{cl}}(\mathcal{C}_x))$;

2. *there exists* $V_x \in \mathcal{N}_{\mathrm{cl}}(x)$ *such that, for all* $y \in V_x \cap \partial_{\mathrm{cl}}(\mathcal{C}_x)$, *we have*

$$y \in \mathrm{cl}(\partial_{\mathrm{cl}}(\mathcal{C}_y) \setminus \mathcal{C}_x).$$

Under these conditions the evolution function of cl *is constantly equal to* ω_1.

In particular $\mathbf{td}_{\mathrm{cl}} \equiv \omega_1$, *and so the topological defect of* cl *is* ω_1. *Furthermore it holds that the sequential order of* $\mathbf{t}(\mathrm{cl})$ *is* ω_1 *and hence* $\mathbf{t}(\mathrm{cl})$ *is a sequential topology for* X *which is not Fréchet at every point of* X.

PROOF. – In order to prove that $\mathbf{ev}_{\mathrm{cl}}(x) = \omega_1$ for each $x \in X$, we must show that, for each x and for each ordinal number $\alpha < \omega_1$, there exists a subset U of X in such way that $U \in \mathcal{N}_{\mathrm{cl}}^{\alpha}(x) \setminus \mathcal{N}_{\mathrm{cl}}^{\alpha+1}(x)$ (because, as we have just recalled, for all Fréchet closure operators u, it holds $\mathbf{ev}_u \leqslant \omega_1$).

In this way it suffices to prove that, *for each* $x \in X$ *and for each ordinal number* $\alpha < \omega_1$, *it holds the following property* $P(x, \alpha)$: *for each* v*-open subset* Ω *of* X *with* $x \in \mathrm{cl}(\partial_{\mathrm{cl}}(\mathcal{C}_x) \cap \Omega) \setminus \Omega$, *there exists a subset* A *of* Ω *such that* $\mathrm{cl}^{\alpha}(A) \subset \Omega$ *and* $\mathrm{cl}^{\beta}(A) \setminus \Omega = \{x\}$ *for each fixed ordinal number* $\beta > \alpha$.

Suppose in fact that $P(x, \alpha)$ holds and let $\Omega = \mathsf{C}\{x\}$ (remember that v is T_1 and so Ω is v-open). By hypothesis 1 we have $x \in \mathrm{cl}(\partial_{\mathrm{cl}}(\mathcal{C}_x))$ and by relation

$\mathcal{C}_x \in \mathcal{N}_{\mathrm{cl}}(x)$ it holds $x \notin \mathrm{cl}(\complement\, \mathcal{C}_x)$ and hence $x \notin \partial_{\mathrm{cl}}(\mathcal{C}_x)$. From these facts it follows that $\mathrm{cl}(\partial_{\mathrm{cl}}(\mathcal{C}_x) \cap \Omega) \setminus \Omega = \mathrm{cl}(\partial_{\mathrm{cl}}(\mathcal{C}_x)) \cap \{x\} = \{x\}$ and so by $P(x, \alpha)$ there exists $A \subset \Omega$ such that $\mathrm{cl}^{\alpha}(A) \not\ni x$ and $\mathrm{cl}^{\alpha+1}(A) \ni x$ or equivalently $\complement A \in \mathcal{N}_{\mathrm{cl}^{\alpha}}(x) \setminus \mathcal{N}_{\mathrm{cl}^{\alpha+1}}(x) = \mathcal{N}^{\alpha}_{\mathrm{cl}}(x) \setminus \mathcal{N}^{\alpha+1}_{\mathrm{cl}}(x)$ which is precisely what we need.

We now proceed by transfinite induction to prove $P(x, \alpha)$.

Let $\alpha = 0$ and let $x \in X$. Fix a v-open subset Ω of X such that $x \in \mathrm{cl}(\partial_{\mathrm{cl}}(\mathcal{C}_x) \cap \Omega) \setminus \Omega$. Since cl is Fréchet there exists a sequence $\{x_n\}_n$ in $\partial_{\mathrm{cl}}(\mathcal{C}_x) \cap \Omega$ which cl-converges to x. Put $A := \bigcup_n \{x_n\}$. By the same argument used in the Proof of Lemma 2, we see at once that $\mathrm{cl}(A) = A \cup \{x\}$ and $\mathrm{cl}(A)$ is cl-closed. Since $A \subset \Omega$ it holds $P(x, 0)$.

Let α be an ordinal number with $0 < \alpha < \omega_1$ and let $x \in X$. Since ω_1 is the first uncountable ordinal and $\alpha < \omega_1$ there exists a non-decreasing sequence $\{\beta_n\}_n$ of ordinal numbers strictly less than α such that $\sup_n \beta_n = \sup\{\beta \mid \beta < \alpha\}$. Now fix a v-open subset Ω of X such that $x \in \mathrm{cl}(\partial_{\mathrm{cl}}(\mathcal{C}_x) \cap \Omega) \setminus \Omega$ and let $\{y_n\}_n$ be a sequence in $\partial_{\mathrm{cl}}(\mathcal{C}_x) \cap \Omega$ cl-converging to x. By Lemma 2 we can choose a subsequence $\{x_k\}_k$ of $\{y_n\}_n$ and, for each k, three v-open sets $\Omega^{(1)}_k$, $\Omega^{(2)}_k$ and D_k with the prescribed properties and such that $\{x_k\}_k$ lies in the cl-neighborhood V_x of x mentioned in hypothesis 2.

For each k we put

$$H_k := \Omega^{(1)}_k \setminus \mathcal{C}_x$$

in such way that H_k is v-open. This choice of $\{\Omega^{(1)}_k\}_k$ is possible because $\Omega^{(1)}_k$ may be taken arbitrarily small about x_k (see the Proof of Lemma 2) and $\mathcal{C}$ is v-loc. closed. By hypothesis 2, for each k, $x_k \in \mathrm{cl}(\partial_{\mathrm{cl}}(\mathcal{C}_{x_k}) \setminus \mathcal{C}_x)$ and hence, being $\Omega^{(1)}_k$ a cl-neighborhood of x_k (for it's a cl-open set containing x_k), x_k lies in the cl-closure of $(\partial_{\mathrm{cl}}(\mathcal{C}_{x_k}) \setminus \mathcal{C}_x) \cap \Omega^{(1)}_k$. On the other hand $(\partial_{\mathrm{cl}}(\mathcal{C}_{x_k}) \setminus \mathcal{C}_x) \cap \Omega^{(1)}_k = \partial_{\mathrm{cl}}(\mathcal{C}_{x_k}) \cap H_k$ and so

$$x_k \in \mathrm{cl}(\partial_{\mathrm{cl}}(\mathcal{C}_{x_k}) \cap H_k) \setminus H_k.$$

Now by inductive assumption we may apply $P(x_k, \beta_k)$ with H_k obtaining, for each k, a subset A_k of H_k such that $\mathrm{cl}^{\beta_k}(A_k) \subset H_k$ and $\mathrm{cl}^{\beta}(A_k) \setminus H_k = \{x_k\}$ for each fixed $\beta > \beta_k$.

We shall prove that the set $A := \bigcup_k A_k$ satisfies $P(x, \alpha)$ with the fixed Ω.

