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Intermediate Domains between a Domain and Some Intersection of Its Localizations.

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Sunto. – *In questo lavoro vengono studiati gli anelli compresi tra un dominio integro R ed un suo sopranello T, definito tramite una intersezione di localizzazioni di R. In particolare, vengono studiate le coppie (R, R_d) ed $(R, \tilde{R})$ dove $R_d = \cap \{R_M \mid M \in \mathrm{Max}(R), htM = \dim R\}$ ed $\tilde{R} = \cap \{R_M \mid M \in \mathrm{Max}(R), htM \geq 2\}$. Si dimostra che, se R è un dominio di Jaffard, allora $(R, R_d[n])$ è una coppia di Jaffard; tale risultato generalizza [5, Théorème 1.9]. Si dimostra anche che, se R è un S-dominio, allora $(R, \tilde{R})$ è una coppia residualmente algebrica (i.e. per ogni dominio intermedio S tra R e $\tilde{R}$ e per ogni ideale primo Q di S, il dominio quoziente S/Q è algebrico su $R/(Q \cap R)$). Inoltre, la coppia $(R, \tilde{R})$ è $\mathcal{P}$ se e soltanto se R è $\mathcal{P}$, per una qualche proprietà $\mathcal{P}$. Infine, viene data una risposta affermativa ad una questione sollevata in [7] da D. F. Anderson e D. N. Elabidine: se R è un dominio locale di Jaffard con ideale massimale M, allora il dominio $R^{\sharp} = \cap \{R_p \mid p \subset M\}$ è un dominio di Jaffard.*

Summary. – *In this paper, we deal with the study of intermediate domains between a domain R and a domain T such that T is an intersection of localizations of R, namely the pair (R, T). More precisely, we study the pair (R, R_d) and the pair $(R, \tilde{R})$, where $R_d = \cap \{R_M \mid M \in \mathrm{Max}(R)$ and $htM = \dim R\}$ and $\tilde{R} = \cap \{R_M \mid M \in \mathrm{Max}(R)$ and $htM \geq 2\}$. We prove that, if R is a Jaffard domain, then $(R, R_d[n])$ is a Jaffard pair, which generalize [5, Théorème 1.9]. We also show that if R is an S-domain, then $(R, \tilde{R})$ is a residually algebraic pair (that is for each intermediate domain S between R and $\tilde{R}$, if Q is a prime ideal of S, then S/Q is algebraic over $R/(Q \cap R)$). Moreover, the pair $(R, \tilde{R})$ is $\mathcal{P}$ if and only if R is $\mathcal{P}$, for some properties $\mathcal{P}$. Lastly, we answer in the positive a question raised in [7] by D. F. Anderson and D. N. Elabidine: we show that if R is a Jaffard local domain with maximal ideal M, then the domain $R^{\sharp} = \cap \{R_p \mid p \subset M\}$ is a Jaffard domain.*

0. – Introduction.

This paper is a sequel to [8]. As in [8], we adopt the conventions that each ring considered is commutative, with unit and an inclusion (extension) of rings signifies that the smaller ring is a subring of the larger and possesses the same multiplicative identity. Throughout this paper, $qf(R)$ denotes the quotient field of an integral domain R and for an extension of integral domains $R \subseteq S$, $\text{tr.deg}[S:R]$ is the transcendence degree of $qf(S)$ over $qf(R)$.

We recall that a ring R of finite Krull dimension is a *Jaffard ring* if its valuative dimension, (the limit of the sequence $(\dim R[X_1, \ldots, X_n] - n,\ n \in \mathbb{N})$), $\dim_v R$, is equal to $\dim R$. R is said to be a *locally Jaffard ring* (resp., a *totally Jaffard ring*) if R_p (resp., R/p) is a Jaffard ring (resp., a locally Jaffard ring) for each prime ideal p of R. For instance Prüfer domains and Noetherian domains are totally Jaffard domains. We assume familiarity with these concepts as in [3, 10].

