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J. Chabrowski

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Mean Curvature and Least Energy Solutions for the Critical Neumann Problem with Weight.

J. CHABROWSKI

Sunto. – *In questo articolo consideriamo il problema di Neumann che richiede un'esponente di Sobolev critico. Noi investighiamo l'effetto combinato del coefficiente della non linearità critica e della curvatura media della frontiera sull'esistenza e sull'inesistenza di soluzioni.*

Summary. – *In this paper we consider the Neumann problem involving a critical Sobolev exponent. We investigate a combined effect of the coefficient of the critical Sobolev nonlinearity and the mean curvature on the existence and nonexistence of solutions.*

1. – Introduction.

In this paper we investigate the existence of solutions of the following nonlinear Neumann problem:

$$\begin{cases} -\Delta u + \lambda u = Q(x)\, u^{2^\star - 1} & \text{in } \Omega\,, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega,\ u > 0 \text{ on } \Omega\,, \end{cases} \tag{1}$$

where $\lambda > 0$ is a parameter, ν is the unit outward normal at the boundary $\partial\Omega$. We assume that $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$. The coefficient Q is Hölder continuous on $\overline{\Omega}$ and $Q(x) > 0$ for $x \in \overline{\Omega}$. Further conditions guaranteeing the solvability of problem (1) will be formulated later. Here $2^\star$ denotes the critical Sobolev exponent, that is, $2^\star = \dfrac{2N}{N-2}$, $N \geq 3$.

By $H^1(\Omega)$ we denote the usual Sobolev space equipped with norm

$$\|u\|^2 = \int_\Omega (|\nabla u|^2 + u^2)\, dx\,.$$

Solutions of (1) will be found as minimizers of the variational problem

$$S_\lambda = \inf\left\{ \int_\Omega (|\nabla u|^2 + \lambda u^2)\, dx;\ u \in H^1(\Omega), \int_\Omega Q(x)\,|u|^{2^\star}\, dx = 1 \right\}.$$

A suitable multiple of a minimizer for S_λ is a solution of problem (1). These solutions are called the least energy solutions. Let $Q_m = \max_{x \in \partial\Omega} Q(x)$ and $Q_M = \max_{x \in \overline{\Omega}} Q(x)$. In a recent paper [8] we investigated problem (1) in two cases: (i) $Q_M \leqslant 2^{2/(N-2)} Q_m$ and (ii) $Q_M > 2^{2/(N-2)} Q_m$. In case (i) we proved that if

$$S_\lambda < \frac{S}{2^{2/N} Q_m^{(N-2)/N}}\,, \tag{2}$$

then S_λ has a minimizer. Here S denotes the best Sobolev constant

$$S = \inf\left\{ \int_{\mathbb{R}^N} |\nabla u|^2\, dx;\ u \in D^{1,2}(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^{2^\star}\, dx = 1 \right\},$$

where $D^{1,2}(\mathbb{R}^N)$ is the Sobolev space obtained as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|^2_{D^{1,2}} = \int_{\mathbb{R}^N} |\nabla u|^2\, dx\,.$$

In the second case (ii) we showed that if

$$S_\lambda < \frac{S}{Q_M^{(N-2)/N}}\,, \tag{3}$$

then S_λ has a minimizer. Obviously both conditions (2) and (3) are satisfied for small $\lambda > 0$. Under some additional assumption on Q the inequality (2) is satisfied for every $\lambda > 0$. Namely, (2) is satisfied for every $\lambda > 0$ if

(A) $|Q(x) - Q(x_0)| = o(|x - x_0|)$ for some $x_0 \in \partial\Omega$, with $Q(x_0) = Q_m$ and $H(x_0) > 0$.

Here $H(y)$ denotes the mean curvature of $\partial\Omega$ at $y \in \partial\Omega$ related to the inner normal to $\partial\Omega$ at y. In case (ii) we showed that there exists $0 < \Lambda < \infty$ such (3) holds for $\lambda \in (0, \Lambda)$ and $S_\lambda = \frac{S}{Q_M^{(N-2)/N}}$ for $\lambda \geqslant \Lambda$. Moreover, we showed that the least energy solution exists also for every $\lambda = \Lambda$ and there are no least energy solutions for $\lambda > \Lambda$.

This contrasts with the case where $Q(x) \equiv 1$. In this case problem (1) always has a nonconstant solution for large $\lambda > 0$ [3], [6], [21]. Further extensions of these results can be found in the papers [14], [22], [23], [24], [25], [26] and [27], where the existence of symmetric and multi-peak solutions have been investigated. The main ingredient in these papers lies in the fact that a bounded and smooth domain in $\mathbb{R}^N$ has points on the boundary with the positive mean curvature. This allows us to show with the aid of instantons that $S_\lambda < \frac{S}{2^{2/N}}$ for every $\lambda > 0$. This inequality ensures the existence of a minimizer for S_λ. In the case of a nonconstant coefficient Q, a new phenomenon arises, namely, the effect of the shape of the graph of this coefficient on the existence of the least energy solutions.

The main purpose of this article is to show that if the assumption (A) in the case (i) is not satisfied, then the least energy solutions exist only for λ belonging to an interval $(0, \bar{\lambda}]$ and $S_\lambda = \frac{S}{2^{2/N} Q_m^{(N-2)/N}}$ for $\lambda \geqslant \bar{\lambda}$. The interesting feature of Theorems 3.4 and 3.5 (in Section 3) is that they give the existence of a minimizer up to the limiting value for λ, which means that the optimal Sobolev inequality has an extremal. This is a result of an interaction of the mean curvature of $\partial\Omega$ and the shape of the graph of the coefficient Q, in the case where $Q|_{\partial\Omega}$ attains its maxiumum Q_m on $\partial\Omega$ only at points where the mean curvature changes the sign. Obviously, this phenomenon does not occur if the coefficient Q is constant [9], [11]. For the existence of extremal functions for optimal Sobolev inequalities we refer to the papers [10] and [15], where the existence of extremals is related to the geometry of manifolds. Throughout this paper we denote strong convergence by «$\rightarrow$» and the weak convergence by «$\rightharpoonup$». The norms in the Lebesgue space $L^q(\Omega)$ are denoted by $\|\cdot\cdot\|_q$.

2. – Estimates of the energy of instantons.