Clearly $x \in \mathrm{cl}^{\alpha+1}(A)$ in fact $x_k \in \mathrm{cl}^{\beta_k+1}(A_k) \subset \mathrm{cl}^{\alpha}(A)$ for each k and $\{x_k\}_k \xrightarrow{\mathrm{cl}} x$. By Lemma 2 it holds

$$\text{(1)} \qquad \mathrm{cl}^{\beta}(A) \subset v^{\beta}(A) = v(A) \subset v\Big(\bigcup_k \Omega^{(1)}_k\Big) \subset v\Big(\bigcup_k v(\Omega^{(1)}_k)\Big) =$$

$$\Big(\bigcup_k v(\Omega^{(1)}_k)\Big) \cup \{x\} \subset \Big(\bigcup_k \Omega^{(2)}_k\Big) \cup \{x\} \subset \Omega \cup \{x\}$$

for each fixed nozero ordinal number β; in particular $\mathrm{cl}^{\beta}(A)\setminus\Omega = \{x\}$ for each fixed $\beta > \alpha$ and hence we have the second part of $P(x, \alpha)$ with Ω.

Remain to be proved that $x \notin \mathrm{cl}^{\alpha}(A)$ in fact by (1) we have $\mathrm{cl}^{\alpha}(A) \subset \Omega \cup \{x\}$ and so we would have $\mathrm{cl}^{\alpha}(A) \subset \Omega$. We observe that $\mathrm{cl}^{\alpha}(A) = \bigcup_k \mathrm{cl}^{\beta_k+1}(A)$ and so, in order to prove that $x \notin \mathrm{cl}^{\alpha}(A)$, it's sufficient to show that $x \notin \mathrm{cl}^{\beta_k+1}(A)$ for each k. On the other hand by the first result in Lemma 3 follows that $\mathrm{cl}^{\beta_k+1}(A) = A \cup \mathrm{Der}_{\mathrm{cl}}(\mathrm{cl}^{\beta_k}(A))$ and hence, being $x \notin A$, we may merely show that $x \notin \mathrm{Der}_{\mathrm{cl}}(\mathrm{cl}^{\beta_k}(A))$, namely we may prove that, for each fixed k, there exists $U \in \mathcal{N}_{\mathrm{cl}}(x)$ (depending on k) such that $(\mathrm{cl}^{\beta_k}(A)\setminus\{x\}) \cap U = \emptyset$.

Fix some k and put $U := \mathcal{C}_x \cap D_k$. By the second result in Lemma 3 we have

$$(\mathrm{cl}^{\beta_k}(A)\setminus\{x\}) \cap U = (\mathrm{cl}^{\beta_k}(A \cap D_k)\setminus\{x\}) \cap U \tag{2}$$

in fact

$$(\mathrm{cl}^{\beta_k}(A)\setminus\{x\}) \cap U = \mathrm{cl}^{\beta_k}(A) \cap D_k \cap \complement\{x\} \cap U \subset (\mathrm{cl}^{\beta_k}(A \cap D_k)\setminus\{x\}) \cap U\,.$$

Using the equation (2) and the inclusion $\mathrm{cl}^{\beta_k}(A \cap D_k) \subset \Big(\bigcup_{n \geq k} \Omega_n^{(2)}\Big) \cup \{x\}$ (which one may easily obtain following the same argument used to prove (1)), we obtain

$$(\mathrm{cl}^{\beta_k}(A)\setminus\{x\}) \cap U \subset \bigcup_{n \geq k} (\Omega_n^{(2)} \cap U)$$

in fact

$$(\mathrm{cl}^{\beta_k}(A)\setminus\{x\}) \cap U = \mathrm{cl}^{\beta_k}(A \cap D_k) \cap (U\setminus\{x\}) \subset \Big(\Big(\bigcup_{n\geq k} \Omega_n^{(2)}\Big) \cup \{x\}\Big) \cap (U\setminus\{x\}) =$$

$$= \bigcup_{n \geq k} (\Omega_n^{(2)} \cap (U\setminus\{x\})) = \bigcup_{n \geq k} (\Omega_n^{(2)} \cap U)\,.$$

Now we recall that, for each n, $\mathrm{cl}^{\beta_n}(A_n) \subset H_n$ and $H_n \cap \mathcal{C}_x = \emptyset$; moreover we observe that from Lemma 2 and the definition of A follows the equation $A \cap \Omega_n^{(2)} = A_n$ for each n. Finally we have

$$(\mathrm{cl}^{\beta_k}(A)\setminus\{x\}) \cap U = \emptyset$$

in fact

$$(\mathrm{cl}^{\beta_k}(A)\setminus\{x\}) \cap U = (\mathrm{cl}^{\beta_k}(A) \cap (U\setminus\{x\})) \cap \bigcup_{n \geq k} (\Omega_n^{(2)} \cap U) =$$

$$= \bigcup_{n \geq k} (\Omega_n^{(2)} \cap U \cap \mathrm{cl}^{\beta_k}(A)) \subset$$

$$\subset \bigcup_{n \geq k} (\mathcal{C}_x \cap \mathrm{cl}^{\beta_k}(A \cap \Omega_n^{(2)})) =$$

$$= \bigcup_{n \geq k} (\mathcal{C}_x \cap \mathrm{cl}^{\beta_k}(A_n)) \subset$$

$$\subset \bigcup_{n \geq k} (\mathcal{C}_x \cap \mathrm{cl}^{\beta_n}(A_n)) \subset$$

$$\subset \bigcup_{n \geq k} (\mathcal{C}_x \cap H_n) = \emptyset \,.$$

In conclusion we prove the last assertion which we have given in the thesis of this theorem.

We have just proved that $\mathbf{ev}_{\mathrm{cl}} = \mathbf{td}_{\mathrm{cl}} \equiv \omega_1$, now we show that this result implies that the sequential order of $\mathbf{t}(\mathrm{cl})$ is ω_1 completing the proof.

We start by making a general observation.

Let u be a Fréchet closure operator for X with Hausdorff topological modification.

It holds that $\mathbf{s}(\mathbf{t}u) = u$ and in particular $\mathbf{t}u$ is sequential.

To prove this assertion we observe that $u = \mathbf{s}u$ (i.e. u is Fréchet) implies that $\mathbf{t}u = \mathbf{ts}u > \mathbf{ts}(\mathbf{t}u) > \mathbf{t}u$ and hence we have $\mathbf{t}u = \mathbf{ts}(\mathbf{t}u)$ that is $\mathbf{t}u$ is sequential. Putting $v := \mathbf{s}(\mathbf{t}u)$ we obtain $\mathbf{t}v = \mathbf{ts}(\mathbf{t}u) = \mathbf{t}u$. Since by hypothesis $\mathbf{t}u$ is Hausdorff we have that u and v are Hausdorff Fréchet closure operators for X with the same topological modification $\mathbf{t}u$. On the other hand every Hausdorff Fréchet closure operator u for X is «topologically greatest» that is if v' is another closure operator for X such that $\mathbf{t}u = \mathbf{t}v'$ then $u > v'$ (see [DG1] section 6, Theorem 6.1 and especially Theorem 6.4). In particular we have $u > v$ and $v > u$ that is $u = v$ and so $u = \mathbf{s}(\mathbf{t}u)$ (for another approch see [Do], section 14).

The preceding comment applies to our case in fact cl is a Fréchet closure operator finer than v and hence $\mathbf{t}(\mathrm{cl}) > v$, on the other hand v is a T_2-topology and so $\mathbf{t}(\mathrm{cl})$ is a T_2-topology too. In this way we have $\mathbf{st}(\mathrm{cl}) = \mathrm{cl}$.