When working with maximal ideals it will frequently be necesssary to distinguish those of rank 1 from those with higher rank; we will call the former «low maximals» and the latter «high maximals». In this paper, we study the domains contained in between R and $\widetilde{R}$, namely the pair $(R, \widetilde{R})$, where R is an integral domain with $\dim R \geqslant 2$, and $\widetilde{R} = \cap R_M$, where the intersection is taken over all the high maximal ideals M of R [20, Definition 2]. Recall from [8] that a pair of rings (R, S) where $R \subseteq S$ is said to be Jaffard (resp., locally Jaffard) if all intermediate rings between R and S are required to be Jaffard (resp., locally Jaffard). Much of the motivation for this paper comes from the result of A. R. Wadsworth [20, Theorem 8] which states that for any Noetherian domain R the pair $(R, \widetilde{R})$ is Noetherian. Our purpose is to determine necessary and sufficient conditions for the pair $(R, \widetilde{R})$ to provide a $\mathcal{P}$-pair (that is each domain in between R and $\widetilde{R}$ satisfies $\mathcal{P}$), where $\mathcal{P}$ denotes respectively Jaffard, locally, (totally) Jaffard, S-domain, (stably) strong S-domain. In [5, Théorème 1.9 (i)] A. Ayache and P.-J. Cahen proved that if R is a Jaffard domain, then $(R, R[n])$ is a Jaffard pair. In Section 1, this result is sharpened in Theorem 1.1, with the aid of the following result: If R is a Jaffard domain, then (R, R_d) is a Jaffard pair, where $R_d = \cap \{R_M \mid M \in \text{Max}(R) \text{ and } htM = \dim R\}$. Notice that always we have $\widetilde{R} \subseteq R_d$ and if $\dim R = 2$, then $\widetilde{R} = R_d$. We show also that $(R, \widetilde{R})$ is a $\mathcal{P}$-pair if and only if R is a $\mathcal{P}$ domain or also if and only if $\overline{R}$ is $\mathcal{P}$, where $\overline{R}$ denotes the integral closure of R in $\widetilde{R}$. In [20, Theorem 10], it was shown that if (R, S) is a Noetherian pair with $\dim R \geqslant 2$, then $S \subseteq \widetilde{R}^*$, where R^* denotes the integral closure of R in S. However, it is easy to use pullback constructions in order to produce an example of a $\mathcal{P}$-pair (R, S) for which the previous condition fails to hold. Hence, this section is ended with the study of pairs (R, S) where $S \subseteq \widetilde{R^*}$. We give necessary and sufficient conditions for such pairs to yield a $\mathcal{P}$-pair, where $\mathcal{P}$ ranges over the above cited properties. Section 2 explores consequences of Lemma 2.1 which presents a sufficient condition that the pair $(R, \widetilde{R})$ is residually algebraic, namely that R is an S-domain. Perhaps the most surprising of these consequences, Theorem 2.2, indicates, that if R is an integral S-domain which is integrally closed in $\widetilde{R}$, then for any ring T in between R and $\widetilde{R}$, $T = \widetilde{R} \cap (\cap \{R_p \mid p \in \mathcal{C}\})$ where $\mathcal{C}$ is a collection of low maximals of R. Section 3 deals with examples and counterexamples illustrating our results and showing their limits. In the Appendix, we answer in the positive a question raised by D. F. Anderson and D. Nour Elabidine [7, Question 3.2]. We show that if R is a Jaffard local domain, then $R^\sharp$ is a Jaffard domain.

Any unexplained terminology is standard, as in [15] and [16].

1. – Jaffard pairs.

Let $R \subset S$ be any extension of rings. Following [8], (R, S) is said to be a *Jaffard pair* (resp., a *locally Jaffard pair*) if any ring T in between R and S is Jaffard (resp., locally Jaffard). In [5] A. Ayache and P.-J. Cahen proved that if R is a Jaffard domain, then $(R, R[n])$ is a Jaffard pair. In what follows we generalize this result.

THEOREM 1.1. – *Let R be a Jaffard domain. Then:*

(i) $(R, R_d[n])$ *is a Jaffard pair;*

(ii) *for each ring T in between R and $R_d[n]$,* $\dim T = \dim R + \mathrm{tr.\,deg}[T : R]$.

To prove this theorem, we need the following lemmas.

LEMMA 1.2. – *Let R be an integral domain and $R_d = \cap\{R_M \mid M \in \mathrm{Max}(R)$ and $htM = \dim R\}$. If R is a Jaffard domain, then (R, R_d) is a Jaffard pair. Moreover, for each ring T in between R and R_d, $\dim T = \dim R$.*

PROOF. – Let T be a ring such that $R \subseteq T \subseteq R_d$. By definition of the valuative dimension, since T is an overring of R and on the other hand, since R is a Jaffard domain, we have

$$\dim_v T \leqslant \dim_v R = \dim R \tag{1}$$

Now let M be a maximal ideal of R such that $\dim R = htM$. We have the containments $R \subseteq T \subseteq R_d \subseteq R_M$. The extension $R \subseteq R_M$ satisfies INC, so does $T \subseteq R_M$. Thus $\dim R_M \leqslant \dim T$. Hence

$$\dim R \leqslant \dim T \tag{2}$$

From (1) and (2), it follows that $\dim_v T = \dim T = \dim R$. Thus T is a Jaffard domain. ∎

LEMMA 1.3. – *Let $R \subset S$ be an extension of integral domains and T a domain contained in between R and $S[n]$. If $qf(S)$ is a finite $qf(R)$- vectorial space, then $ht((X_1, \ldots, X_n)\, S[n] \cap T) = \mathrm{tr.deg}[T : R]$.*

PROOF. – The proof [5, Proposition 1.8] adapts easily. By localization of R in the multiplicative subset complement of $\{0\}$ in R, we can assume that R is a field. Under these assumptions, the domain $A = qf(S)[n]$ is a Noetherian finitely generated domain over T. Hence the extension $T \subseteq A$ satisfies the altitude inequality formula [5, Théorème 1.2]. In particular if $Q = (X_1, \ldots, X_n)A$, then we have:

$$(1) \qquad htQ + \text{tr.deg}[A/Q : T/P] \leqslant htP + \text{tr.deg}[A : T]$$

where $P = Q \cap T$. One check easily that Q is of height n, $T/P \subseteq A/Q$ is an algebraic extension and $\text{tr.deg}[A : T] = n - \text{tr.deg}[T : R]$. By (1), we conclude that $\text{tr.deg}[T : R] \leqslant htP$. On the other hand since T_P contains the field $qf(R)$, then we have $htP \leqslant \dim_v T_P = ht_v P \leqslant \text{tr.deg}[T : R]$ [4, Lemme 1.1]. ■