The best Sobolev constant S is achieved by

$$U(x) = \left[\frac{N(N-2)}{N(N-2) + |x|^2} \right]^{(N-2)/2}.$$

The function U, called an instanton, satisfies the equation

$$-\Delta U = U^{2^\star - 1} \text{ in } \mathbb{R}^N.$$

We also have $\int_{\mathbb{R}^N} |\nabla U|^2 \, dx = \int_{\mathbb{R}^N} U^{2^\star} \, dx = S^{N/2}$. We set

$$U_{\varepsilon,y}(x) = \varepsilon^{-(N-2)/2} U\left(\frac{x-y}{\varepsilon} \right)$$

for $y \in \mathbb{R}^N$, $\varepsilon > 0$. If $y = 0$ we write $U_\varepsilon = U_{\varepsilon,0}$. Let

$$J_\lambda(u) = \int_\Omega (|\nabla u|^2 + \lambda u^2) \, dx$$

for $u \in H^1(\Omega)$. In what follows we denote by $H(y)$ the mean curvature of $\partial\Omega$ at $y \in \Omega$ related to the inner normal to $\partial\Omega$ at y.

PROPOSITION 2.1. – *Suppose that $N \geqslant 5$. Let $y \in \partial\Omega$ be such that $H(y) < 0$. Then there exist constants $\alpha < 0$, $\varepsilon_0 > 0$ and $C > 0$ such that*

$$J_\lambda\left(\frac{U_{\varepsilon,y}}{\|U_{\varepsilon,y}\|_{2^\star}} \right) \geqslant \frac{S}{2^{2/N}} - \alpha H(y)\, \varepsilon + C\lambda\varepsilon^2 + O(\varepsilon^2)$$

for all $0 < \varepsilon \leqslant \varepsilon_0$. If $H(y) < 0$ for $y \in B(y_0, \varrho) \cap \partial\Omega$, $y_0 \in \partial\Omega$, for some $\varrho > 0$, then the constants α and C can be chosen independently of y.

PROOF. – A related estimate from above in the case $H(y) > 0$ can be found in the paper [1]. This estimate was proved under the assumption that all principal curvatures are positive. We follow some ideas from the paper [21]. Without loss of generality we may assume that $y = 0$ and that near 0 the boundary $\partial\Omega$ is represented, changing coordinates if needed, by

$$x_N = h(x') = \frac{1}{2} \sum_{i=1}^{N-1} \alpha_i x_i^2 + O(|x'|^3)$$

for $x' = (x_1, \ldots, x_{N-1}) \in D(0, a)$ for some $a > 0$, where $D(0, a) = B(0, a) \cap (x_N = 0)$. Then the mean curvature $H(0)$ is given by $H(0) = \frac{1}{N-1} \sum_{i=1}^{N-1} \alpha_i$. Let $g(x') = \frac{1}{2} \sum_{i=1}^{N-1} \alpha_i x_i^2$. Then

$$\int_\Omega |\nabla U_\varepsilon|^2 \, dx = \frac{1}{2} \int_{\mathbb{R}^N_+} |\nabla U_\varepsilon|^2 \, dx - \int_{D(0,a) \cap g(x') > 0} dx' \int_0^{g(x')} |\nabla U_\varepsilon|^2 \, dx_N + \int_{D(0,a) \cap g(x') < 0} dx' \int_{g(x')}^0 |\nabla U_\varepsilon|^2 \, dx_N \tag{4}$$

$$+\int_{D(0,\,a)} dx' \int_{g(x')}^{h(x')} |\nabla U_\varepsilon|^2\, dx_N + o(\varepsilon^{N-2})$$

$$= \frac{1}{2}K_1 - \int_{\mathbb{R}^{N-1}\cap g(x')>0} dx' \int_0^{g(x')} |\nabla U_\varepsilon|^2\, dx_N$$

$$+ \int_{\mathbb{R}^{N-1}\cap g(x')<0} dx' \int_{g(x')}^{0} |\nabla U_\varepsilon|^2\, dx_N$$

$$+\int_{D(0,\,a)} dx' \int_{g(x')}^{h(x')} |\nabla U_\varepsilon|^2\, dx_N + o(\varepsilon^{N-2}),$$

where $K_1 = \int_{\mathbb{R}^N} |\nabla U|^2\, dx$. We now estimate the last integral on the right side of (4). Since $N \geqslant 5$, we get

(5) $$\left| \int_{D(0,\,a)} dx' \int_{g(x')}^{h(x')} |\nabla U_\varepsilon|^2\, dx_N \right| \leqslant$$

$$\varepsilon^{N-2} a_N \int_{\mathbb{R}^{N-1}} \left[\frac{1}{[\varepsilon^2 N(N-2) + |x'|^2]^{N-1}} + \frac{\varepsilon^2 (N-2)\, N}{[\varepsilon^2 N(N-2) + |x'|^2]^{N}} \right] \times$$

$$|g(x') - h(x')|\, dx' = 0(\varepsilon^2),$$

where $a_N = (N-2)^2[N(N-2)]^{N-2}$. Setting

$$I^-(\varepsilon) = \int_{\mathbb{R}^{N-1}\cap g(x')<0} dx' \int_{g(x')}^{0} |\nabla U_\varepsilon(x)|^2\, dx_N$$

and

$$I^+(\varepsilon) = \int_{\mathbb{R}^{N-1}\cap g(x')>0} dx' \int_0^{g(x')} |\nabla U_\varepsilon(x)|^2\, dx_N,$$

we rewrite (2) as

(6) $$K_1(\varepsilon) = \int_\Omega |\nabla U_\varepsilon|^2\, dx = \frac{1}{2}K_1 + I^-(\varepsilon) - I^+(\varepsilon) + O(\varepsilon^2).$$