Follows that the sequential order of $\mathbf{t}(\mathrm{cl})$ is $\mathbf{td}(\mathrm{cl})$ that is ω_1 (compare the latter result with Theorem 7.6 in [DG2]). In conclusion we may apply Theorem 7.1 of [DG1] which tells us that $\mathbf{t}(\mathrm{cl})$ is not Fréchet at every x of X. This completes the proof. ■

We proceed with some observations about Theorem 6.

Let X, cl and v be as in Theorem 6. We remind that a subset A of X is v-loc. closed iff there exists a v-closed subset F and a v-open subset G of X such that $A = F \cap G$. Follows that a section $\mathcal{C}$ of $\mathcal{N}_{\mathrm{cl}}$ is v-loc. closed iff there exists a section $\mathcal{B}$ of $\mathcal{N}_{\mathrm{cl}}$ and a section $\mathcal{H}$ of $\mathcal{N}_v$ with values in the set of all v-open subsets of X in such way that

$$\mathcal{C}(x) = v(\mathcal{B}(x)) \cap \mathcal{H}(x)$$

for each $x \in X$. In this manner we have an explicit way to construct v-loc.

closed section of $\mathcal{N}_{\mathrm{cl}}$ from arbitrary sections of $\mathcal{N}_{\mathrm{cl}}$. About condition 2 of the preceding theorem we may say that, in one sense, «v furnishes the necessary space to cl so that it can evolve many times (in fact ω_1 times)». Condition 1 is a geometric version of the following one: $\mathbf{ev}_{\mathrm{cl}} \geqslant 2$ (which is obviously necessary to obtain $\mathbf{ev}_{\mathrm{cl}} \equiv \omega_1$), more precisely we have

LEMMA 7. – *Let (X, u) be a space and let x be a point of X. It holds: $\mathbf{ev}_u(x) \geqslant 2$ iff there exists $U \in \mathcal{N}_u(x)$ such that $x \in u(\partial_u(U))$.*

PROOF. – First assume that there exists $U \in \mathcal{N}_u(x) \setminus \mathcal{N}_u^2(x)$ (i.e. $\mathbf{ev}_u(x) \geqslant 2$). By definition of $\mathcal{N}_u^2(x)$ we have that, for each fixed $V \in \mathcal{N}_u(x)$ contained in U and for each $V_y \in \mathcal{N}_u(y)$ with $y \in V$, $\bigcup_{y \in V} V_y \not\subset U$. In particular there exists $y \in V$ such that, for all $V_y \in \mathcal{N}_u(y)$, $V_y \not\subset U$ that is $y \in u(\complement U)$. In this manner for every u-neighborhood V of x contained in U we have $V \cap \partial_u(U) = V \cap u(\complement U) \neq \emptyset$ and so $x \in u(\partial_u(U))$.

Now suppose $x \in u(\partial_u(U))$ for some u-neighborhood U of x. We have $x \in u(\partial_u(U)) \subset u^2(\complement U)$ therefore $U \notin \mathcal{N}_{u^2}(x) = \mathcal{N}_u^2(x)$. This completes the proof. ■

We now present a corollary of Theorem 6 which we'll use in the third section.

COROLLARY 8. – *Let X be a set, let v be a T_3^1-topology for X and let* cl *and* cl′ *be two Fréchet closure operators for X such that $\mathrm{cl} > \mathrm{cl}' > v$. Suppose that there exists a v-loc. closed section $\mathcal{C}$ of $\mathcal{N}_{\mathrm{cl}'}$ such that, for each $x \in X$, putting $\mathcal{C}_x := \mathcal{C}(x)$ it holds*

1. $x \in \mathrm{cl}(\partial_{\mathrm{cl}}(\mathcal{C}_x))$;
2. *there exists $V_x \in \mathcal{N}_{\mathrm{cl}'}(x)$ such that if $y \in V_x \cap \partial_{\mathrm{cl}'}(\mathcal{C}_x)$ then*

$$y \in \mathrm{cl}(\partial_{\mathrm{cl}}(\mathcal{C}_y) \setminus \mathcal{C}_x).$$

Each Fréchet closure operator cl* *for X such that $\mathrm{cl} > \mathrm{cl}^* > \mathrm{cl}'$ fulfils the conditions of Theorem* 6 *by section $\mathcal{C}$ and hence $\mathbf{ev}_{\mathrm{cl}^*} \equiv \omega_1$.*

PROOF. – We observe at once that if u and v are two closure operators for X such that $u > v$ then $\partial_u(A) \subset \partial_v(A)$ for each $A \subset X$ and we remember that in this case $\mathcal{N}_v(x) \subset \mathcal{N}_u(x)$ for each $x \in X$. Follows that $\mathcal{C}$ is also a section for $\mathcal{N}_{\mathrm{cl}^*}$ and, for each $x \in X$, $V_x \in \mathcal{N}_{\mathrm{cl}^*}(x)$ and $\partial_{\mathrm{cl}}(\mathcal{C}_x) \subset \partial_{\mathrm{cl}^*}(\mathcal{C}_x)$, therefore

$$x \in \mathrm{cl}(\partial_{\mathrm{cl}}(\mathcal{C}_x)) \subset \mathrm{cl}^*(\partial_{\mathrm{cl}^*}(\mathcal{C}_x))$$

and for every $y \in V_x \cap \partial_{\mathrm{cl}^*}(\mathcal{C}_x) \subset V_x \cap \partial_{\mathrm{cl}'}(\mathcal{C}_x)$

$$y \in \mathrm{cl}(\partial_{\mathrm{cl}}(\mathcal{C}_y) \setminus \mathcal{C}_x) \subset \mathrm{cl}^*(\partial_{\mathrm{cl}^*}(\mathcal{C}_y) \setminus \mathcal{C}_x).$$

In this way cl* fulfils the mentioned conditions of Theorem 6 and so the proof is complete. ■

Let X be a set and let u be a closure operator for X. We say that (X, u) is a **perfect closure space** if there is not u-isolated points in X (use the usual condition on u-neighborhoods to define u-isolated points in X) or equivalently if $x \in \mathrm{Der}_u(X)$ for every $x \in X$.

A map $\mathcal{D}$ from X to $\mathcal{P}(X)$ will be called **set distribution** in X if $x \in \mathcal{D}(x)$ for each $x \in X$ (remember that $\mathcal{P}(X)$ is the family of all subsets of X); moreover if v is a topology for X then we say that $\mathcal{D}$ is a set distribution of v-loc. closed subsets of X if $\mathcal{D}$ is a set distribution in X such that $\mathcal{D}(x)$ is a v-loc. closed subset of X for each $x \in X$.

We now present a very simple and effective version of Theorem 6.

THEOREM 9. – *Let* (X, cl) *be a perfect Fréchet space and let* v *be a* T_3^1*-topology for* X *such that* $\mathrm{cl} > v$. *Suppose that there exists a set distribution* $\mathcal{D}$ *of* v*-loc. closed subsets of* X *such that*

1. $\mathrm{int}_{\mathrm{cl}}(\mathcal{D}(x)) = \{x\}$ *for each* $x \in X$.

Then $\mathbf{ev}_{\mathrm{cl}} = \mathbf{td}_{\mathrm{cl}} \equiv \omega_1$ *and so both the topological defect of* cl *and the sequential order of* $\mathbf{t}(\mathrm{cl})$ *are* ω_1.