PROOF OF THEOREM 1.1. – We proceed as in [5]. Since R is a Jaffard domain, then so is R_d and $\dim R_d = \dim R$ [Lemma 1.2]. Thus $\dim_v R_d[n] = \dim_v R_d + n = \dim_v R + \text{tr.deg}[R_d[n] : R]$. Hence, $\dim_v T = \dim_v R + \text{tr.deg}[T : R] = \dim R + \text{tr.deg}[T : R]$ for each ring T in between R and $R_d[n]$ [8, Lemma 1.2]. To obtain the desired conclusion, it suffices to show that $\dim T \geqslant \dim R + \text{tr.deg}[T : R]$. Set $P = (X_1, \ldots, X_n) R_d[n] \cap T$. We have $R \subseteq T/P \subseteq R_d$ and $\dim T/P = \dim R$ [Lemma 1.2]. Hence $\dim T \geqslant \dim T/P + htP \geqslant \dim R + htP$. By Lemma 1.3, $htP = \text{tr.deg}[T : R]$. It follows that $\dim T \geqslant \dim_v T$ and clearly T is a Jaffard domain. ■

REMARK 1.4. – It may be that R_d is a Jaffard domain, while R is not (Example 3.1(b)).

Let R be a domain. Following [16], we say that R is an S-domain if, for each height 1 prime ideal P of R, the extended prime $P[X]$ has height 1 in the polynomial ring $R[X]$; and R is said to be a strong S-domain if R/P is an S-domain for each prime ideal P of R. Despite the above material, the class of strong S-domains is not very stable, for instance with respect to polynomial extension. Following [17], we say that R is stably strong S-domain if $R[X_1, \ldots, X_n]$ is a strong S-domain for each nonnegative integer n.

We introduce now a useful terminological device. If $\mathcal{P}$ is a property which may be possessed by ring (extensions), we say that $\mathcal{P}$ is a «good» property if it satisfies the following conditions:

(i) $\mathcal{P}$ is a local property: That is R is a ring satisfying $\mathcal{P}$ if and only if R_p satisfies $\mathcal{P}$ for each prime ideal p of R.

(ii) If R satisfies $\mathcal{P}$, then it is an S-domain.

(iii) If $R \subseteq S$ is an integral extension and S is $\mathcal{P}$, then so is R.

(iv) For a one dimensional ring, the properties $\mathcal{P}$ and S-domain are equivalent. For instance $\mathcal{P}$ = locally (totally) Jaffard, S-domain, (stably) strong S-domain.

In the following we determine necessary and sufficient conditions for the pair $(R, \widetilde{R})$ to provide a $\mathcal{P}$-pair, where $\mathcal{P}$ is a good property.

THEOREM 1.5. – *Let $\mathcal{P}$ be a good property and R an integral domain with* $\dim R \geqslant 2$, *then the following statements are equivalent.*

(*i*) $(R, \widetilde{R})$ *is a $\mathcal{P}$-pair;*

(*ii*) $\overline{R}$ *is $\mathcal{P}$, where $\overline{R}$ is the integral closure of R in $\widetilde{R}$.*

(*iii*) R *is $\mathcal{P}$.*

To prove this theorem we need the following lemma.

LEMMA 1.6. – *Let R be an integral domain with* $\dim R \geqslant 2$ *and T a domain in between R and $\widetilde{R}$. Then for each high maximal ideal M of R, $T_M = R_M = \widetilde{R}_M$.*

PROOF. – If T is an intermediate ring between R and $\widetilde{R}$ and M is a high maximal ideal of R, we have $R_M \subseteq T_M \subseteq \widetilde{R}_M \subseteq (R_M)_M = R_M$. Hence $T_M = R_M = \widetilde{R}_M$. ■

PROOF OF THEOREM 1.5. – (*i*) $\Rightarrow$ (*ii*) $\Rightarrow$ (*iii*). Trivial.

(*iii*) $\Rightarrow$ (*i*). Let T be a ring in between R and $\widetilde{R}$. Our task is to show that, for any nonzero prime ideal q of T, T_q is $\mathcal{P}$. Set $p = q \cap R$, there exists a maximal ideal M of R containing p. If $htM \geqslant 2$, then $p_M = q_M$ since $R_M = T_M$ [Lemma 1.6]. Thus $T_q = R_p$ which is $\mathcal{P}$.

If $htM \leqslant 1$, then $M = p$ since $p \neq (0)$. Hence $htp = 1$. But R is an S-domain. Thus $ht_v p = 1$. On the other hand the extension $R \subset T$ always satisfies the valuative altitude inequality formula [4, Théorème 1.3]. Therefore

$$1 \leqslant htq + \text{tr.deg}[T/q : R/p] \leqslant ht_v q + \text{tr.deg}[T/q : R/p] \leqslant ht_v p = 1 .$$

Hence $htq = ht_v q = 1$. Thus T_q is a one dimensional Jaffard domain, so an S-domain [3]. ■

REMARK 1.7. – We construct an example of a domain R such that R and R_d are $\mathcal{P}$ domains for $\mathcal{P}$ = locally (totally) Jaffard, strong S or stably strong S and the pair (R, R_d) is not $\mathcal{P}$ (Example 3.2).