In a similar way we check that

$$K_2(\varepsilon) = \int_\Omega U_\varepsilon^{2^\star}\,dx = \frac{1}{2}\int_{\mathbb{R}^N_+} U^{2^\star}\,dx \tag{7}$$

$$- \int_{D(0,\,a)\,\cap\, g(x')>0} dx' \int_0^{g(x')} U_\varepsilon^{2^\star}\,dx_N$$

$$+ \int_{D(0,\,a)\,\cap\, g(x')<0} dx' \int_{g(x')}^0 U_\varepsilon^{2^\star}\,dx_N + \int_{D(0,\,a)} dx' \int_{g(x')}^{h(x')} U_\varepsilon^{2^\star}\,dx_N$$

$$+ o(\varepsilon^N) = \frac{1}{2}K_2 - \int_{\mathbb{R}^{N-1}\,\cap\, g(x')>0} dx' \int_0^{g(x')} U_\varepsilon^{2^\star}\,dx_N$$

$$+ \int_{\mathbb{R}^{N-1}\,\cap\, g(x')<0} dx' \int_{g(x')}^0 U_\varepsilon^{2^\star}\,dx_N + O(\varepsilon^2)$$

$$= \frac{1}{2}K_2 - \Pi^+(\varepsilon) + \Pi^-(\varepsilon) + O(\varepsilon^2),$$

where $K_2 = \int_{\mathbb{R}^N} U^{2^\star}\,dx$,

$$\Pi^+(\varepsilon) = \int_{\mathbb{R}^{N-1}\,\cap\, g(x')>0} dx' \int_o^{g(x')} U_\varepsilon^{2^\star}\,dx_N$$

and

$$\Pi^-(\varepsilon) = \int_{\mathbb{R}^{N-1}\,\cap\, g(x')<0} dx' \int_{g(x')}^0 U_\varepsilon^{2^\star}\,dx_N.$$

We now observe that

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}(\Pi^-(\varepsilon) - \Pi^+(\varepsilon)) =$$

$$- \int_{\mathbb{R}^{N-1}\,\cap\, g(x')<0} g(x')\,U^{2^\star}(x')\,dx' - \int_{\mathbb{R}^{N-1}\,\cap\, g(x')>0} g(x')\,U^{2^\star}(x')\,dx' =$$

$$- \int_{\mathbb{R}^{N-1}} g(x')\,U^{2^\star}(x')\,dx'.$$

Since

$$\int_{\mathbb{R}^{N-1}} x_1^2 U^{2^\star}(x')\, dx' = \ldots = \int_{\mathbb{R}^{N-1}} x_{N-1}^2 U(x')^{2^\star} dx',$$

we can write the above limit as

$$\lim_{\varepsilon \to 0} \varepsilon^{-1}(\Pi^-(\varepsilon) - \Pi^+(\varepsilon)) = -\alpha_N H(0), \tag{8}$$

for some positive constant α_N depending only on N. Following the argument from the paper [21] we show that

$$\lim_{\varepsilon \to 0} \frac{I^-(\varepsilon) - I^+(\varepsilon)}{\Pi^-(\varepsilon) - \Pi^+(\varepsilon)} = \frac{\int_{\mathbb{R}^{N-1}} |\nabla U(x')|^2 (-g(x'))\, dx'}{\int_{\mathbb{R}^{N-1}} U(x')^{2^\star}(-g(x'))\, dx'} \tag{9}$$

$$> \frac{(N-2)\, K_1}{NK_2}.$$

Indeed, changing variables, we obtain

$$\frac{\int_{\mathbb{R}^{N-1}} (-g(x'))|\nabla U(x')|^2 dx'}{\int_{\mathbb{R}^{N-1}} (-g(x'))\, U^{2^\star}(x')\, dx'} = (N-2)^2 \frac{\int_0^\infty \frac{r^{N+2}}{(1+r^2)^N} dr}{\int_0^\infty \frac{r^N}{(1+r^2)^N} dr}.$$

Hence, using the formula (3.18) from the paper [21] we get

$$\lim_{\varepsilon \to 0} \frac{I^-(\varepsilon) - I^+(\varepsilon)}{\Pi^-(\varepsilon) - \Pi^+(\varepsilon)} > (N-2)^2 \frac{N+1}{N-3}.$$

On the other hand, as demonstrated in [21], we have

$$\frac{(N-2)\, K_1}{NK_2} = (N-2)^2$$

and inequality (9) follows. It follows from (6) and (7) that

$$\frac{K_1(\varepsilon)}{(K_2(\varepsilon))^{(N-2)/N}} = \frac{\frac{1}{2}K_1 + I^-(\varepsilon) - I^+(\varepsilon) + O(\varepsilon^2)}{(\frac{1}{2}K_2 + \Pi^-(\varepsilon) - \Pi^+(\varepsilon) + O(\varepsilon^2))^{(N-2)/N}} = \tag{10}$$

$$\left(\frac{1}{2}K_1 + I^-(\varepsilon) - I^+(\varepsilon) + O(\varepsilon^2)\right) \times$$

$$\left[\left(\frac{1}{2}K_2\right)^{-(N-2)/N} - \frac{N-2}{N}\left(\frac{1}{2}K_2\right)^{(-2N+2)/N}(\Pi^-(\varepsilon)-\Pi^+(\varepsilon))+O(\varepsilon^2)\right]=$$

$$\frac{\frac{1}{2}K_1}{(\frac{1}{2}K_2)^{(N-2)/N}}+\left(\frac{1}{2}K_2\right)^{-(N-2)/N}(I^-(\varepsilon)-I^+(\varepsilon))-$$

$$\frac{(N-2)\,K_1}{2N}\left(\frac{K_2}{2}\right)^{-(2N+2)/N}(\Pi^-(\varepsilon)-\Pi^+(\varepsilon))+O(\varepsilon^2).$$

According to (9) there exist constants $\varrho>0$ and $\varepsilon_0>0$ such that

$$I^-(\varepsilon)-I^+(\varepsilon)>\left(\frac{(N-2)\,K_1}{NK_2}+\varrho\right)(\Pi^-(\varepsilon)-\Pi^+(\varepsilon))$$

for $0<\varepsilon\leqslant\varepsilon_0$. Using this and the fact that $\frac{K_1}{K_2^{(N-2)/N}}=S$, we derive from (10) that

$$\frac{K_1(\varepsilon)}{K_2(\varepsilon)^{(N-2)/N}}>\frac{S}{2^{2/N}}+\left[\frac{(N-2)\,K_1}{NK_2}\left(\frac{1}{2}K_2\right)^{-(N-2)/N}+\varrho\left(\frac{1}{2}K_2\right)^{-(N-2)/N}\right.$$

$$\left.-\frac{(N-2)\,K_1}{2N}\left(\frac{K_2}{2}\right)^{-(2N+2N)/N}\right](\Pi^-(\varepsilon)-\Pi^+(\varepsilon))+O(\varepsilon^2)$$

$$=\frac{S}{2^{2/N}}+\varrho\left(\frac{K_2}{2}\right)^{-(N-2)/N}(\Pi^-(\varepsilon)-\Pi^+(\varepsilon))+O(\varepsilon^2).$$

Combining this with (8) the result readily follows. ∎

We now establish an analogue of Proposition 2.1 in the case when $\partial\Omega$ has a flat part. We assume that

($\mathbf{H}_0$) $D(0,a)\subset\partial\Omega$ for some $a>0$, where $D(0,a)=B(0,a)\cap\{x_N=0\}$.