PROOF. – As we have already seen in the Proof of Theorem 6 it suffices to prove by transfinite induction that, *for each* $x \in X$ *and for each ordinal number* $\alpha < \omega_1$, *it holds the following property* $R(x, \alpha)$: *for each* v*-open subset* Ω *of* X *with* $x \in \mathrm{cl}(\Omega)\setminus\Omega$, *there exists* $A \subset \Omega$ *such that* $\mathrm{cl}^\alpha(A) \subset \Omega$ *and* $\mathrm{cl}^\beta(A)\setminus\Omega = \{x\}$ *for each fixed ordinal number* $\beta > \alpha$.

In fact suppose that $R(x, \alpha)$ holds for every $\alpha < \omega_1$ and for every $x \in X$. Fix $x \in X$ and let $\Omega := \complement\{x\}$. Since (X, cl) is perfect we have that $x \in \mathrm{cl}(\Omega)\setminus\Omega$ and so we can apply $R(x, \alpha)$ with such Ω obtaining $\mathbf{ev}_{\mathrm{cl}}(x) \geqslant \alpha$ for each $\alpha < \omega_1$ and hence the thesis.

We start with the transfinite induction.

Let $\alpha = 0$ and $x \in X$. One can easily obtain the demonstration following the corresponding part of the Proof of Theorem 6.

Let α be an ordinal number strictly less than ω_1 and let $x \in X$. As in the Proof of the quoted theorem we fix: a non-decreasing sequence $\{\beta_n\}_n$ of ordinal numbers strictly less than α such that $\sup_n \beta_n = \sup\{\beta \mid \beta < \alpha\}$ and a v-open subset Ω of X such that $x \in \mathrm{cl}(\Omega)\setminus\Omega$. We observe that hypothesis 1 of this theorem implies that $\mathcal{D}(x) \in \mathcal{N}_{\mathrm{cl}}(x)$. In this manner there exists a sequence $\{y_n\}_n$ in $\Omega \cap \mathcal{D}(x)$ cl-converging to x. By Lemma 2 we can choose a subsequence $\{x_k\}_k$ of $\{y_n\}_n$ and, for each k, three v-open sets $\Omega_k^{(1)}$, $\Omega_k^{(2)}$ and D_k with the prescribed properties and such that, putting $H_k := \Omega_k^{(1)}\setminus\mathcal{D}(x)$ for each k, H_k is υ-open. On the other hand, for each k, we have $x_k \in \mathrm{cl}(H_k)\backslash H_k$ in fact

$$\mathrm{cl}(H_k) = \mathrm{cl}(\Omega_k^{(1)} \cap \complement\mathcal{D}(x)) \supset \Omega_k^{(1)} \cap \mathrm{cl}(\complement\mathcal{D}(x)) = \Omega_k^{(1)} \cap \complement\{x\} = \Omega_k^{(1)} \ni x_k$$

(observe that in the second step of the preceding expression we use Lemma 3 and in the third step we use hypothesis 1 of this theorem). Now we may apply $R(x_k, \beta_k)$ with H_k and we can conclude the proof as in the Proof of Theorem 6. ∎

We underline that the set distribution $\mathcal{D}$ in X mentioned in Theorem 9 is a section of $\mathcal{N}_{\mathrm{cl}}$ in fact by condition 1 we have $x \in \mathrm{int}_{\mathrm{cl}}(\mathcal{D}(x))$ for each $x \in X$.

The next result is a corollary of Theorem 9 which corresponds to Corollary 8 of Theorem 6.

COROLLARY 10. – *Let X be a set, let υ be a T_3^1-topology for X and let* cl *and* cl′ *be two Fréchet closure operators for X such that* $\mathrm{cl} > \mathrm{cl}' > \upsilon$ *and* (X, cl) *is perfect. Suppose that there exists a υ-loc. closed section $\mathcal{D}$ of $\mathcal{N}_{\mathrm{cl}'}$ such that, for each $x \in X$, it holds*

1. $\mathrm{int}_{\mathrm{cl}}(\mathcal{D}(x)) = \{x\}$.

Each Fréchet closure operator cl* *for X such that* $\mathrm{cl} > \mathrm{cl}^* > \mathrm{cl}'$ *fulfils the conditions of Theorem 9 by the set distribution $\mathcal{D}$ and hence* $\mathbf{ev}_{\mathrm{cl}^*} \equiv \omega_1$.

PROOF. – Since $\mathrm{cl}^* > \mathrm{cl}'$ the set distribution $\mathcal{D}$ is a υ-loc. closed section of $\mathcal{N}_{\mathrm{cl}^*}$. Since $\mathrm{cl} > \mathrm{cl}^*$ we have that (X, cl^*) is perfect and it holds

$$x \in \mathrm{int}_{\mathrm{cl}^*}(\mathcal{D}(x)) \subset \mathrm{int}_{\mathrm{cl}}(\mathcal{D}(x)) = \{x\}$$

and so $\mathrm{int}_{\mathrm{cl}^*}(\mathcal{D}(x)) = \{x\}$ for each $x \in X$. ∎

3. – Some applications.

3.1. *The space S_ω of Arhangel'skiĭ and Franklin.*

In 1968 Arhangel'skiĭ and Franklin furnished (see [AF]) the first example of countable homogeneous Hausdorff sequential topological space, called S_ω, with sequential order equal to ω_1. For completeness we recall the definition of S_ω (following strictly pp. 314, 315 and 316 of [AF]) and we explain how to obtain the result of Arhangel'skiĭ and Franklin from Theorem 9. We underline that all the following topological notions are usual.

Let $S := \{0\} \cup \left\{ \frac{1}{n} \,\middle|\, n \in \mathbf{N} \right\} \subset \mathbf{R}$ with the usual relative topology. For each $n \in \mathbf{N}$, let $(X_n)_{x_n}$ be a T_1-topological space with a base point $x_n \in X_n$. If X is the disjoint topological sum of the spaces X_n then $A := \{x_n \mid n \in \mathbf{N}\}$ is a closed subspace of X and the map $f: A \to S$ such that $f(x_n) := \frac{1}{n}$ is continuous. One defines the sequential sum as the adjunction space $X \cup_f S$.

Now we construct the spaces S_n and we define the level $l_n(x)$ for points x of S_n. Let $S_0 := \{0\} \subset S$ and, having just defined S_{n-1} with base point $0 \in S$, we put S_n equal to the sequential sum of countably many copies of $(S_{n-1})_0$, choosing $0 \in S$ again as a base point. Let $l_0(0) := 0$ and, having just defined $l_{n-1}(x)$ for each $x \in S_{n-1}$, we put $l_n(0) := 0$ and if $x \in S_n \setminus \{0\}$ then x lies in one copy of S_{n-1} and so we may put $l_n(x) := l_{n-1}(x) + 1$. Now, for all points x of level n in S_n (which are denumerable in number), take a copy $(S_x)_{0_x}$ of $(S)_0$ and let Y be the disjoint topological sum of the spaces S_x. If $B := \{0_x \in Y \mid x \in S_n,\ l_n(x) = n\}$ and $g: B \to S_n$ is defined by $g(0_x) := x$ then the adjunction space $Y \cup_g S_n$ is homeomorphic to S_{n+1}. In this manner, for each n, we have a natural embedding $\phi_n^{n+1}: S_n \to S_{n+1}$; furthermore for each pair of integers m and n such that $m < n$ we may define ϕ_m^n putting $\phi_m^n := \phi_{n-1}^n \circ \ldots \circ \phi_m^{m+1}$ and hence we have an inductive system of spaces S_n and maps ϕ_m^n. Finally (S_ω, τ) is defined as the inductive limit of this system and so it's sequential (see [Fr1], Corollary 1.7).

