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Weak Bases in P-adic Spaces.

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Sunto. – *Si studiano spazi polari localmente convessi su un non trivialmente valutato campo completo non archimedeo con una debole base topologica. Dimostriamo due teoremi di completezza e un teorema tipo Hahn-Banach per spazi localmente convessi con una debole base di Schauder.*

Summary. – *We study polar locally convex spaces over a non-archimedean non-trivially valued complete field with a weak topological basis. We prove two completeness theorems and a Hahn-Banach type theorem for locally convex spaces with a weak Schauder basis.*

0. – Introduction and preliminary facts.

Throughout K denotes a non-archimedean non-trivially valued complete field with a valuation $|\,.\,|$. For general properties of locally convex spaces (lcs) over K we refer to [15] and [13]. If E is a lcs, E' and E^* denote its *topological* and *algebraic* dual, respectively, and $\sigma(E, E')$ and $\sigma(E', E)$ the weak topology of E and E', respectively. A metrizable and complete lcs is called a *Fréchet space*. By $\mathcal{T}(E)$ we denote the set of all (non-archimedean) continuous seminorms on E; following [13] we shall require that $p(x) \in \overline{|K|}$ (the closure of the set $|K|$), $x \in E$, $p \in \mathcal{T}(E)$.

Let $E = (E, \tau)$ be a lcs. A sequence (x_n) in E is called a (topological) *basis* of E if every $x \in E$ can be written uniquely as $x = \sum_n t_n x_n$ with $t_n \in K$, $n \in \mathbb{N}$. (x_n) is called *regular* if there exists a continuous seminorm p on E such that $\inf_n p(x_n) > 0$. If the coefficient functionals $f_n \colon x \to t_n$, $n \in \mathbb{N}$, are continuous, (x_n) is called a *Schauder basis*. As in the real or complex case one proves that every basis in a Fréchet space is a Schauder basis. Clearly every lcs E with a basis is (strictly) of countable type, i.e. there is a countable set whose K-linear span is dense in E. Conversely, any infinite dimensional Banach space of countable type is linearly homeomorphic to the space c_0 of all sequences in K

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converging to zero (with the sup-norm topology), hence it has a Schauder basis, cf. [12]. It is unknown whether a Fréchet space of countable type (or equivalently strictly of countable type) has a Schauder basis. Nevertheless, as we proved in [3] every non-Montel Fréchet space has an «orthogonal» basic sequence. Very recently Sliwa has proved [14] that every infinite-dimensional Fréchet space has an «orthogonal» basic sequence. A basis (x_n) is called *«orthogonal»* [1] if the topology of E can be determined by a subfamily $\mathcal{U}$ of $\mathcal{T}(E)$ such that $p(x) = \max_n p(f_n(x)\, x_n)$, $p \in \mathcal{U}$, $x = \sum_n f_n(x)\, x_n \in E$. For a lcs E, a basis (x_n) is «orthogonal» iff the maps $T_n\colon x \to f_n(x)\, x_n$, $x \in E$, form an equicontinuous family, cf. [3].

Recall, cf. [3], that

($*$) *if (x_n) is an «orthogonal» basis in a lcs E, then it is an «orthogonal» basis in any lcs containing E as a dense subspace.* This conclusion fails when «orthogonal» is replaced by «Schauder».

Recall that for a lcs E with a Schauder basis in $\sigma(E, E')$ we say that the *weak basis theorem holds* if every weak Schauder basis, i.e. a Schauder basis in the weak topology $\sigma(E, E')$, is a Schauder basis in the original topology. If K is spherically complete, the weak basis theorem holds for any lcs over K, cf. [8]. If K is not spherically complete, the space l^∞ of all bounded sequences in K with the sup-norm topology has a weak Schauder basis (the unit vectors (e_n)) but l^∞ is not even of countable type. Nevertheless, the vectors e_n, $n \in \mathbb{N}$, form a basic sequence in l^∞, i.e. (e_n) is a Schauder basis in the closed linear span of the $\{e_1, e_2, \ldots\}$ in l^∞.

It turns out, see [9], that a in polar Fréchet space every weak Schauder basis is an «orthogonal» basic sequence. Hovewer, it seems to be unknown if a Fréchet space with a weak Schauder basis for which every weak Schauder basis is a basic sequence is polar.

In this note we prove two completeness theorems (extending Kalton's [6] Webb's [16] and Kamthan-Gupta's [7] results to the p-adic case) and a Hahn-Banach type theorem for lcs with a weak Schauder basis.