As in Remark 1.4, if $\widetilde{R}$ is a $\mathcal{P}$ domain, R may be not $\mathcal{P}$ (Example 3.1. (b)).

COROLLARY 1.8. – *Let $\mathcal{P}$ be a good property and R an integral domain with* $\dim R \geqslant 2$ *such that R is $\mathcal{P}$ (resp., Jaffard). Let C be the set of all elements of R which are contained in no high maximal ideal. Then for any multiplicative subset N of R such that $N \subseteq C$, the pair $(R, N^{-1}R)$ is $\mathcal{P}$ (resp., Jaffard).*

PROOF. – Let M be a high maximal ideal of R, and let $\frac{r}{s}$ be an element of $N^{-1}R$. Since $N \subseteq C$, then $s \in R \setminus M$. Thus $N^{-1}R \subseteq R_M$. Hence $N^{-1}R \subseteq \widetilde{R}$ and $(R, N^{-1}R)$ is a $\mathcal{P}$-pair (resp., a Jaffard pair) by the previous theorem (resp., by Lemma 1.2). ■

REMARK 1.9. – We claim that there exists a domain R and a multiplicative subset N of R such that $(R, N^{-1}R)$ is a $\mathcal{P}$-pair where $\mathcal{P}$ is a good property and $N \nsubseteq C$, where C is the set of all elements of R which are contained in no high maximal ideal (see Example 3.3).

It was shown in [20] that if (R, S) is a Noetherian pair (that is any domain A, $R \subseteq A \subseteq S$ is Noetherian) and R^* is the integral closure of R in S then $S \subseteq \widetilde{R^*}$. Contrary to this fact, there exists a $\mathcal{P}$-pair (R, S) (for $\mathcal{P}$ = locally (totally) Jaffard, S-domain, (stably) strong S-domain) such that $S \nsubseteq \widetilde{R^*}$. For this, let K be a field, and X, Y two indeterminates over K and let $L = K(X, Y)$. Consider the domains $V_1 = K(X) + M_1$ and $V_2 = K + M_2$. V_1 is a rank 1 (discrete) valuation domain of L, with maximal ideal $M_1 = YK(X)[Y]_{(Y)}$ while V_2 is a rank 2 valuation domain of L, with maximal ideal $M_2 = XK[X]_{(X)} + YK(X)[Y]_{(Y)}$. V_1 and V_2 are incomparable. Thus $S = V_1 \cap V_2$ is a Prüfer semi local domain with $M_1' = M_1 \cap S$ and $M_2' = M_2 \cap S$ as maximal ideals. Let $R = \{x \in S \mid \overline{x} = x + M_1' \in K[X]\}$. According to [8, Theorem 2.2], (R, S) is a $\mathcal{P}$-pair. On the other hand, R is integrally closed in S. Hence $R^* = R$. Since any maximal ideal of R is of height $\geqslant 2$, then $\widetilde{R} = R = \widetilde{R^*}$. Thus $S \nsubseteq \widetilde{R^*}$.

THEOREM 1.10. – *Let $\mathcal{P}$ be a good property and let $R \subseteq S$ be an extension of integral domains such that $S \subseteq \widetilde{R^*}$ and $\dim R \geqslant 2$. Then, the following hold:*

(*a*) *If R^* is $\mathcal{P}$, then (R, S) is a $\mathcal{P}$-pair.*

(*b*) *If the integral closure R' of R has no low maximals and S is $\mathcal{P}$ (resp., Jaffard), then (R, S) is a $\mathcal{P}$-pair (resp., a Jaffard pair).*

PROOF. – (a) By Theorem 1.5, $(R^*, \widetilde{R^*})$ is a $\mathcal{P}$-pair. So is (R^*, S) since $S \subseteq \widetilde{R^*}$. Let T in between R and S. Then T^* (the integral closure of T in S) is contained between R^* and S and $T \subseteq T^*$ is an integral extension. Therefore T is $\mathcal{P}$.

(b) Let $(R^*)'$ the integral closure of R^* and R' that of R. Suppose that R^* has a low maximal N. Since $(R^*)'$ is integral over R^* there is a prime p' of $(R^*)'$ lying over N, and p' must also be a low maximal. By the going-down theorem [19, 10.13], $p' \cap R'$ is a low maximal of R', contradicting the hypothesis. Thus R^* can have no low maximal. It then follows that $\widetilde{R^*} = R^*$. Thus $S = R^*$. We complete the proof by using assertion (a). ■

REMARK 1.11. – In Theorem 1.10, we claim that if we assume only that R is $\mathcal{P}$, the pair (R, S) may be not $\mathcal{P}$, for $\mathcal{P}$= locally Jaffard, totally Jaffard. For this, consider an integral extension $R \subset S$ such that R is $\mathcal{P}$ and S is not $\mathcal{P}$. Clearly, we have $S \subseteq \widetilde{R} = \widetilde{R^*}$ while (R, S) is not a $\mathcal{P}$-pair.

2. – Residually algebraic pairs.

Recall from [12] that a ring extension $R \subseteq S$ of integral domains is said to be residually algebraic if for any prime ideal Q of S, S/Q is algebraic over $R/(Q \cap R)$. A pair of rings (R, S) is said to be *residually algebraic* if for any intermediate ring T in between R and S, the extension $R \subseteq T$ is residually algebraic [6].