PROPOSITION 2.2. – *Let $N\geqslant 5$ and suppose that (H_0) holds. Then there exist constants $C>0$ and $\varepsilon_0>0$ such that*

$$J_\lambda\left(\frac{U_{\varepsilon,y}}{\|U_{\varepsilon,y}\|_{2^\star}}\right)\geqslant\frac{S}{2^{2/N}}+\lambda C\varepsilon^2$$

for all $y\in D\left(0,\frac{a}{2}\right)$ and $0<\varepsilon\leqslant\varepsilon_0$.

PROOF. – Using notations from Proposition 2.1 we have

$$K_1(\varepsilon)=\int_\Omega|\nabla U_{\varepsilon,y}|^2\,dx=\frac{1}{2}K_1+O(\varepsilon^{N-2})$$

and

$$K_2(\varepsilon) = \int_\Omega U_{\varepsilon,y}^{2^\star}\,dx = \frac{1}{2}K_2 + O(\varepsilon^N)\,.$$

Since

$$\frac{K_1(\varepsilon)}{K_2(\varepsilon)^{(N-2)/N}} = \frac{\frac{1}{2}K_1}{(\frac{1}{2}K_2)^{(N-2)/N}} + O(\varepsilon^{N-2})\,,$$

the result easily follows. ∎

3. – Existence and nonexistence results for least energy solutions.

We commence by showing that $S_\lambda < \dfrac{S}{2^{2/N} Q_m^{(N-2)/N}}$ for $\lambda \in (0,\bar\lambda)$ and $S_\lambda = \dfrac{S}{2^{2/N} Q_m^{(N-2)/N}}$ for $\lambda \geqslant \bar\lambda$. We proceed by contradiction and use a blow-up technique. By rescaling we may assume that $Q_m = 1$. We need some technical lemmas:

LEMMA 3.1. – *Suppose that $Q_M < 2^{2/(N-2)} Q_m$. Let $\lambda_k > 0$ and let $\{u_k\} \subset H^1(\Omega)$ be a sequence, weakly convergent to zero, of positive solutions of*

$$-\Delta u + \lambda_k u = \mu_k Q(x) u^{2^\star - 1}, \qquad \frac{\partial u}{\partial \nu} = 0\,,$$

with

$$\int_\Omega Q(x)\, u_k^{2^\star}\,dx = 1\,, \qquad 0 < \mu_0 \leqslant \mu_k \leqslant \frac{S}{2^{2/N} Q_m^{(N-2)/N}}$$

for all k and some constant μ_0. Then there exist a sequence of points $x_k \to y_0 \in \partial\Omega$, $x_k \in \partial\Omega$ and a sequence of numbers $\varepsilon_k \to 0$, $\varepsilon_k > 0$ such that $Q(y_0) = Q_m$ and

$$\lim_{k\to\infty} \int_\Omega |\nabla(\mu_k^{1(2^\star-2)} u_k - U_{\varepsilon_k, x_k})|^2\,dx = 0\,.$$

PROOF. – The function $v_k = \mu_k^{1/(2^\star-2)} u_k$ satisfies

$$\begin{cases} -\Delta v_k + \lambda_k v_k = Q(x)\, v_k^{2^\star-1} \text{ in } \Omega\,, \\ v_k > 0 \text{ on } \Omega \text{ and } \dfrac{\partial v_k}{\partial \nu} = 0 \text{ on } \partial\Omega\,. \end{cases}$$

We let $M_k = \max\limits_{x\in\Omega} v_k = v_k(x_k)$ for some $x_k \in \bar\Omega$. It is easy to check that $M_k \to \infty$

as $k \to \infty$. We set $\varepsilon_k = M_k^{-1/(N-2)}$ and $\Omega_k = \dfrac{\Omega - x_k}{\varepsilon_k}$ and

$$w_k(x) = \varepsilon_k^{(N-2)/2} v_k(\varepsilon_k x + x_k) \text{ for } x \in \Omega_k.$$

Thus we have

$$\begin{cases} -\Delta w_k + \lambda_k \varepsilon_k^2 w_k & = Q(\varepsilon_k x + x_k)\, w_k^{2^\star - 1} \text{ in } \Omega_k, \\ 0 < w_k(x) \leqslant w_k(0) & = 1 \text{ in } \Omega_k \text{ and } \dfrac{\partial w_k}{\partial \nu} = 0 \text{ on } \partial\Omega_k. \end{cases}$$

Since $\lambda_k \varepsilon_k^2 \leqslant Q_M$, we may assume that $\lambda_k \varepsilon^2 \to a$. Furthermore, we may assume that $x_k \to y_0$ and $\dfrac{\operatorname{dist}(x_k, \partial\Omega)}{\varepsilon_k} \to \alpha$. Let $\Omega_\infty = \{x \in \mathbb{R}^N;\, x_N > -\alpha\}$. By the elliptic regularity theory we have $w_k \to w$ in $C^2_{\mathrm{loc}}(\Omega_\infty)$ and

$$\begin{cases} -\Delta w + aw & = Q(y_0)\, w^{2^\star - 1} \text{ in } \Omega_\infty \\ 0 \leqslant w(x) \leqslant w(0) & = 1 \text{ in } \Omega_\infty, \ \dfrac{\partial w}{\partial \nu} = 0 \text{ on } \partial\Omega_\infty. \end{cases}$$

Since $\{w_k\}$ is bounded in $H^1(\Omega)$ we have

$$\int_{\Omega_\infty} |\nabla w|^2\, dx \leqslant \lim_{k \to \infty} \int_{\Omega_k} |\nabla w_k|^2\, dx = \lim_{k \to \infty} \int_{\Omega} |\nabla v_k|^2\, dx < \infty$$

and

$$\int_{\Omega_\infty} |w|^{2^\star}\, dx \leqslant \lim_{k \to \infty} \int_{\Omega_k} |w_k|^{2^\star}\, dx = \lim_{k \to \infty} \int_{\Omega} |v_k|^{2^\star}\, dx < \infty .$$