1. – A Hahn-Banach type theorem.

It is known that the unit vectors (e_n) in the space l^∞ (if K is not spherically complete) form a sequence which weakly converges to zero in l^∞ [13] and (clearly) is regular in the original norm. We start with the following purely non-archimedean

THEOREM 1.1. – *Let $E = (E, \tau)$ be a lcs with a weak Schauder basis (x_n) and its associated sequence (f_n) of coefficient functionals. Let G be the closed* *linear span of the* x_n, $n \in \mathbb{N}$. *Then the following assertions are equivalent.*

(a) (x_n) *is a Schauder basis.*

(b) E *is of countable type.*

(c) E *has the Orlicz-Pettis property ((OP)-property,* [11]), *i.e. every weakly convergent sequence in* E *converges in* E.

(d) *Every weak Schauder basis in* E *is a Schauder basis.*

(e) E *is polar and* $f_n(x)\, x_n \to 0$, $x \in E$.

(f) E *is polar and has the Hahn-Banach property.*

(g) E *is polar and the closed linear span of any regular «orthogonal» basic sequence in* E *is a Hahn-Banach subspace of* E.

(h) E *is polar and* G *is a Hahn-Banach subspace of* E.

If in addition E *is a normed* l^∞*-barrelled space, i.e. every* $\sigma(E', E)$*-bounded sequence in* E' *is equicontinuous, then properties* (a)$\to$(h) *are equivalent to*

(i) $\lim_n g_n(x)\, y_n = 0$ *in* E *for all* $x \in E$ *and for every weak Schauder basis* (y_n) *of* E *with coefficient functionals* (g_n) *in* E'.

Recall that a subspace F of a lcs E is a *Hahn-Banach subspace* (HB-subspace) of E if every continuous linear functional over F can be extended to a continuous linear functional over E. A lcs E has the Hahn-Banach property (HB-property) if every subspace of E is a HB-subspace of E. If K is spherically complete, then every lcs over K has the HB-property, cf. [15], [2]. Note also that every Fréchet space (or even barrelled space) is l^∞-barrelled; cf. [10].

PROOF. – Clearly (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e). Now we prove (e) $\Rightarrow$ (a): It is enough to show that $x = \sum_n f_n(x)\, x_n$ in E for all $x \in E$. Observe that, since $\lim_n f_n(x)\, x_n = 0$, there exists y in the completion Y of E such that $y = \sum_n f_n(x)\, x_n$ in Y for all $x \in E$. Since $x = \sum_n f_n(x)\, x_n$ weakly, then $f(x) = f(y)$ for any continuous linear functional f on Y. But Y is polar [13], 5.5, and hence is dual-separating. So $x = y$ and the conclusion follows.

(b) $\Rightarrow$ (f): E is of countable type, so it has the HB-property, cf. [13]. (f) $\Rightarrow$ (g): Obvious. (g) $\Rightarrow$ (c): Assume that (c) fails. Let (z_n) be a sequence in E which weakly converges to zero in E and $p(z_n) \geq c > 0$, $n \in \mathbb{N}$, for some $c > 0$ and $p \in \mathcal{P}(E)$. By 3.2 of [3] there exists a subsequence (y_n) of (z_n) which is an «orthogonal» basic sequence in E. Let F be the closed linear span of elements $y_1, y_2, \ldots$ in E. By (g) the subspace F has the HB-property and consequently $\sigma(F, F') = \sigma(E, E')\,|\,F$, cf. [2], 2.3. Hence $\lim_n y_n = 0$ in $\sigma(F, F')$. On the other hand, since F is of countable type, $\lim_n y_n = 0$, a contradiction.