We now present two equivalent descriptions of the sequential modification $(S_\omega, \mathbf{s}\tau)$ of (S_ω, τ) (N.B. The first of these descriptions and the idea to obtain the second have been essentially extracted from [AF] page 318).

First of all we observe that we may define the level $l(x)$ of a point x in S_ω in fact every S_n is canonically embedded in S_ω and we may unambiguously put $l(x) := l_n(x)$ where n is a fixed integer such that $x \in S_n$. It's easy to see that the subset of S_ω formed by the points of level n corresponds bijectively with the set of all finite sequences of nozero integers of lengh n. In this manner there is a correspondence between S_ω and all finite sequences of nozero integers included the empty sequence. A sequence of n nozero integers $a_1, \ldots, a_n$ will be indicate with the symbol $[a_1, \ldots, a_n]$ and the empty sequence with the symbol $[\emptyset]$. The preceding correspondence may be choosen in such way that the convergence of the sequences in S_ω can be described as follows. Let $\mathbf{a}_k = [a_1^{(k)}, \ldots, a_{n_k-1}^{(k)}, a_{n_k}^{(k)}]$ for each k. $\{\mathbf{a}_k\}_k$ converges if it's eventually constant or if it's not eventually constant but, eventually in k, n_k is constantly equal to some n, each of the first $n-1$ coordinates is constant (suppose $a_i^{(k)} = a_i$ with $i = 1, \ldots, n-1$) and $a_{n_k}^{(k)} \to +\infty$ if $k \to +\infty$. If $\{\mathbf{a}_k\}_k$ is eventually equal to some $\mathbf{a}$ then $\{\mathbf{a}_k\}_k \to \mathbf{a}$, while if $\{\mathbf{a}_k\}_k$ converges but it's not eventually constant then $\{\mathbf{a}_k\}_k \to [a_1, \ldots, a_{n-1}]$ (use the preceding assumption). This is the first description we need.

Now we see the second description.

Let $(S_\omega, \mathbf{s}\tau)$ be the above space understood as in the preceding representation, let $\boldsymbol{l}^2$ be the Hilbert space of the real-valued sequences $\mathbf{x} =$

$(x_1, \ldots, x_n, \ldots)$ such that $\|\mathbf{x}\|_2 := (\sum_n x_n^2)^{1/2} < +\infty$ and let $\{\mathbf{e}_n\}_n$ be the standard orthonormal basis for $\boldsymbol{l}^2$.

We define the map $\varphi : S_\omega \to \boldsymbol{l}^2$ putting

$$\varphi([\emptyset]) := \underline{0}$$

and

$$\varphi([a_1, \ldots, a_n]) := \sum_{k=1}^{n} \left(\frac{1}{a_k}\right) \mathbf{e}_k$$

for each $[a_1, \ldots, a_n] \in S_\omega \setminus \{[\emptyset]\}$. Evidently φ is one-to-one and it's also sequentially continuous in fact if $\{\mathbf{a}_k\}_k \to \mathbf{a}$ in S_ω (we may suppose $\mathbf{a} = [a_1, \ldots, a_{n-1}]$, $\mathbf{a}_k = [a_1, \ldots, a_{n-1}, a_n^{(k)}]$ for each k and $\{a_n^{(k)}\}_k \to +\infty$) then

$$\|\varphi(\mathbf{a}_k) - \varphi(\mathbf{a})\|_2 = \frac{1}{a_n^{(k)}}$$

and so $\{\varphi(\mathbf{a}_k)\}_k \to \varphi(\mathbf{a})$ in $\boldsymbol{l}^2$. Observe that $(S_\omega, \mathbf{s}\tau)$ and $\boldsymbol{l}^2$ equipped with the usual metric topology σ associated with $\|\ldots\|_2$ are Fréchet closure spaces, so the sequential continuity of φ and the continuity of φ coincide (use the usual condition on neighborhoods to define the continuity of a map between closure spaces).

We put $L_\omega := \varphi(S_\omega)$ and we indicate with the symbol v the relativization of σ to L_ω. Using the injectivity of φ it's easy to verify that there exists a (unique) Fréchet closure operator cl for L_ω such that φ is a homeomorphism of the closure space $(S_\omega, \mathbf{s}\tau)$ onto the closure space (L_ω, cl) (see also Theorem 35 B.13 in [Če]). Moreover, since φ is continuous from $(S_\omega, \mathbf{s}\tau)$ to $(\boldsymbol{l}^2, \sigma)$, it follows that $\mathrm{cl} > v$.

We now work with the latter equivalent description of $(S_\omega, \mathbf{s}\tau)$ that is (L_ω, cl).

Suppose $a_1, \ldots, a_n \in \mathbf{N}$, we use the symbol $\langle a_1, \ldots, a_n \rangle$ to indicate $\varphi([a_1, \ldots, a_n])$. We define a set distribution $\mathcal{D}$ in L_ω putting

$$\mathcal{D}(\mathbf{x}) := \{\langle x_1, \ldots, x_n, j\rangle \in L_\omega \mid j \in \mathbf{N}\} \cup \{\mathbf{x}\}$$

for each $\mathbf{x} = \langle x_1, \ldots, x_n\rangle \in L_\omega \setminus \{\underline{0}\}$ and $\mathcal{D}(\underline{0}) := \{\langle j\rangle \in L_\omega \mid j \in \mathbf{N}\} \cup \{\underline{0}\}$.

Observe that $\mathcal{D}(\mathbf{x})$ is v-closed for every $\mathbf{x} \in L_\omega$ because $\mathcal{D}(\mathbf{x})$ is clearly a closed subset of $\boldsymbol{l}^2$.

We prove that, for each fixed $\mathbf{x} \in L_\omega$, $\mathrm{cl}(\complement\mathcal{D}(\mathbf{x})) = L_\omega \setminus \{\mathbf{x}\}$ that is $\mathrm{int}_{\mathrm{cl}}(\mathcal{D}(\mathbf{x})) = \{\mathbf{x}\}$. Since every sequence cl-converging to $\{\mathbf{x}\}$ may be eventually contained in $\mathcal{D}(\mathbf{x})$ we have that $\mathbf{x} \notin \mathrm{cl}(\complement\mathcal{D}(\mathbf{x}))$. Suppose $\mathbf{x} = \langle x_1, \ldots, x_n\rangle$. Fix $j \in \mathbf{N}$ and put $y := \langle x_1, \ldots, x_n, j\rangle$, we have that $y \in \mathcal{D}(\mathbf{x}) \setminus \{\mathbf{x}\}$ and the sequence $\{\langle x_1, \ldots, x_n, j, k\rangle\}_k$ is contained in $\complement\mathcal{D}(\mathbf{x})$ and it converges to y with respect to cl (similarly if $\mathbf{x} = \underline{0}$ then we fix $j \in \mathbf{N}$, we put $y := \langle j \rangle$ and we observe that the sequence $\{\langle j, k \rangle\}_k$ lies in $\complement \mathcal{D}(\underline{0})$ and it cl-converges to y); in particular we have the required property.