By Pohozaev's identity $a = 0$, $w(x) = Q(y_0)^{-1/(2^\star - 2)} U_{\varepsilon, z}$. Since $w(0) = 1$, we deduce that $z = 0$, $\varepsilon = Q(y_0)^{-1/2}$ and $\alpha = 0$ or $\alpha = \infty$. If $\alpha = \infty$, then

$$Q(y_0)^{-2/(2^\star - 2)} S^{N/2} = Q(y_0)^{-2/(2^\star - 2)} \int_{\mathbb{R}^N} |\nabla U|^2\, dx \leqslant \lim_{k \to \infty} \int_{\Omega_k} |\nabla w_k|^2\, dx \leqslant \frac{S^{N/2}}{2 Q_m^{(N-2)/2}} .$$

This yields that $Q(y_0)^{2/(2^\star - 2)} \geqslant 2 Q_m^{(N-2)/2}$, that is, $Q_M \geqslant Q(y_0) \geqslant 2^{2/(N-2)} Q_m$, which is impossible. Therefore, $\alpha = 0$, $\Omega_\infty = \mathbb{R}^N_+$, $y_0 \in \partial\Omega$ with $Q(y_0)^{-2/(2^\star - 2)} \dfrac{S^{N/2}}{2} \leqslant \dfrac{S^{N/2}}{2 Q_m^{(N-2)/2}}$ implying that $Q(y_0) \geqslant Q_m$, so $Q(y_0) = Q_m = 1$. As in the paper [4] we show that $x_k \in \partial\Omega$ for large k. This completes the proof. ■

To proceed further we some results from the paper [4] (see also [28]). We define a set

$$\mathcal{M} = \{C U_{\varepsilon, y};\, C \in \mathbb{R},\, \varepsilon > 0,\, y \in \partial\Omega\} .$$

LEMMA 3.2. – *Let $\delta > 0$ and $\{z_n\} \subset H^1(\Omega)$ be such that $z_n \rightharpoonup 0$ in $H^1(\Omega)$ and*

$$d(z_n, \mathcal{M}) \leqslant \|\nabla z_n\|_2^2 - 2\delta .$$

where $d(u, \mathcal{M}) = \inf\{\|\nabla(u-z)\|_2^2;\ z \in H^1(\Omega)\}$. Then there exists $l_0 > 0$, such that for all $l \geqslant l_0$, $d(z_l, \mathcal{M})$ is attained by some $C_l U_{\varepsilon_l, y_l}$. Moreover, if w_l is defined by

$$z_l = C_l U_{\varepsilon_l, y_l} + w_l,$$

then for a subsequence

(*i*) $\lim_{l \to \infty} \varepsilon_l = 0$,

(*ii*) *if $d(z_l, \mathcal{M}) \to 0$ as $l \to \infty$, then $\lim_{l \to \infty} C_l = C_0 \neq 0$,*

(*iii*) *we also have*

$$\int_\Omega w_l U_{\varepsilon_l, y_l}^{2^\star - 1}\, dx = \beta(\varepsilon_l)\|w_l\| ,$$

where

$$\beta(\varepsilon) = \begin{cases} \varepsilon^{1/2} & \text{if } N = 3 \\ \varepsilon\left(\log \dfrac{1}{\varepsilon}\right)^{2/3} & \text{if } N = 4 \\ \varepsilon & \text{if } N \geqslant 5 . \end{cases}$$

Suppose that for each $\lambda > 0$ there exists the least energy solution u_λ. Let $\lambda_k \to \infty$ and set $u_k = u_{\lambda_k}$. We apply Lemma 3.1 with $\mu_k = S_{\lambda_k}$ and set $v_k = S_{\lambda_k}^{1/(2^\star - 2)} u_k$. Since

$$\|\nabla(v_k - U_{\varepsilon_k, x_k})\|_2^2 \to 0 ,$$

it follows from Lemma 3.2 that there exist sequences $\{\delta_k\}$ and $\{y_k\}$ such that $\delta_k \to 0$, $y_k \in \partial\Omega$ and

$$v_k = C_k U_{\delta_k, y_k} + w_k. \tag{11}$$

As in Lemma 2.3 of the paper [4] we check that $C_k \to 1$, and $\frac{\varepsilon_k}{\delta_k} \to 1$. Therefore, we may assume that (11) holds with $\delta_k = \varepsilon_k$ and $x_k = y_k$.

LEMMA 3.3. – *There exists a constant $\delta > 0$ such that*

$$\int_\Omega (|\nabla w_k|^2 + \lambda_k w_k^2)\,dx \geqslant (2^\star - 1 + \delta)\int_\Omega U_{\varepsilon_k, y_k}^{2^\star - 2} w_k^2\,dx + O(\beta(\varepsilon_k)^2 \|w_k\|^2)$$

for every k.

For the proofs of Lemmas 3.2 and 3.3 we refer the reader to the paper [4]. We are now in a position to prove the main result of this paper. We need the following assumptions:

($\mathbf{Q}_1$) $|Q(x_0) - Q(x)| = o(|x - x_0|)$ as $|x - x_0| \to 0$ for every $x_0 \in \partial\Omega$ such that $Q_m = Q(x_0)$.