In [11], E. Davis proved that if R in an integrally closed Noetherian domain, then for each ring T in between R and $\widetilde{R}$, $T = R \cap (\cap \{R_p \mid p \in \mathcal{C}\})$ where $\mathcal{C}$ is a collection of low maximals of R. In Theorem 2.2, we show that it is enough to suppose that R is an S-domain integrally closed in $\widetilde{R}$. But first a key lemma.

LEMMA 2.1. – *If R is an integral S-domain with* $\dim R \geqslant 2$, *then* $(R, \widetilde{R})$ *is a residually algebraic pair.*

PROOF. – By [6, Proposition 2.4] and Lemma 1.6, it suffices to show that $(R_M, \widetilde{R}_M)$ is a residually algebraic pair for each low maximal ideal M of R. Let T_1 be a ring such that $R_M \subseteq T_1 \subseteq \widetilde{R}_M$, then $T_1 = T_M$, where T is such that $R \subseteq T \subseteq \widetilde{R}$. Let q_1 be a nonzero prime ideal of T_1, then $q_1 = q_M$, where $q \in \mathrm{Spec}(T)$. We have $q_M \cap R_M = MR_M$. By the valuative altitude inequality formula, we deduce that $ht_v q_M + \mathrm{tr.deg}[T_M/q_M \colon R_M/MR_M] \leqslant ht_v MR_M + \mathrm{tr.deg}[T_M \colon R_M] = 1$. Hence $\mathrm{tr.deg}[T/q : R(q \cap R)] = \mathrm{tr.deg}[T/q : R/M] = 0$. ∎

THEOREM 2.2. – *Let R be an integral S-domain with* $\dim R \geqslant 2$ *such that R is integrally closed in $\widetilde{R}$, then for each ring T in between R and $\widetilde{R}$, $T = \widetilde{R} \cap (\cap \{R_p \mid p \in \mathcal{C}\})$ where $\mathcal{C}$ is a collection of low maximals of R.*

PROOF. – By Lemma 2.1, the pair $(R, \widetilde{R})$ is residually algebraic. Since R is integrally closed in $\widetilde{R}$, then for any ring T in between R and $\widetilde{R}$, $T = \cap_{i \in I} T_{M_i}$ where $\{M_i,\, i \in I\}$ is the set of maximal ideals of R [6, Lemma 3.1]. Denoting by $I_1 = \{i \in I \mid htM_i \geqslant 2\}$ and $I_2 = \{i \in I \mid htM_i = 1\}$. Then $T = (\cap_{i \in I_1} T_{M_i}) \cap (\cap_{i \in I_2} T_{M_i})$. But for each $i \in I_1$, we have $T_{M_i} = R_{M_i}$ [Lemma 1.6]. By [6, Theorem 2.5], for each $i \in I_2$, there exists a divided ideal Q_i of R contained in M_i such that $T_{M_i} = R_{Q_i}$. Since $htM_i = 1$, then either $Q_i = M_i$ or $Q_i = (0)$. Therefore $T = \widetilde{R} \cap (\cap \{R_{Q_i} \mid Q_i \subseteq M_i,\, i \in I_2\})$. ∎

REMARK 2.3. – If R is not an S-domain, then (R, R) may be not a residually algebraic pair (Example 3.1(a)).

If R is an S-domain, even stably strong S, the pair (R, R_d) may be not residually algebraic (Example 3.2).

The following proposition gives another kind of residually algebraic pairs. Recall that for a given ring R, the Nagata ring $R(X)$ is equal to $N^{-1}R[X]$, where $N = R[X] \setminus \cup \{M[X] \mid M \in Max(R)\}$ [18].

PROPOSITION 2.4. – *Let R be an integral S-domain with* $\dim R \geq 2$, *then* $(R(X), \widetilde{R}(X))$ *is a residually algebraic pair.*

PROOF. – First we prove that $\widetilde{R}(X) \subseteq \widetilde{R(X)}$. Indeed we have $\widetilde{R(X)} = \cap \{R(X)_{M(X)} \mid M \in \mathrm{Max}(R)$ and $htM(X) \geq 2\}$. Since R is an S-domain, then $htM(X) \geq 2$ if and only if $htM \geq 2$. Hence $\widetilde{R(X)} = \cap \{R(X)_{M(X)} \mid M \in \mathrm{Max}(R)$ and $htM \geq 2\}$. We have $\widetilde{R}(X) \subseteq R(X)_{M(X)} = R[X]_{M[X]}$ for each high maximal ideal M of R. Indeed, let M be a high maximal ideal of R and $f_1 \in \widetilde{R}(X)$. Then $f_1 = \frac{g_1}{h_1}$ with $g_1 \in \widetilde{R}[X]$, and $h_1 \in \widetilde{R}[X] \setminus \cup \{m[X] \mid m \in \mathrm{Max}(\widetilde{R})\}$. But $\widetilde{R}[X] = \cap \{R_M[X] \mid M \in \mathrm{Max}(R)$ and $htM \geq 2\}$. Hence $g_1 = \frac{1}{s}g$, where $g \in R[X]$ and $s \in R \setminus M$. Moreover, $h_1 = \frac{1}{s'}h$, where $h \in R[X]$ and $s' \in R \setminus M$. We verify that $h \notin M[X]$. Indeed, let $q = MR_M \cap \widetilde{R}$. Then $htq = htM \geq 2$ and $q \in \mathrm{Max}\, \widetilde{R}$. Therefore $h_1 \notin MR_M[X]$. Thus $h \notin M[X]$. Now by Lemma 2.1, the pair $(R(X), \widetilde{R(X)})$ is residually algebraic since $R(X)$ is always an S-domain with $\dim R(X) \geq 2$. Thus $(R(X), \widetilde{R}(X))$ is a residually algebraic pair because $\widetilde{R}(X) \subseteq \widetilde{R(X)}$. ∎

For the polynomial case, the previous proposition fails to be true. Indeed, we establish that $(R[X], \widetilde{R}[X])$ is a residually algebraic pair if and only if $R \subseteq \widetilde{R}$ is an integral extension. More generally, we have the following.