Summarizing L_ω, cl, v and $\mathcal{D}$ fulfil the conditions of Theorem 9 and hence we have $\mathbf{td}(\mathbf{s}\tau) = \mathbf{td}(\mathrm{cl}) = \omega_1$. On the other hand by definition $\mathbf{td}(\mathbf{s}\tau)$ is the sequential order of (S_ω, τ) and so we have obtained again Arhangel'skiĭ and Franklin's result (see [1], Theorem 5.1 page 316).

3.2. *Féron cross and radiolar topologies for the Euclidean plane.*

Let $\mathbf{R}^2$ be the Euclidean plane equipped with the usual Euclidean topological closure operator τ and with the usual Cartesian coordinates (x, y).

We now define the **Féron cross closure operator** F (or briefly **cross closure operator** F) for $\mathbf{R}^2$ given a base system $\mathcal{B}_F$ for F.

For each positive real number ε and for each point $p := (a, b)$ of $\mathbf{R}^2$ we put

$$\mathcal{C}(p) := \{x = a\} \cup \{y = b\}$$

and

$$\mathcal{C}_\varepsilon(p) := \mathcal{C}(p) \cap B_\varepsilon(p)$$

where $B_\varepsilon(p)$ is the usual τ-open ball of radius ε centered in p. $\mathcal{C}(p)$ is called **cross** centered in p and $\mathcal{C}_\varepsilon(p)$ is called ε-**cross** centered in p. We define $\mathcal{B}_F$ putting

$$\mathcal{B}_F(p) := \{\mathcal{C}_{1/n}(p) \mid n \in \mathbf{N}\}$$

for each $p \in \mathbf{R}^2$ and so we have

$$F(A) = \{p \in \mathbf{R}^2 \mid A \cap \mathcal{C}_{1/n}(p) \neq \emptyset \ \ \forall n \in \mathbf{N}\}$$

for each $A \subset \mathbf{R}^2$. We underline that, for each $p \in \mathbf{R}^2$, $\mathcal{B}_F(p)$ is countable and so F is a Fréchet closure operator for $\mathbf{R}^2$. The space $(\mathbf{R}^2, F)$ is called **Féron pretopological plane** or **cross closure plane** and the space $(\mathbf{R}^2, \mathbf{t}F)$ is called **cross topological plane.**

In order to define the radiolar topological plane we may first define the closure operator R' and the radiolar closure operator R for $\mathbf{R}^2$.

We fix $p \in \mathbf{R}^2$ and we say that a subset A of $\mathbf{R}^2$ containing p is a p-**radiolar** of $\mathbf{R}^2$ iff each straight line r through p intersects A in a subset A_r of r such that p lies in the τ_r-interior of A_r where τ_r is the relativization of τ to r. For each $p \in \mathbf{R}^2$ we define $\mathcal{N}_{R'}(p)$ as the collection of all p-radiolars of $\mathbf{R}^2$. It's immediate to verify that $\mathcal{N}_{R'}$ is a neighborhood system of a (unique) closure operator R' for $\mathbf{R}^2$ (observe that $\mathcal{N}_{R'}$ is a filter system on $\mathbf{R}^2$ such that $p \in \bigcap_{U \in \mathcal{N}_{R'}(p)} U$ for each $p \in \mathbf{R}^2$).

Now we put the **radiolar closure operator** R for $\mathbf{R}^2$ equal to the sequential modification of R' (i.e. $R = \mathbf{s}R'$); in particular a sequence S converges to p with respect to R iff S is eventually in each p-radiolar of $\mathbf{R}^2$. The space $(\mathbf{R}^2, R)$ will be called **radiolar closure plane** and the space $(\mathbf{R}^2, \mathbf{t}R)$ will be called **radiolar topological plane.**

Evidently $F > R > \tau$ and since τ is a metric topology for $\mathbf{R}^2$ it's also a T_3^1-topology for $\mathbf{R}^2$.

Now let $D := \overline{B}_1((1, 0)) \cap \{y \geqslant 0\}$ (where $\overline{B}_1((1, 0))$ is the τ-closed ball centered in $(1, 0)$ with radius 1), for each $\varphi \in [0, 2\pi)$ let D_φ be the image of D under the counterclockwise φ-rotation around the origin $\underline{0}$ of $\mathbf{R}^2$, let

$$\mathcal{S}_{\underline{0}} := D_0 \cup D_{\pi/2} \cup D_\pi \cup D_{3\pi/2}$$

and, for each $p \in \mathbf{R}^2$, let $\mathcal{S}_p := \mathcal{S}_{\underline{0}} + p$ (i.e. $\mathcal{S}_p$ is the image of $\mathcal{S}_{\underline{0}}$ under the translation of $\mathbf{R}^2$ that assigns to $\underline{0}$ the point p; see Figure 1).

We observe that the set distribution $\mathcal{S}$ defined putting $\mathcal{S}(p) := \mathcal{S}_p$ for each $p \in \mathbf{R}^2$ is a τ-closed section for R in fact $\mathcal{S}_p$ is a τ-closed p-radiolar for every $p \in \mathbf{R}^2$ (remember that $R > R'$). From elementary geometric considerations it's immediate to establish that, for each $p \in \mathbf{R}^2$, it holds

$$\partial_F(\mathcal{S}_p) = \partial_R(\mathcal{S}_p) = \partial_\tau(\mathcal{S}_p) \setminus \{p\} \text{ (see Figure 1)},$$

$$p \in F(\partial_F(\mathcal{S}_p)) = \partial_\tau(\mathcal{S}_p)$$

and for each $q \in \partial_R(\mathcal{S}_p)$

$$q \in F(\partial_F(\mathcal{S}_q) \setminus \mathcal{S}_p) \text{ (see Figure 2)}.$$

In this manner we may apply Corollary 8 obtaining that the conclusions of Theorem 6 hold for each Fréchet closure operator cl for $\mathbf{R}^2$ such that $F > \mathrm{cl} > R$. In particular we have: $\mathbf{td}(F) = \mathbf{td}(R) = \omega_1$, both the cross topology $\mathbf{t}F$ and the radiolar topology $\mathbf{t}R$ are sequential but not Fréchet at every point of $\mathbf{R}^2$ and their sequential order is ω_1. We point out that this subsection may be repeated word for word using the rational plane

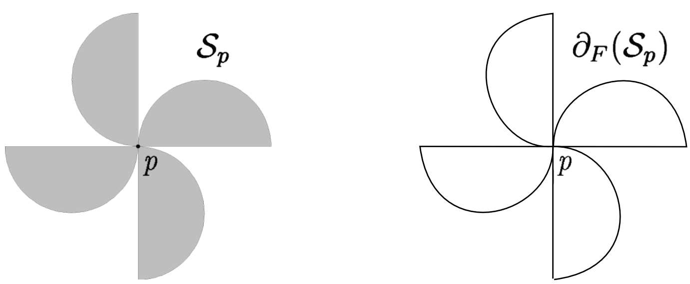

Figure 1. – $\mathcal{S}_p$ and $\partial_F(\mathcal{S}_p)$.

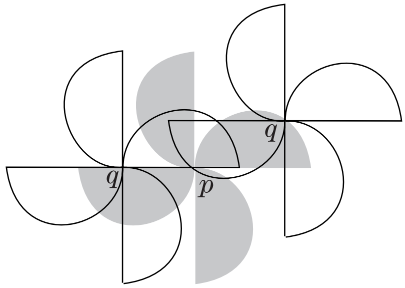

Figure 2. – $q \in F(\partial_F(\mathcal{S}_q) \setminus \mathcal{S}_p)$.