(**H**) $\{x \in \partial\Omega;\ H(x) < 0\} \neq \emptyset$ and $\{x \in \partial\Omega;\ Q(x) = Q_m\} \subset \{x \in \partial\Omega;\ H(x) < 0\}$

THEOREM. – 3.4. – *Let $N \geqslant 5$ and $Q_M \leqslant 2^{2/(N-2)} Q_m$. Suppose that (H) and (Q_1) hold. Then there exists $\bar\lambda > 0$ such that $S_\lambda < \dfrac{S}{2^{2/N} Q_m^{(N-2)/N}}$ for all $0 < \lambda < \bar\lambda$ and $S_\lambda = \dfrac{S}{2^{2/N} Q_m^{(N-2)/N}}$ for $\lambda \geqslant \bar\lambda$. Furthermore, S_λ is achieved iff $\lambda \in (0, \bar\lambda]$.*

PROOF. – First we consider the case $Q_M < 2^{2/(N-2)} Q_m$. Assuming that the least energy solutions exist for each $\lambda > 0$ we define a sequence v_k described in the paragraph preceding Lemma 3.3. Using the decomposition (11) we have

$$J_{\lambda_k}\left(\frac{u_k}{(\int_\Omega Q|u_k|^{2^\star}\,dx)^{1/2^\star}}\right) = J_{\lambda_k}\left(\frac{v_k}{(\int_\Omega Q|v_k|^{2^\star}}\,dx)^{1/2^\star}\right) = \tag{12}$$

$$\frac{1}{(\int_\Omega Q|v_k|^{2^\star}\,dx)^{2/2^\star}}\left[C_k^2 J_{\lambda_k}(U_{\varepsilon_k, y_k}) + \|\nabla w_k\|_2^2 + \lambda_k \|w_k\|_2^2 + 2\lambda_k C_k \int_\Omega w_k U_{\varepsilon_k, y_k}\,dx\right].$$

It follows from (Q_1) that

$$\int_\Omega Q(x)\, U_{\varepsilon_k, y_k}^{2^\star}\,dx = Q(y_k)\int_\Omega U_{\varepsilon_k, y_k}^{2^\star}\,dx + o(\varepsilon_k)\,. \tag{13}$$

We now apply Lemma 3.5 from [4] and deduce that

$$\left(\int_\Omega Q|v_k|^{2^\star}\,dx\right)^{-2/2^\star} = C_k^{-2}\left(\int_\Omega Q U_{\varepsilon_k, y_k}^{2^\star}\,dx\right)^{-2/2^\star} \times$$

$$\left[1+\frac{2^\star(2^\star-1)\int_\Omega QU_{\varepsilon_k,y_k}^{2^\star-2}w_k^2\,dx}{2C_k^2\int_\Omega QU_{\varepsilon_k,y_k}^{2^\star}\,dx}+O(\beta(\varepsilon_k)\|w_k\|+\|w_k\|^r)\right]^{-2/2^\star}=$$

$$C_k^{-2}\left(\int_\Omega QU_{\varepsilon_k,y_k}^{2^\star}\,dx\right)^{-2/2^\star}\left\{1-\frac{(2^\star-1)\int_\Omega QU_{\varepsilon_k,y_k}^{2^\star-2}w_k^2\,dx}{C_k^2\int_\Omega QU_{\varepsilon_k,y_k}^{2^\star}\,dx}+O(\beta(\varepsilon_k)\|w_k\|+\|w_k\|^r)\right\}$$

for some $2<r<2^\star$. This, combined with (12), gives

$$J_{\lambda_k}\left(\frac{u_k}{(\int_\Omega Q|u_k|^{2^\star}\,dx)^{1/2^\star}}\right)= \tag{14}$$

$$\left\{J_{\lambda_k}\left(\frac{U_{\varepsilon_k,y_k}}{(\int_\Omega QU_{\varepsilon_k,y_k}\,dx)^{1/2^\star}}\right)+\frac{\|\nabla w_k\|_2^2+\lambda_k\|w_k\|_2^2+2C_k\lambda_k\int_\Omega U_{\varepsilon_k,y_k}w_k\,dx}{C_k^2(\int_\Omega(QU_{\varepsilon_k,y_k}^{2^\star}\,dx))^{2/2^\star}}\right\}\times$$

$$\times\left\{1-\frac{(2^\star-1)\int_\Omega QU_{\varepsilon_k,y_k}^{2^\star-2}w_k^2\,dx}{C_k^2\int_\Omega QU_{\varepsilon_k,y_k}^{2^\star}\,dx}+O(\beta(\varepsilon_k)\|w_k\|+\|w_k\|^r)\right\}.$$

Let $N\geqslant 7$, then

$$\int_\Omega U_{\varepsilon_k,y_k}w_k\,dx=O(\varepsilon_k^2\|w_k\|)\,. \tag{15}$$

(see estimate (2.32) in [4]). Hence by (14) we get

$$J_{\lambda_k}\left(\frac{u_k}{(\int_\Omega Qu_k^{2^\star}\,dx)^{1/2^\star}}\right)=J_{\lambda_k}\left(\frac{U_{\varepsilon_k,y_k}}{(\int_\Omega QU_{\varepsilon_k,y_k}^{2^\star}\,dx)^{1/2^\star}}\right)- \tag{16}$$

$$\frac{(2^\star-1)}{C_k^2}J_{\lambda_k}\left(\frac{U_{\varepsilon_k,y_k}}{(\int_\Omega QU_{\varepsilon_k,y_k}^{2^\star}\,dx)^{1/2^\star}}\right)\frac{\int_\Omega QU_{\varepsilon_k,y_k}^{2^\star-2}w_k^2\,dx}{\int_\Omega QU_{\varepsilon_k,y_k}^{2^\star}\,dx}+$$

$$\frac{\|\nabla w_k\|_2^2+\lambda_k\|w_k\|_2^2+O(\lambda_k\varepsilon_k^2\|w_k\|)}{C_k^2(\int_\Omega QU_{\varepsilon_k,y_k}^{2^\star}\,dx)^{2/2^\star}}+$$

$$O(\|w_k\|^2+\beta(\varepsilon_k)\|w_k\|+\|w_k\|^r)(\|\nabla w_k\|_2^2+\lambda_k\|w_k\|_2^2+O(\lambda_k\varepsilon_k^2\|w_k\|))+$$

$$O(\beta(\varepsilon_k)\|w_k\|+\|w_k\|^r)+O(\beta(\varepsilon_k)\|w_k\|+\|w_k\|^r)(O(\varepsilon_k)+\lambda_kC_k\varepsilon_k^2)\,.$$

It follows from Proposition 2.1 and (13) that

$$(17)\qquad J_{\lambda_k}\left(\frac{U_{\varepsilon_k, y_k}}{(\int_\Omega QU^{2^\star}_{\varepsilon_k, y_k}\,dx)^{1/2^\star}}\right) \geqslant \frac{S}{2^{2/N}Q(y_k)^{(N-2)/N}} - \tilde{\alpha}H(y_k)\,\varepsilon_k + \lambda_k C\varepsilon_k^2$$

for some constant $\tilde{\alpha} > 0$ and $\varepsilon_k > 0$ sufficiently small. According to Lemma 3.3 we can find $0 < \varrho < 1$ and $0 < \delta_1 < \delta$ such that

$$(1-\varrho)\int_\Omega (|\nabla w_k|^2 + \lambda_k w_k^2)\,dx \geqslant (2^\star - 1 + \delta_1)\int_\Omega U^{2^\star-2}_{\varepsilon_k, y_k} w_k^2\,dx + O(\beta(\varepsilon_k)^2\|w_k\|^2)\,.$$