(f) $\Rightarrow$ (h): Obvious. (h) $\Rightarrow$ (e): Similarly as (g) $\Rightarrow$ (c). We conclude the proof by showing (i) $\Rightarrow$ (e): Let $E = (E, \|.\|)$ be a normed l^∞-barrelled space. It is enough to show that E is polar. Suppose E is not polar. We can assume that $\lim_n \|x_n\| = 0$. Let τ_p be the polar topology on E associated with the norm topology defined by $\|.\|$ and let $\|.\|_p$ be a polar norm on E defining τ_p. Then $(E, \|.\|_p)$ is a polar normed space for which (x_n) is a weak Schauder basis with coefficient functionals (f_n) and $\lim_n \|f_n(x)\, x_n\|_p = 0$, $x \in E$. By (e) $\Rightarrow$ (a), (x_n) is a Schauder basis in (E, τ_p). Since (E, τ_p) is l^∞-barrelled, (x_n) is an «orthogonal» basis in τ_p, see 3.3 below. By (*) (x_n) is an «orthogonal» basis of the completion Y of (E, τ_p) having as coefficient functionals the canonical extensions g_n, $n \in \mathbb{N}$, of f_n, $n \in \mathbb{N}$, to Y with their canonical norms $\|g_n\|_p$, $n \in \mathbb{N}$, as defined in [12]. Since $\tau_p < \tau$, given $0 < \alpha < 1$, there exists a sequence (z_n) in E such that $\|z_n\|_p \leq \min\{\alpha \|g_n\|_p^{-1}, \alpha^n\}$ for all $n \in \mathbb{N}$ and $\inf_n \|z_n\| > 0$. For each $n \in \mathbb{N}$ put $y_n = x_n + z_n$. Then $\|g_n\|_p \|y_n - x_n\|_p < \alpha$, $n \in \mathbb{N}$, and $\inf_n \|y_n\| > 0$. It follows from [8] that (y_n) is a Schauder basis in Y and hence in (E, τ_p). Therefore it is a weak Schauder basis in E. Let (h_n) be the sequence of its associated coefficient functionals in E'. By (i) we have $\lim_n \|h_n(x)\, y_n\| = 0$ and also $h_n(x) \to 0$ for each $x \in E$. Since (E, τ_p) is l^∞-barrelled one gets that $\sup_n \|h_n\|_p < \infty$. On the other hand, since $\|z_n\|_p \to 0$ and $\|x_n\|_p \to 0$ we deduce that $\|y_n\|_p \to 0$. But then $h_n(y_n) \to 0$, a contradiction since $h_n(y_n) = 1$ for all $n \in \mathbb{N}$.

COROLLARY 1.2. – *Let E be a polar lcs with a weak Schauder basis (x_n). Then (x_n) contains a basic sequence.*

PROOF. – If E has the (OP)-property, then (x_n) is a Schauder basis in E. Assume that E does not have the (OP)-property. By 1.1 there exists $x \in E$ such that $y_n = f_n(x)\, x_n$ does not converge to zero in E, where (f_n) is the sequence of coefficient functionals in E' associated with (x_n). But $y_n \to 0$ in $\sigma(E, E')$. By [3] (y_n) contains a subsequence which is an «orthogonal» basic sequence in E.

For the rest of the paper we will ASSUME (if nothing more will be mentioned) that $E = (E, \tau)$ is a Hausdorff polar lcs over K and (x_n) is a weak basis (not necessarily Schauder) with its associated sequence (f_n) of coefficient functionals. Since (E, τ) is polar, for every $x \in E$ the set $\{f_n(x)\, x_n : n \in \mathbb{N}\}$ is $\sigma(E, E')$-bounded, so it is τ-bounded, [13], 7.5. Following [1] we define $p^*(x) = \sup_n |f_n(x)|\, p(x_n)$, $x \in E$, $p \in \mathcal{P}(E)$. Let τ^* be the locally convex topology on E defined by the seminorms $\{p^* : p \in \mathcal{P}(E)\}$.

We will prove that $\tau \leq \tau^*$ and (x_n) is an «orthogonal» basic sequence for (E, τ^*). If (E, τ) is (sequentially) complete, then so is (E, τ^*). If (E, τ) is weakly sequentially complete, then (E, τ^*) is complete. We show that $\tau = \tau^*$ if (x_n) is a weak Schauder basis and E is l^∞-barrelled. In particular $\tau = \tau^*$ for every polar Fréchet space with a weak basis.

2. – More about the topology τ^*.

Our first result introduces the basic properties of the topology τ^*.

PROPOSITION 2.1. – (a) *(E, τ^*) is a polar space.*

(b) *The coefficient functionals $f_1, f_2, \ldots$, are τ^*-continuous.*

(c) *$\tau \leq \tau^*$; in particular (E, τ^*) is Hausdorff.*

(d) *If E is metrizable, then (E, τ^*) is metrizable.*

(e) *(x_n) is an «orthogonal» basic sequence in (E, τ^*).*

(f) *(x_n) is an «orthogonal» basis in τ^* iff (x_n) is a topological basis in E. In this case, τ^* is the smallest locally convex topology on E greater that τ for which (x_n) is an «orthogonal» basis.*

(g) *Let (x_n) be a weak Schauder basis in E and E' is $\sigma(E', E)$-sequentially complete. Then τ^* is compatible with the duality $\langle E, E' \rangle$ iff (x_n) is a weak Schauder basis in (E, τ^*).*

(h) *If (x_n) is a Schauder basis in E and E' is $\sigma(E', E)$-sequentially complete, then τ^* is compatible with the duality $\langle E, E' \rangle$.*

(i) *If E is bornological, then τ^* is compatible with the duality $\langle E, E' \rangle$ iff $\tau = \tau^*$.*

PROOF. – Properties (a), (b) and (d) follow from the definition of τ^*.