$\mathbf{Q}^2$ instead of $\mathbf{R}^2$ and hence we have discovered again the results of the papers [Gr] and [Fč].

3.3. *Limited topological spaces.*

Let (X, v) be a topological closure space and let $\mathcal{D}\colon X \to \mathcal{P}(X)$ be a set distribution in X. We define the closure operator $v_{\mathcal{D}}$ for X as the unique closure operator for X having the following base system $\mathcal{B}_v^{(\mathcal{D})}$ on X as one of its base system:

$$\mathcal{B}_v^{(\mathcal{D})}(x) := \{U \cap \mathcal{D}(x) \,|\, U \in \mathcal{N}_v(x)\}$$

for each $x \in X$. The closure operator $v_{\mathcal{D}}$ will be called **$\mathcal{D}$-limited topology** v for X and the closure space $(X, v_{\mathcal{D}})$ will be called **$\mathcal{D}$-limited topological space** (X, v).

By definition it follows that $v_{\mathcal{D}} > v$ and it holds

$$v_{\mathcal{D}}(A) = \{x \in X \,|\, U \cap \mathcal{D}(x) \cap A \neq \emptyset \ \ \forall U \in \mathcal{N}_v(x)\}$$

for every $A \subset X$. More precisely in the preceding expression for «$v_{\mathcal{D}}(A)$» we may replace the neighborhood system $\mathcal{N}_v$ with any base system $\mathcal{B}$ for v, in particular if v is a topology for X associated with a distance d and if $B_\varepsilon(x)$ is the subset $\{y \in X \,|\, d(x, y) < \varepsilon\}$ of X then it holds

$$v_{\mathcal{D}}(A) = \{x \in X \,|\, B_{1/n}(x) \cap \mathcal{D}(x) \cap A \neq \emptyset \ \ \forall n \in \mathbf{N}\}$$

for every $A \subset X$. In this way the cross closure operator for $\mathbf{R}^2$ is the $\mathcal{C}$-limited Euclidean plane topology where $\mathcal{C}$ is the «cross distribution» in $\mathbf{R}^2$ (see above).

We observe that if (X, v) is a T_3^1-topological space then each closure operator cl for X obtained limiting v (i.e. $\mathrm{cl} = v_{\mathcal{D}}$ for some set distribution $\mathcal{D}$ in X) is finer than v and it's a Fréchet closure operator for X because it admits a base system $\mathcal{B}$ such that $\mathcal{B}(x)$ is countable for each $x \in X$ (remember that v fulfils the first axiom of countability). In particular X, cl and v fulfil the preliminary hypotheses of each results obtained in the preceding section (i.e. (X, cl) is a Fréchet space, v is a T_3^1-topology for X, $\mathrm{cl} > v$); furthermore $\mathcal{D}$ is always a section of $\mathcal{N}_{\mathrm{cl}}$. For these facts the limited topological spaces form a natural ambient to apply our results.

In this direction we present a theorem and a simple consequence of its demonstration which study those limited topological spaces «similar to» the cross closure plane.

THEOREM 11. – *Let* (E, v) *be a metrizable topological real* (*or complex*) *vector space and let* $\mathcal{D}$ *be a set distribution in* E. *If* $\mathcal{D}$ *is invariant under translations and it assigns to the origin* $\underline{0}$ *of* E *a finite union of* v-*closed vectorial subspaces of* E *then only one of the two following situations must occur*:

1. $\mathcal{D}(\underline{0})$ *is a vectorial subspace of* E *and hence* $v_{\mathcal{D}}$ *is a topological closure operator for* E;

2. $\mathcal{D}(\underline{0})$ *is not a vectorial subspace of* E *and hence* $\mathbf{ev}_{v_{\mathcal{D}}} \equiv \omega_1$.

PROOF. – For convenience we put $u := v_{\mathcal{D}}$, $V := \mathcal{D}(\underline{0})$ (hence $V + x = \mathcal{D}(x)$) and we use the symbol $\mathbf{K}$ to indicate both the set of real numbers and the set of complex numbers equipped with the usual topologies.

First suppose V equal to a vectorial subspace of E. By definition of u the collection

$$\{(V+x) \cap U \mid U \text{ is a } v\text{-open neighborhood of } x\}$$

is a base of the filter $\mathcal{N}_u(x)$ for each $x \in E$.

Fix a v-open neighborhood U of $\underline{0}$ and fix a point $y \in U \cap V$. Since V is a vectorial subspace of E and U is v-open we respectively have that $(V+y) \cap U = V \cap U$, $(V+y) \cap U \in \mathcal{N}_u(y)$ and so $y \in \mathrm{int}_u(U \cap V)$; in particular $U \cap V = \mathrm{int}_u(U \cap V)$. Follows that $U \cap V$ is u-open for each fixed v-open neighborhood U of $\underline{0}$ and hence the collection of all u-open neighborhoods of $\underline{0}$ is a base of the filter $\mathcal{N}_u(\underline{0})$. Since $\mathcal{D}$ is invariant under translations we have that u is homogeneous and so, for each $x \in E$, the collection of all u-open neighborhoods of x is a base of the filter $\mathcal{N}_u(x)$; hence u is topological.

We now prove that if V is not a vectorial subspace of E then (E, u), v and the section $\mathcal{D}$ of $\mathcal{N}_u$ fulfil the hypotheses of Theorem 6. By homogeneity of u it suffices to verify that $\underline{0} \in u(\partial_u(V))$ and, for each $y \in \partial_u(V)$, $y \in u(\partial_u(V+y) \setminus V)$. These conditions are respectively equivalent to $\underline{0} \in v(\partial_u(V))$ and, for each $y \in \partial_u(V)$, $y \in v(\partial_u(V+y) \setminus V)$. To see this we observe that since $u > v$ it holds $u(\partial_u(V)) \subset v(\partial_u(V))$ and $u(\partial_u(V+y) \setminus V) \subset v(\partial_u(V+y) \setminus V)$ for each $y \in \partial_u(V)$; in particular $\underline{0} \in u(\partial_u(V))$ implies $\underline{0} \in v(\partial_u(V))$ and $y \in u(\partial_u(V+y) \setminus V)$ implies $y \in v(\partial_u(V+y) \setminus V)$ for each $y \in \partial_u(V)$. On the other hand one has $\underline{0} \in v(\partial_u(V))$

iff, for each v-neighborhood U of $\underline{0}$, it holds $\partial_u(V) \cap U \neq \emptyset$; now $\partial_u(V) \subset V$ and so $\partial_u(V) \cap (V \cap U) = \partial_u(V) \cap U \neq \emptyset$ for each v-neighborhood U of $\underline{0}$ that is $\underline{0} \in u(\partial_u(V))$. Since $\partial_u(V+y) \setminus V \subset V+y$ for each $y \in \partial_u(V)$ we obtain as above that if $y \in v(\partial_u(V+y)\setminus V)$ then $y \in u(\partial_u(V+y)\setminus V)$.
In this way we must only verify that

$$\underline{0} \in v(\partial_u(V))$$

and, for each $y \in \partial_u(V)$,

$$y \in v(\partial_u(V+y)\setminus V).$$

First we calculate $\partial_u(V)$.