PROPOSITION 2.5. – *An extension $R \subseteq S$ of integral domains is integral if and only if $(R[X], S[X])$ is a residually algebraic pair.*

PROOF. – Of course the «only if» half is immediate, since $R[X] \subseteq S[X]$ is an integral extension. For the «if» half, we can assume that R is local and integrally closed in S. Then by [15, Theorem 10.7], $R[X]$ is integrally closed in $S[X]$. Our task is to show that $S = R$. Consider the ring $T = R + XS[X]$, we have $R[X] \subseteq T \subseteq S[X]$. Denote by M the maximal ideal of R, then $Q = M + XS[X]$ is a prime ideal of T. Let $P = Q \cap R[X]$, we have $P = M + XR[X]$. Let $a \in S$, the element $aX \in T_Q$. By [6, Theorem 2.10], $T_Q = R[X]_P$. Thus there exist $f \in R[X]$ and $g \in R[X] \setminus P$ such that $\frac{f}{g} = aX$. Write $f = \sum_{i=0}^{n} a_i X^i$ and $g = \sum_{j=0}^{m} b_j X^j$. The equality $f = aXg$ shows that $n = m + 1$ and $a_1 = ab_0$. But

$b_0 \in R \backslash M$. Hence b_0 is a unit in R. Therefore $a = a_1 b_0^{-1} \in R$. Hence $S = R$. ■

Now, we turn our attention to compute the number of domains between a given domain R and $\widetilde{R}$. First, recall from [8], that for an extension of integral domains $R \subseteq S$, if we set $[R, S]$ to be the set of all intermediate domains between R and S, then $|[R, S]|$ denotes the cardinal of the set $[R, S]$.

PROPOSITION 2.6. – *Let R be a semilocal integral S-domain with maximal ideals $M_1, \ldots, M_r$ such that* $\dim R \geq 2$ *and R is integrally closed in $\widetilde{R}$. Then* $|[R, \widetilde{R}]| \leq 2^s$ *where* $s = \text{card}\{M_i \mid htM_i = 1\}$.

PROOF. – By Lemma 2.1, $(R, \widetilde{R})$ is a residually algebraic pair. Thus for each $i \in \{1, \ldots, r\}$, there exists a prime ideal q_i of R such that $q_i \subseteq M_i$ and $R_{q_i} = \widetilde{R}_{M_i}$ [6, Theorem 2.10]. Set $h_i = ht(M_i/q_i)$, $i \in \{1, \ldots, r\}$. If $htM_i \geq 2$, then $\widetilde{R}_{M_i} = R_{M_i}$ [Lemma 1.6]. Thus $R_{M_i} = R_{q_i}$ which gives $q_i = M_i$. Therefore $h_i = 0$. Evidently, if $htM_i = 1$, then $h_i \leq 1$. By [6, Theorem 3.3 (i)], $|[R, \widetilde{R}]| \leq \prod_{i=1}^{r} \cdot (h_i + 1) \leq 2^s$ where $s = \text{card}\{M_i \mid htM_i = 1\}$. ■

REMARK 2.7. – Notice that if there exists a high maximal ideal M of R which is not Jaffard (In particular, if R is not Jaffard), then by Lemma 1.6, for each T in between R and $\widetilde{R}$, $T_M = R_M$ which is not Jaffard. Thus there is no locally Jaffard domain in between R and $\widetilde{R}$. Example 3.4 shows that the last result holds, even if R is a Jaffard domain.

3. – Examples and counterexamples.

This section is concerned with examples showing limits of the results established in the previous sections. First, recall some terminology from [3], [9] and [10]. Specifically, let S be an integral domain, I a nonzero ideal of S, $\varphi : S \to S/I$ the natural epimorphism, D a subring of S/I and $R = \varphi^{-1}(D)$ the pullback of the following diagram:

$$\begin{array}{ccc} R & \to & D \\ \downarrow & & \downarrow \\ S & \to & S/I \end{array}$$

We say that R is the ring of the (S, I, D) construction ([9]).

We next recall a few wellknown properties about pullbacks to be used in examples throughout this paper (they may easily be proved directly, or see [3], [9], [10] and [14]). First, I is a common ideal to both R and S, and $R/I \cong D$. For each $p \in Spec(R)$ with $I \not\subseteq p$, there is a unique $q \in \text{Spec}(S)$ such that $q \cap R = p$. If in addition $I \in \text{Max}(S)$ and $p \in \text{Spec}(R)$ such that $I \subseteq p$, then there is a unique $q \in \operatorname{Spec}(D)$ such that $\varphi^{-1}(q) = p$; and moreover $\varphi^{-1}(D_q) = R_p$. R is local if and only if D is local and $I \subseteq RadS$ (the Jacobson radical of S).