Thus,

$$\frac{(1-\varrho)\|\nabla w_k\|_2^2 + \lambda_k\|w_k\|_2^2}{C_k^2(\int_\Omega QU^{2^\star}_{\varepsilon_k, y_k}\,dx)^{2/2^\star}} - \frac{2^\star-1}{C_k^2}J_{\lambda_k}\left(\frac{U_{\varepsilon_k, y_k}}{(\int_\Omega QU^{2^\star}_{\varepsilon_k, y_k}\,dx)^{1/2^\star}}\right)\times$$

$$\frac{\int_\Omega QU^{2^\star-2}_{\varepsilon_k, y_k} w_k^2\,dx}{\int_\Omega QU^{2^\star}_{\varepsilon_k, y_k}\,dx} \geqslant \int_\Omega QU^{2^\star-2}_{\varepsilon_k, y_k} w_k\,dx \times$$

$$\left[\frac{2^\star - 1 + \delta_1}{C_k^2(\int_\Omega QU^{2^\star}_{\varepsilon_k, y_k}\,dx)^{2/2^\star}} - \frac{(2^\star-1)}{C_k^2\int_\Omega QU^{2^\star}_{\varepsilon_k, y_k}\,dx}J_{\lambda_k}\left(\frac{U_{\varepsilon_k, y_k}}{(\int_\Omega QU^{2^\star}_{\varepsilon_k, y_k})^{1/2^\star}}\right)\right] +$$

$$O(\beta(\varepsilon_k)^2\|w_k\|^2) = D_k + O(\beta(\varepsilon_k)^2\|w_k\|^2)\,,$$

where $D_k > 0$ for large k. Since $\|w_k\| \to 0$, by (16) and (17) we have for some constants $\alpha^* > 0$, $\tilde{C} > 0$ and $\tilde{\varrho} > 0$

$$J_{\lambda_k}\left(\frac{u_k}{(\int_\Omega Qu_k^{2^\star}\,dx)^{1/2^\star}}\right) \geqslant \frac{S}{2^{2/N}} - \alpha^* H(y_k)\,\varepsilon_k + \lambda_k\tilde{C}\varepsilon_k^2$$

$$+ D_k + \tilde{\varrho}\,\frac{\|\nabla w_k\|_2^2 + \lambda_k\|w_k\|_2^2}{C_k^2(\int_\Omega QU^{2^\star}_{\varepsilon_k, y_k}\,dx)^{2/2^\star}} + O(\beta(\varepsilon_k)\|w_k\|)$$

for large k. Using the Young inequality, we deduce from this that

$$J_{\lambda_k}\left(\frac{u_k}{(\int_\Omega Qu_k^{2^\star}\,dx)^{1/2^\star}}\right) > \frac{S}{2^{2/N}}\,,$$

which is impossible. If $N = 5, 6$, we use, instead of (15), the following estimate (see [23] or [24]): for $q \in \left(\frac{N}{N-2}, 2\right) \cap \left(\frac{2N}{N+2}, 2\right)$ there exist constants $C(q) > 0$

and $a = a(q) \in [0, 1)$ with $a(q) = \dfrac{Nq - 2N + 2q}{2q}$ such that for every $\gamma > 1$

$$\left| \int_\Omega U_{\varepsilon_k, y_k} w_k \, dx \right| \leq \left(1 - \frac{a}{2}\right) C(q) \, \gamma^{2/(2-a)} \varepsilon_k^2 \|w_k\|_{2^\star}^{q(1-a)/(2-a)} + \frac{a}{2} \frac{1}{\gamma^{2/a}} \|w_k\|_2^2 .$$

We choose γ so that $\dfrac{a}{2\gamma^{2/a}} < \varrho$ and an obvious modification of the previous argument leads to the inequality (18). This completes the proof in the case $Q_M < 2^{2/(N-2)} Q_m$. If $Q_M = 2^{2/(N-2)} Q_m$, then the sequence of instantons which is close to the sequence $\{v_k\}$ concentrates either on the boundary or at a point $y \in \Omega$ with $Q(y) = Q_M$. In the first case we argue as in the first part of the proof. If the concentration occurs in Ω we apply Proposition 4.2 of the paper [24] to get a contradiction. The existence of the least energy solutions for $\lambda \in (0, \bar\lambda)$ follows from [8] (see also Introduction). Finally, we show that the least energy solution exists for $\lambda = \bar\lambda$. First, we consider the case $Q_M < 2^{2/(N-2)} Q_m$. Let $\lambda_k \to \bar\lambda$ and $\lambda_k < \bar\lambda$. By the first part of the proof for each k there exists a least energy solution $u_k = u_{\lambda_k}$. Since $\{u_k\}$ is bounded in $H^1(\Omega)$, then up to a subsequence $u_k \rightharpoonup u$ in $H^1(\Omega)$. It is sufficient to show that $u \not\equiv 0$. Then by the concentration-compactness principle u is the least energy solution corresponding to $\bar\lambda$. Arguing by contradiction assume that $u \equiv 0$. Let $M_k = \max_{x \in \bar\Omega} u_k(x) = u_k(x_k)$, $x_k \in \bar\Omega$. Then $M_k \to \infty$. We set

$$v_k = \varepsilon_k^{(N-2)/2} u_k(\varepsilon_k x + x_k) \quad \text{for} \quad x \in \Omega_k = \frac{\Omega - x_k}{\varepsilon_k} ,$$

where $\varepsilon_k = M_k^{-2/(N-2)}$. The function v_k satisfies

$$\begin{cases} -\Delta v_k + \lambda_k \varepsilon_k^2 v_k & = S_{\lambda_k} Q(\varepsilon_k x + x_k) \, v_k^{2^\star - 1} \text{in} \\ \dfrac{\partial v_k}{\partial \nu} = 0 \ \text{ on } \ \Omega & \text{and} \ \ v_k(0) = 1 . \end{cases}$$