By assumption we may express V as union of some vectorial subspaces $V_1, \ldots, V_n$ in such way that $V_j \not\subset \bigcup_{i \neq j} V_i$ for each $j = 1, \ldots, n$.

We prove that $\partial_u(V) = V \setminus \bigcap_j V_j$.

If $x \in \bigcap_j V_j$ then $V_j + x = V_j$ for each j therefore $V + x = \bigcup_j (V_j + x) = V$ and so $x \notin \partial_u(V)$ in fact $x \notin u(\complement V)$. Assume now that $x \in V \setminus \bigcap_j V_j$. We may suppose that $x \notin \bigcup_{j=1}^{p} V_j$ and $x \in \bigcap_{j=p+1}^{n} V_j$ for some $p \in \{1, \ldots, n-1\}$. We fix $y \in V_1 \setminus \bigcup_{j=2}^{n} V_j$ and we observe that, for each $j = 1, p+1, \ldots, n$ and each $t \in \mathbf{K}\setminus\{0\}$, $x + ty \notin V_j$ (otherwise if $x + ty \in V_1$ then since $y \in V_1$ it would follow $x = x + ty - ty \in V_1$ which contradicts our assumptions; in the same way if $x + ty \in V_j$ for some $j = p+1, \ldots, n$ then since $x \in \bigcap_{j=p+1}^{n} V_j$ it would follow $y = \frac{1}{t}(x + ty - x) \in V_j$ which again contradicts our assumptions). On the other hand the map $\gamma : \mathbf{K} \to E$ defined putting $\gamma(t) := x + ty$ is continuous (because (E, v) is a topological vector space), $\bigcup_{j=2}^{p} V_j$ is v-closed and $\gamma(0) = x \notin \bigcup_{j=2}^{p} V_j$ therefore there exists a positive real number ε such that if $|t| < \varepsilon$ then $x + ty \notin \bigcup_{j=2}^{p} V_j$. Follows that, for each fixed $t \in \mathbf{K}\setminus\{0\}$ with $|t| < \varepsilon$, $x + ty \in (V_1 + x)\setminus V \subset (V + x)\setminus V = (V + x) \cap \complement V$ therefore $x \in u(\complement V) \cap V = \partial_u(V)$ and hence $\partial_u(V) = V \setminus \bigcap_j V_j$.

Now $V_1 \setminus \bigcup_{j=2}^{n} V_j \subset \partial_u(V)$ and it's obvious that $\underline{0} \in v(V_1 \setminus \bigcup_{j=2}^{n} V_j)$ therefore $\underline{0} \in v(\partial_u(V))$.

Remain to be shown that for each $y \in \partial_u(V)$ it holds $y \in v(\partial_u(V+y)\setminus V)$. We may suppose that $y \notin \bigcup_{j=1}^{p} V_j$ and $y \in \bigcap_{j=p+1}^{n} V_j$ for some $p \in \{1, \ldots, n-1\}$; by the above reasoning we have $y \in v((V_1 + y)\setminus V)$. Since $\partial_u(V+y) = \bigcup_j (V_j +$

$y)\setminus\bigcap_j(V_j+y)$ and $\bigcap_j(V_j+y)\subset V_{p+1}\subset V$ we have

$$\partial_u(V+y)\setminus V\supset(V_1+y)\setminus V$$

and so $y\in v(\partial_u(V+y)\setminus V)$. ■

COROLLARY 12. – *Let (E, v) and $\mathcal{D}$ be as in the Theorem 11 and let $\mathcal{C}$ be a set distribution in E invariant under translations which assigns to the origin of E a union of two indipendent straight lines r and s such that $\mathcal{D}(\underline{0})\cap\mathcal{C}(\underline{0})=\{\underline{0}\}$. Let also* cl' *be the infimum of $\{v_{\mathcal{D}}, v_{\mathcal{C}}\}$ in the ordered class of all closure operators for E.*

Follows that each Fréchet closure operator cl^* *for E such that* $v_{\mathcal{C}}>\mathrm{cl}^*>\mathrm{cl}'$ *has evolution function constantly equal to ω_1.*

PROOF. – First of all we remember that cl' exists and it's characterized by the relation

$$\mathcal{N}_{\mathrm{cl}'}(x)=\mathcal{N}_{v_{\mathcal{D}}}(x)\cap\mathcal{N}_{v_{\mathcal{C}}}(x)$$

for each $x\in E$ (see [Če], section 31) and so it coincides with the $(\mathcal{D}\cup\mathcal{C})$-limited topology v for E where $\mathcal{D}\cup\mathcal{C}$ is the set distribution in E defined putting $(\mathcal{D}\cup\mathcal{C})(x):=\mathcal{D}(x)\cup\mathcal{C}(x)$ for each $x\in E$.

For short we put $\mathrm{cl}:=v_{\mathcal{C}}$, $V:=\mathcal{D}(\underline{0})$ and $W:=(\mathcal{D}\cup\mathcal{C})(\underline{0})=V\cup r\cup s$.

As above we may express V as union of vectorial subspaces $V_3,\dots,V_n$ such that $V_j\not\subset\bigcup_{i\neq j}V_i$ for each $j=3,\dots,n$ and so, putting $V_1:=r$ and $V_2:=s$, we have $W=\bigcup_{j=1}^{n}V_j$ and $V_j\not\subset\bigcup_{i\neq j}V_i$ for each $j=1,\dots,n$.

Following the same argument used in the Proof of Theorem 11 we see that $\partial_{\mathrm{cl}}(W)=W\setminus\{\underline{0}\}$ that is $\mathrm{int}_{\mathrm{cl}}(W)=\{\underline{0}\}$. On the other hand it's immediate to verify that E is perfect with respect to cl and $\mathcal{D}\cup\mathcal{C}$ is a v-closed section of $\mathcal{N}_{\mathrm{cl}'}$; hence we may apply Corollary 10 and so we conclude the proof. ■

Since all applications seen above concern homogeneous closure spaces one may believe that our results work only in these cases. This is not true as we can see with the following simple example.

Let $\mathcal{C}$ be a set distribution in $\mathbf{R}^2$ (equipped with the usual Euclidean topology τ) defined putting

$$\mathcal{C}(p):=(\{a\}\times\mathbf{Q})\cup(\mathbf{Q}\times\{b\})\cup\{p\}$$

if $p=(a,b)\in\mathbf{R}^2\setminus\{\underline{0}\}$ and $\mathcal{C}(\underline{0}):=(\{0\}\times\mathbf{R})\cup(\mathbf{R}\times\{0\})$ (where $\mathbf{Q}$ is the subset of $\mathbf{R}$ formed by all rational numbers) and let $\overline{\mathcal{C}}$ be the set distribution which assigns to each point p of $\mathbf{R}^2$ the τ-closure of $\mathcal{C}(p)$ (i.e. the cross distribution in $\mathbf{R}^2$). It's easy to prove that $(\mathbf{R}^2, \tau_{\mathcal{C}})$, τ and $\overline{\mathcal{C}}$ fulfil the conditions of Theorem 9 but $\tau_{\mathcal{C}}$ is not homogeneous. Otherwise there would exist a injective map from a set of the form $\mathcal{C}(\underline{0}) \cap B_\varepsilon(\underline{0})$ (for some $\varepsilon > 0$) to a set of the form $\{(a, b)\} \cup (\{a\} \times \mathbf{Q}) \cup (\mathbf{Q} \times \{b\})$ which is impossible for a question of cardinality.

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