As stated before, if we leave out the assumption «R is an S-domain» in Lemma 2.1, the following example shows, among other facts, that $(R, \widetilde{R})$ may be not residually algebraic.

EXAMPLE 3.1. – This example provides:

(a) A domain R such that $(R, \widetilde{R})$ is not a residually algebraic pair.

(b) A non Jaffard domain T but such that T_d is Jaffard.

Let K be a field, $S_1 = K[X, Y]$ the polynomial ring in two indeterminates over K, $M_1 = XS_1$ and $M_2 = (X-1, Y-1)S_1$. If N_1 is the multiplicative subset complement of $M_1 \cup M_2$, then $S_2 = N_1^{-1}S_1$ is a two-dimensional semilocal domain with two maximal ideals, $M_1' = N_1^{-1}M_1$ and $M_2' = N_1^{-1}M_2$ such that $htM_1' = 1$, $htM_2' = 2$ and $S_2/M_1' \cong K(Y)$. Let R be the ring of the (S_2, M_1', K) construction. The rings R and S_2 share the ideal M_1'. We have $\dim R = \dim_v R = 2$, $\dim R_{M_1'} = 1$ and $\dim_v R_{M_1'} = 2$. In this example $\widetilde{R} = R_{M_2' \cap R} = (S_2)_{M_2'}$. Since $S_2 \in [R, \widetilde{R}]$ and $R \subseteq S_2$ is not a residually algebraic extension, then the pair $(R, \widetilde{R})$ is not residually algebraic. On the other hand R is a Jaffard domain. Hence the condition «R is an S-domain» in Lemma 2.1 can not be omitted or replaced by «R is a Jaffard domain». Now consider the multiplicative subset of R, $N = R \backslash (M_2' \cap R)$. Then the set of all elements of R which are contained in no high maximal ideal is C equal to N. The pair $(R, N^{-1}R) = (R, \widetilde{R})$ is not $\mathcal{P}$ because R is not a $\mathcal{P}$-domain (for $\mathcal{P}$ = locally (totally) Jaffard, stably strong S). Thus the condition «R is $\mathcal{P}$» is essential in Corollary 1.8. By [8, Proposition 1.3], (R, S_2) is a Jaffard pair. Since $R/M_1' \cong K$ is integrally closed in $S_2/M_1' \cong K(Y)$, then the integral closure of R in S_2 is equal to R. Also we have $S_2 \subsetneq (S_2)_{M_2'} = \widetilde{R^*} = \widetilde{R}$. But (R, S_2) is not a $\mathcal{P}$-pair. Notice that R is integrally closed and has M_1' as a low maximal, while S_2 is not integral over R. This shows that the condition «R' has no low maximals» in Theorem 1.10 (b) can not be deleted.

(b) We assume now that K is of the form $k(x_1, x_2, \dots)$ where k is a field and $x_1, x_2, \dots,$ are countably many indeterminates over k. We have $S_2/M_1' \cong K(Y) \cong k(Y, x_1, x_2, \dots)$. Consider the k-monomorphism

$$\theta : K = k(x_1, x_2, \dots) \rightarrow S_2/M_1' = k(Y, x_1, x_2, \dots) = k(Z_1, Z_2, \dots).$$

$$x_n \rightarrow Z_{n+2}.$$

Let $D = \theta(K)$. Then the ring T of the (S_2, M_1', D) construction is semilocal with maximal ideals M_1' and $M_2' \cap T$. We have $ht(M_2' \cap T) = ht_v(M_2' \cap T) = 2$, $ht_T M_1' = 1$ and $\dim_v T\, M_1' = 3$. Thus $\dim T = 2 < \dim_v T = 3$. Hence T is not a Jaffard domain, while $T_d = T_{M_2' \cap T}$ is a Jaffard domain. Notice that T is not an

S-domain, a fortiori T is not a $\mathcal{P}$ domain for $\mathcal{P}$ = locally (totally) Jaffard, stably (strong) S-domain. However T_d is $\mathcal{P}$, since $T_d = (S_2)_{M_2'}$.

The next example supplies a stably strong S-domain R but such that the pair (R, R_d) is neither residually algebraic nor locally Jaffard.

EXAMPLE 3.2. – Let K be a field and S be a semilocal Prüfer domain with two maximal ideals M and N such that $htM = 1$, $htN \geqslant 4$ and S/M is isomorphic to the field $K_1 = K(X, Y)$ where X, Y are two indeterminates over K. Let $D = K[X, Y]$, $T = K[X, Y/(X+1)]$, $Q = (X+1)T$ and consider the ring D_1 of the (T_Q, QT_Q, K) construction. Notice that $D_1 = K[X, Y] + QT_Q$. Thus D_1 is an overring of D which is not Jaffard (since $\dim D_1 = 1 < \dim_v D_1 = 2$ [3]). Denoting by $\varphi : S \to S/M$ the natural epimorphism and let $R = \varphi^{-1}(D)$ and $R_1 = \varphi^{-1}(D_1)$. We have $R_d = R_{N \cap R} = S_N$, since N does not contain M. R and R_d are stably strong S-domains, so $\mathcal{P}$ domains. However (R, R_d) is not a locally Jaffard pair. Hence (R, R_d) is not a $\mathcal{P}$-pair, since R_1 is in between R and R_d and R_1 is not a locally Jaffard domain. Since D' is not a Prüfer domain, then by [6, Proposition 5.1] the pair (R, S) is not residually algebraic, a fortiori the pair (R, R_d) is not residually algebraic.