As in the proof of Lemma 3.1 we show, after scaling $Q_m = 1$, that

$$\lim_{k \to \infty} \int_\Omega |\nabla (S_{\lambda_k}^{1/(2^\star - 2)} u_k - U_{\varepsilon_k, x_k})|^2 \, dx = 0 ,$$

with $x_k \to x_0$, $Q(x_0) = Q_m$, $x_0 \in \partial\Omega$. Moreover, $x_k \in \partial\Omega$ for large k. Using the decomposition (11) we show that, as in the proof of Theorem 3.4, that

$$J_{\lambda_k}\left(\frac{u_k}{(\int_\Omega Q u_k^{2^\star} \, dx)^{1/2^\star}} \right) > \frac{S}{2^{2/N}}$$

for large k, which is impossible. If $Q_M = 2^{2/(N-2)} Q_m$ we distinguish two cases: (i) $\{u_k\}$ is close to instantons concentrating on the boundary $\partial\Omega$ or (ii) $\{u_k\}$ is close to instantons concentrating at interior point x_0 with $Q_M = Q(x_0)$. If (i) occurs we argue as in the first part of the proof. If the case (ii) prevails we modify the previous argument by considering the set of instantons

$$\mathcal{M}_1 = \{CU_{\varepsilon, y};\, C > 0,\ \varepsilon > 0,\ y \in \overline{\Omega}\}$$

and apply Proposition 4.2 from the paper [24] to arrive at a contradiction. ■

To illustrate Theorem 3.4 we consider the following example. Let $\Omega = B(0, R) - \overline{B(0, r)}$, $0 < r < R$. Then $H = \frac{1}{R^2}$ on $\partial B(0, R)$ and $H = -\frac{1}{r^2}$ on $\partial B(0, r)$. If $N \geqslant 5$, $\{x \in \partial\Omega;\, Q(x) = Q_m\} \subset \partial B(0, r)$ and $Q_M < 2^{2/(N-2)} Q_m$, then, by virtue of Theorem 3.4, there exists a number $\overline{\lambda} > 0$ such that problem (1) has the least energy solutions only for $\lambda \in (0, \overline{\lambda}]$. However, if $\Omega = B(0, R)$ and $Q_M < 2^{2/(2-N)} Q_m$. then problem (1) has the least energy solutions for every $\lambda > 0$.

To establish a similar result in the flat case we need the following assumption:

($\mathbf{Q}_2$) $\{x \in \partial\Omega;\, Q(x) = Q_m\} \subset D\left(0, \frac{a}{2}\right)$ and for every $x_0 \in D\left(0, \frac{a}{2}\right)$, $Q(|x - x_0|) = o(|x - x_0|^2)$ for x near x_0

THEOREM 3.5. – *Let $N \geqslant 4$ and let $Q_M \leqslant 2^{2(N-2)} Q_m$. Suppose that (H_0) and (Q_2) hold. Then there exists $\lambda^* > 0$ such that*

$$S_\lambda < \frac{S}{2^{2/N} Q_m^{(N-2)/N}} \ \textit{for}\ 0 < \lambda < \lambda^*$$

and

$$S_\lambda = \frac{S}{2^{2/N} Q_m^{(N-2)/N}} \ \textit{for}\ \lambda \geqslant \lambda^*.$$

Furthermore, the least energy solutions exist for $\lambda \in (0, \lambda^*]$ and there are no least energy solutions for $\lambda > \lambda^*$.

PROOF. – We may assume that $Q_m = 1$. Let $Q_M < 2^{2/(N-2)}$. Arguing indirectly assume that $S_\lambda < \frac{S}{2^{2/N}}$ for each $\lambda > 0$. Let $\lambda_k \to \infty$ and let $u_k = u_{\lambda_k}$ be the corresponding sequence of the least energy solutions. Applying Lemmas 3.1 and 3.2 we obtain the decomposition (11). As in the proof of Theorem 3.4 we get the relation (16). Since $y_k \to y_0 \in D\left(0, \frac{a}{2}\right)$, the inequality (17) in this case takes the form

$$J_{\lambda_k}\left(\frac{U_{\varepsilon_k, y_k}}{(\int_\Omega QU_{\varepsilon_k, y_k}^{2^\star}\, dx)^{1/2^\star}}\right) \geqslant \frac{S}{2^{2/N}} + \lambda C_1 \varepsilon_k^2$$

for large k (see Proposition 2.2). Hence as in the proof of Theorem 3.4 we obtain the estimate (18), however without the term $\tilde{a} H(y_k)\, \varepsilon_k$ on the right side. From this estimate, using the Hölder inequality, we deduce that

$$J_{\lambda_k}\left(\frac{u_k}{(\int_\Omega Qu_k^{2^\star}\, dx)^{1/2^\star}}\right) > \frac{S}{2^{2/N}}$$

for large k, which is a contradiction. If $Q_M = 2^{2/(N-2)}$, we argue as in the second part of Theorem 3.4. ■

4. – Remark on Sobolev inequalities.

Theorem 3.4 and 3.5 give rise to the following inequalities

COROLLARY 4.1. – *Let $N \geqslant 5$. Suppose that $Q_M \leqslant 2^{2/(N-2)} Q_m$ and that (H) and (Q_1) hold. Then there exists a constant $\Lambda_1 = \Lambda_1(\Omega) > 0$ such that*

$$\left(\int_\Omega Q|u|^{2^\star} dx\right)^{2/2^\star} \leqslant \frac{2^{2/N} Q_m^{(N-2)/N}}{S} \int_\Omega |\nabla u|^2 dx + \Lambda_1 \int_\Omega u^2 dx$$

for every $u \in H^1(\Omega)$.

COROLLARY 4.2. – *Let $N \geqslant 5$. Suppose that $Q_M \leqslant 2^{2/N_2} Q_m$ and that (H_0) and (Q_2) hold. Then there exists a constant $\Lambda_2(\Omega) > 0$ such that*

$$\left(\int_\Omega Q|u|^{2^\star} dx\right)^{2/2^\star} \leqslant \frac{2^{2/N} Q_m^{(N-2)/N}}{S} \int_\Omega |\nabla u|^2 dx + \Lambda_2 \int_\Omega u^2 dx$$

for every $h \in H^1(\Omega)$.

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The University of Queensland, Department of Mathematics
St. Lucia 4072, Qld, Australia

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