(c) Let p be a polar τ-continuous seminorm on E and let $f \in E^*$ be such that $|f| \leq p$. Since $x = \sum_n f_n(x)\, x_n$ weakly, $x \in E$, one gets $f(x) = \sum_n f_n(x)\, f(x_n)$. Then

$$|f(x)| \leq \sup_n |f_n(x)| |f(x_n)| \leq \sup_n |f_n(x)| p(x_n) = p^*(x).$$

Hence $|f| \leq p^*$ as soon as $|f| \leq p$. Then $p = \sup\{|f| : f \in E^*, |f| \leq p\} \leq p^*$ and consequently p is τ^*-continuous.

(e) Let D be the linear span of the elements x_n, $n \in \mathbb{N}$. Take $x \in D$. Then $x = \sum_{j=1}^{n} f_j(x)\, x_j$ for some $n \in \mathbb{N}$ and for all $p \in \mathcal{F}(E)$

$$p^*(x) = \sup_m |f_m(x)| p(x_m) = \sup_{1 \leq m \leq n} |f_m(x)| p^*(x_m).$$

Since the expression of each $x \in D$ as $\sum_n t_n x_n$ is unique, this proves that (x_n) is an «orthogonal» basis in (D, τ^*). Now $(*)$ (from the Introduction) completes the proof.

(f) First assume that (x_n) is an «orthogonal» basis for (E, τ^*). Since $\sigma(E, E') \leq \tau \leq \tau^*$, the coefficient functionals are the f_n and every $x \in E$ can be written uniquely as $x = \sum_n f_n(x)\, x_n$ in τ. For the converse, it is enough to prove (by (e)) that $x = \sum_n f_n(x)\, x_n$ in τ^*, $x \in E$. Since (x_n) is a basis in τ, for $x \in E$, $m \in \mathbb{N}$ and $p \in \mathcal{F}(E)$ one has $p^*(x - \sum_{n=1}^{m} f_n(x)\, x_n) = \sup_{n>m} |f_n(x)|\, p(x_n) \to 0$. Now we prove the second part of (f). Let τ_0 be a locally convex topology on E, $\tau \leqslant \tau_0$, such that (x_n) is an «orthogonal» basis for τ_0 (with coefficient functionals (f_n)) and let $\mathcal{F}_0(E)$ be a basis of τ_0-continuous seminorms on E for which $x_1, x_2, \ldots$, are orthogonal. For every $p \in \mathcal{F}(E)$ there exists $p_0 \in \mathcal{F}_0(E)$ such that $p \leqslant p_0$ and hence $p^* \leqslant p_0$, so $\tau^* \leqslant \tau_0$.

(g) The «only if» is obvious. In order to show the «if» suppose that (x_n) is a weak Schauder basis for τ^*. Clearly $(E, \tau)' \subset (E, \tau^*)'$. By Lemma 3 of [1] the coefficient functionals (f_n) associated to (x_n) form a Schauder basis of $(E, \tau^*)'$ (endowed with the canonical weak* topology). Since $f_n \in E'$ for all $n \in \mathbb{N}$ and E' is $\sigma(E', E)$-sequentially complete we conclude that $(E, \tau^*)' \subset (E, \tau)'$.

(h) It follows directly from (f) and (g).

(i) If τ^* is compatible with the duality $\langle E, E' \rangle$ and (E, τ) is bornological, then $\tau^* \leqslant \tau$, [13], 7.9.

It seems to be natural to ask whether (x_n) is a weak Schauder basis for (E, τ^*). If K is spherically complete, the answer is «yes». Indeed, since E has the (OP)-property, cf. [11], it is a basis for E. Finally 2.1 (f) completes the proof.