EXAMPLE 3.3. – Consider two incomparables valuation domains V and W such that $\dim V$ and $\dim W$ are greater than 2. Let M_1 and M_2 respectively the maximal ideals of V respectively W. The ring $R = V \cap W$ is Prüfer with $M_1' = M_1 \cap R$ and $M_2' = M_2 \cap R$ as maximal ideals. Denoting by $N = R \backslash M_1'$ a multiplicative subset of R. It is obvious that the pair $(R, N^{-1}R)$ is $\mathcal{P}$ for $\mathcal{P}$ = Jaffard, locally Jaffard, totally Jaffard and stably strong S, whereas $N \not\subseteq C$ since $C = R \backslash (M_1' \cup M_2')$.

EXAMPLE 3.4. – We construct an integral domain R such that $(R, \widetilde{R})$ is a Jaffard pair and $|[R, \widetilde{R}]_{l.J}| = 0$.

Let K be a field, X, Y, Z three indeterminates over K. $S_0 = K[X, Y, Z]$, $M = XS_0$, $N = (X-1, Y)S_0$ and $N' = (X-1, Y-1, Z-1)S_0$. Consider the multiplicative subset of S_0: $N_0 = S_0 \backslash (M \cup N \cup N')$. Let $S = N_0^{-1}S_0$. Set $M' = N_0^{-1}M$, $M'' = N0^{-1}N$, $N'' = N_0^{-1}N'$ and R the ring of the (S, M'', K) construction. One shows easily that R is a 3-dimensional Jaffard domain. Hence $(R, \widetilde{R})$ is a Jaffard pair [Theorem 1.1]. Notice that R is an integrally closed S-domain and M'' is a high maximal ideal of R such that $htM'' = 2$ and $ht_v M'' = 3$. By Remark 2.7, there is no locally Jaffard domain in between R and $\widetilde{R}$.

We close this section by an example showing that if $(\widetilde{R})'$ is a Prüfer domain, then R' may be not.

EXAMPLE 3.5. – Let K be a field, X and Y two indeterminates over K. Set $V_1 = K + M_1$ and $V_2 = K + M_2$, where $M_1 = YK(X)[Y]_{(Y)}$ and $M_2 = XK[X]_{(X)} + (Y+1)K(X)[Y]_{(Y+1)}$. Consider the ring $R = V_1 \cap V_2$. R is semilocal with $M_1' =$

$M_1 \cap T$ and $M_2' = M_2 \cap T$ as maximal ideals. Moreover we have $R_{M_1'} = V_1$ and $R_{M_2'} = V_2$. Thus $\widetilde{R} = V_2$ is a valuation domain. Hence $(\widetilde{R})'$ is a Prüfer domain. The ring $R_{M_1'} = V_1$ is a one dimensional non Jaffard domain [3, Proposition 2.5], hence $R_{M_1'}$ is not an S-domain [3, Theorem 1.10] and so is R. Therefore R' is not a Prüfer domain.

Appendix.

Let R be an integral domain with quotient field K.

$R^\sharp = \cap\{R_x \,|\, x$ is a nonzero non unit of $R\}$. If R is not local, then $R^\sharp = R$; and $R^\sharp = K$ when R is a 1-dimensional local integral domain [2, Theorem 1.2]. Hence, we will be interested in the case where R is a local integral domain with maximal ideal M and $\dim R \geqslant 2$. In the local case, $R^\sharp = \cap\{R_p \,|\, p \in \mathrm{Spec}(R) \setminus \{M\}\}$ [2, Proposition 1.3]. We pause to answer a question which was left open in [7]: If R is a Jaffard domain is $R^\sharp$ a Jaffard domain? However, it is still open the question: If R is a Jaffard domain is $(R, R^\sharp)$ a Jaffard pair?

THEOREM. – *Let R be a local Jaffard domain, then $R^\sharp$ is a Jaffard domain.*

PROOF. – By [7, Corollary 2.2 and Proposition 3.1], we have $\dim R - 1 \leqslant \dim R^\sharp \leqslant \dim R$. If $\dim R^\sharp = \dim R$, then $\dim_v R^\sharp \leqslant \dim_v R = \dim R = \dim R^\sharp$. Thus $R^\sharp$ is a Jaffard domain. If $\dim R^\sharp = \dim R - 1$, then we get $MR^\sharp = R^\sharp$ since if not (that is if $MR^\sharp \neq R^\sharp$), then $\dim R^\sharp \geqslant \dim R$ by [2, page 27]. In this case $R^\sharp{}_q = R_{q \cap R}$ for each $q \in \mathrm{Spec}(R^\sharp)$. Hence $\dim_v R^\sharp{}_q = ht_v q = ht_v(q \cap R) \leqslant \dim_v R - 1 = \dim R - 1 = \dim R^\sharp$. Therefore

$$\dim_v R^\sharp = \sup\{ht_v q \,|\, q \in \mathrm{Spec}(R^\sharp)\} \leqslant \dim R^\sharp.$$

Thus $R^\sharp$ is a Jaffard domain. ∎

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