EXAMPLE 2.2. – Let c_{00} be the subspace of c_0 consisting of all sequences (ξ_n) such that $\xi_n = 0$ for large $n \in \mathbb{N}$. Let $\alpha \in K$, $0 < |\alpha| < 1$, $E = c_{00}$. Put $x_n := (1, \alpha, \alpha^2, \ldots, \alpha^{n-1}, 0, 0, \ldots)$, $n \in \mathbb{N}$. It is easy to see that (x_n) is a Schauder (Hamel) basis of E but not «orthogonal». (a) This shows that not every Schauder basis is «orthogonal». (b) The topology τ^* on E associated with (x_n) on E is defined by the norm $\|x\|^* = \sup_n |(\alpha\xi_n - \xi_{n+1})\, \alpha^{-n}|$, $x = (\xi_n)$, not equivalent to the original one of E. Therefore (by 2.1 (i)) the topology τ^* is not compatible with the duality $\langle E, E' \rangle$. This example shows that if E' is not $\sigma(E', E)$-sequentially complete, then properties (g) and (h) of 2.1 are not true in general.

It turns out that for a wide class of lcs with a weak Schauder basis the equality $\tau = \tau^*$ holds.

PROPOSITION 2.3. – *Let (x_n) be a weak Schauder basis for an l^∞-barrelled space (E, τ) with its associated sequence (f_n) in E'. Then $\tau = \tau^*$.*

PROOF. – By 2.1 we need only to show that $\tau^* \leqslant \tau$. Take $p \in \mathcal{F}(E)$. Since $p(x_n) \in \overline{|K|}$, for each $n \in \mathbb{N}$, there exists a sequence $(t_m^n)_m$ in K such that $|t_1^n| \leqslant |t_2^n| \leqslant \ldots$ and $\sup_m |t_m^n| = \lim_m |t_m^n| = p(x_n)$, $n \in \mathbb{N}$. Hence for each $x \in E$ one gets $p^*(x) = \sup_{n,m} |f_n(x)\, t_m^n|$. Then $(t_m^n f_n)_{n,m}$ is a pointwise bounded sequence in E'. By assumption this sequence is τ-equicontinuous. Hence p^* is τ-continuous, so $\tau^* \leqslant \tau$.

3. – Two completeness theorems.

Theorem 3.2 below extends (to the p-adic case) results of [7], 2.5, 2.6. We need the following simple

LEMMA 3.1. – *Let $(y_t)_{t \in T}$ be a Cauchy net in (E, τ^*). Then $z_n := \lim_t f_n(y_t)$ exists for every $n \in \mathbb{N}$.*

Now we are ready to prove the following

THEOREM 3.2. – (a) *If (E, τ) is (sequentially) complete, then so is (E, τ^*).*
(b) *If (E, τ) is weakly sequentially complete, then (E, τ^*) is complete.*

PROOF. – (a) Assume that (E, τ) is complete and let $(y_t)_{t \in T}$ be a Cauchy net in (E, τ^*). Since $\tau \leqslant \tau^*$, $y := \lim_t y_t$ exists in τ. It is enough to prove that if a τ^*-Cauchy net $(y_t)_{t \in T}$ converges to zero in τ, then it converges to zero in τ^*. Let z_n be as in 3.1, $n \in \mathbb{N}$. We show that $z_n = 0$, $n \in \mathbb{N}$. To prove this it is enough to show that $\sum_n z_n x_n = 0$ weakly. Let $f \in E'$. For each $m \in \mathbb{N}$, $t \in T$ we have

$$\left| \sum_{n=1}^{m} z_n f(x_n) \right| \leqslant \max \left\{ |f(y_t)|, \left| f(y_t) - \sum_{n=1}^{m} f_n(y_t) f(x_n) \right|, \left| \sum_{n=1}^{m} (f_n(y_t) - z_n) f(x_n) \right| \right\}.$$

Let $\varepsilon > 0$. There exists $t_0 \in T$ such that for all t, $r \geqslant t_0$, $n \in \mathbb{N}$ one gets $|f_n(y_t) - f_n(y_r)| \, |f(x_n)| < \varepsilon$; note that $|f|^*$ is a polar τ^*-continuous seminorm. Take $t \geqslant t_0$ such that $|f(y_t)| < \varepsilon$. Then $\left| \sum_{n=1}^{m} (f_n(y_t) - z_n) f(z_n) \right| \leqslant \varepsilon$, $m \in \mathbb{N}$ since $y_t = \lim_m \sum_{n=1}^{m} f_n(y_t) z_n$ weakly we have

$$f(y_t) = \lim_m \sum_{n=1}^{m} f_n(y_t) f(x_n).$$

Therefore

$$\left| f(y_t) - \sum_{n=1}^{m} f_n(y_t) f(x_n) \right| < \varepsilon$$

for sufficiently large $m \in \mathbb{N}$. Hence $\sum_n z_n f(x_n) = 0$, so $z_n = 0$, $n \in \mathbb{N}$. Finally we prove that $y_t \to 0$ in τ^*. Take $p \in \mathcal{P}(E)$, $\varepsilon > 0$. There exists t_0 in T such that for t, $r \geqslant t_0$ we have

$$|f_n(y_t) - f_n(y_r)| \, p(x_n) \leqslant p^*(y_t - y_r) < \varepsilon ,$$

$n \in \mathbb{N}$. Then $|f_n(y_t) - z_n| p(x_n) \leqslant \varepsilon$, $t \geqslant t_0$, $n \in \mathbb{N}$. Hence

$$p^*(y_t) = \sup_n |f_n(y_t)| p(x_n) \leqslant \varepsilon$$

for $t \geqslant t_0$.

(b) Assume that (E, τ) is weakly sequentially complete. Let $(y_t)_{t \in T}$ be a Cauchy net in (E, τ^*). By 3.1, $z_n := \lim_t f_n(y_t)$ exists for $n \in \mathbb{N}$. We show that $\sum_n z_n x_n$ exists in $\sigma(E, E')$. By weak sequential completeness this is the same as showing that $z_n x_n \rightarrow 0$ weakly. For $t \in T$ and $n \in \mathbb{N}$ we note the following

$$|z_n f(x_n)| \leqslant \max\{|(z_n - f_n(y_t)) f(x_n)|, |f_n(y_t) f(x_n)|\},$$

$f \in E'$. Let $\varepsilon > 0$. There exists $t_0 \in T$ such that for $t, r \geqslant t_0$ we have $|(f_n(y_t - y_r)) f(x_n)| < \varepsilon$, $n \in \mathbb{N}$. Choose $t \geqslant t_0$. There exists $n_0 \in \mathbb{N}$ such that $|f_n(y_t) f(x_n)| < \varepsilon$ for all $n \geqslant n_0$. Therefore $|z_n f(x_n)| \leqslant \varepsilon$ for $n \geqslant n_0$. Let $a := \sum_n z_n x_n$ in $\sigma(E, E')$. We show that $a = \lim_t y_t$ in τ^*. Let $p \in \mathcal{T}(E)$, $\varepsilon > 0$. Then

$$p^*(a - y_t) = p^*\Big(\sum_n z_n x_n - \sum_n f_n(y_t) x_n\Big) = \sup_n |z_n - f_n(y_t)| p(x_n).$$

Since $(y_t)_{t \in T}$ is τ^*-Cauchy, the last expression can be made less than $\varepsilon > 0$ for enough large $t \in T$.

We complete with the following conclusions.

COROLLARY 3.3. – *Let (E, τ) be a polar lcs with a weak basis (x_n). Assume that $\tau = \tau^*$. Then*

(a) *(x_n) is a weak Schauder basis.*
(b) *(x_n) is an «orthogonal» basic sequence in τ.*
(c) *If (E, τ) is weakly sequentially complete, then (E, τ) is complete.*
(d) *If (x_n) is a basis in (E, τ), it is an «orthogonal» basis.*

The next conclusion extends (to the p-adic case) Kalton's result of [6], Corollary 2, cf. also [7], [16].

COROLLARY 3.4. – *Let (E, τ) be a lcs with a topological basis. If $\tau = \tau^*$ and (E, τ) is sequentially complete, then (E, τ) is complete.*

PROOF. – By assumption (E, τ) is of countable type, so it is polar and has the (OP)-property. Hence the space (E, τ) is weakly sequentially complete and 3.3 applies.

3.4 suggests the following

EXAMPLE 3.5. – Let K_0 be the linear subspace of the product space K^K of the elements $x = (x_t)_{t \in K}$ such that $\{t \in K : x_t \neq 0\}^{\#} \leq \aleph_0$, where $A^{\#}$ denotes the cardinality of a set A. Then K_0 is a sequentially complete non-complete subspace of K^K of countable type (without a weak Schauder basis) and K_0 is l^{∞}-barrelled. Indeed, following as in [4], the space K_0 is the inductive limit of the Fréchet spaces K^J for all countable subsets J of K. Then the inductive limit space K_0 is l^{∞}-barrelled, cf. [10].

Applying 2.1, 3.2 and the closed graph theorem of [5], 2.2, one deduces also the following

COROLLARY 3.6. – *If* (E, τ) *is a polar Fréchet space with a weak basis, then* $\tau = \tau^*$.

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