Bollettino
Unione Matematica Italiana

A. L. Gilotti, U. Tiberio

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Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **5-B** (2002), n.3, p. 735–746.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2002_8_5B_3_735_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2002.

Bollettino U. M. I.
(8) 5-B (2002), 735-746

Intersecting Maximals.

A. L. GILOTTI (*) - U. TIBERIO (**)

Sunto. – *Data una classe $\mathcal{X}$ di gruppi finiti e un gruppo finito G gli autori studiano il sottogruppo $\mathcal{X}(G)$ intersezione dei sottogruppi massimali non appartenenti a $\mathcal{X}$.*

Summary. – *Given a class $\mathcal{X}$ of finite groups and a finite group G, the authors study the subgroup $\mathcal{X}(G)$ intersection of maximal subgroups that do not belong to $\mathcal{X}$.*

Introduction.

Let $\mathcal{X}$ be a class of finite groups and let G be a finite group.

Let us denote by $\mathcal{X}(G)$ the intersection of all maximal subgroups of G not belonging to $\mathcal{X}$. If G is a group of $\mathcal{X}$ or if G is minimal non-$\mathcal{X}$, set $\mathcal{X}(G) = G$.

With this notation $S(G)$ will denote the intersection of the insoluble maximal subgroups of G, $H_p(G)$ will denote the intersection of the non p-nilpotent maximal subgroups of G and $\Sigma(G)$ the intersection of the non-supersoluble maximal subgroups of G.

Further $H(G)$, $M(G)$, $C(G)$ will denote respectively the intersection of the non-nilpotent, non-abelian, non-cyclic maximal subgroups of G.

Most of the time these subgroups coincide among themselves and very often they coincide with the Frattini subgroup of G. However if they do not coincide and if at least one of them contains properly the Frattini subgroup then there are consequences on the structure of G. Problems of this type and the characterization of the structure of these subgroups have been studied in various papers and by various authors (cf. [1], [2], [3], [4], [5]).

With the usual notation, let $F^*(G)$ be the generalized Fitting subgroup and $E(G)$ the maximal normal semisimple subgroup of the finite group G. If the class $\mathcal{X}$ is a formation, $G^{\mathcal{X}}$ will denote the $\mathcal{X}$-residual of G.

The main results of the first section of this paper are the following:

a) *Suppose that* $\Phi(G) \subsetneq \Sigma(G) \subsetneq H_2(G) \subsetneq G$, *then* $\Sigma(G)$ *is nilpotent,* $G^{\mathcal{X}_2} = G^{\Sigma}$ *and* $\Sigma(G) = G^{\Sigma}\Phi(G)$.

(*) Member of the G.N.S.A.G.A. of C.N.R.
(**) Research partially supported by ex 40%, 60% MURST funds.

(where Σ is the formation of the supersoluble groups.)

b) *If G is insoluble and if* $\Phi(G) \subsetneq S(G) \subseteq F^*(G)$ *then* $S(G) = E(G)\,\Phi(G) = G^S\,\Phi(G)$.

Note that b) extends to an insoluble group G and its subgroup $S(G)$ the results on $\Sigma(G)$ of theorem 4 of [3].

The results of the second section deal with $C(G)$ and $M(G)$. In particular we characterize finite groups in which $\Phi(G) \subsetneq C(G)$ and nilpotent groups such that $\Phi(G) \subsetneq M(G)$.

The non nilpotent case for $M(G)$ was already studied in [4]. Precisely we prove that if G is a p-group $\Phi(G) \subsetneq M(G)$ implies $M(G) = G$. If G is nilpotent but not a p-group, then there exists a prime p dividing the order of G such that the Sylow p-subgroup P of G is minimal non-abelian and every other Sylow q-subgroup of G is abelian.

Notation and preliminaries.

All groups considered in this paper are finite and notation is usually standard (cfr [6])

DEFINITION 1. – *Let* $\mathcal{X}$ *be a class of groups. Denote by* $\mathcal{X}(G)$ *the intersection of the maximal subgroups of G not belonging to* $\mathcal{X}$.

If no such a subgroup exists, i.e., if G belongs to $\mathcal{X}$ *or if G is minimal non-*$\mathcal{X}$ *let us set* $\mathcal{X}(G) = G$.

Let Σ be the class of supersoluble groups, $\mathcal{S}$ be the class of soluble groups, $\mathcal{H}_p$ be the class of p-nilpotent groups (p a prime), $\mathcal{H}$ be the class of nilpotent groups, $\mathcal{M}$ be the class of abelian groups and $\mathcal{C}$ be the class of cyclic groups. For the convenience of the reader later on we point out that $\mathcal{H}_2 \subseteq \mathcal{S}$. Correspondingly, according to the Definition 1, we will get the subgroups $\Sigma(G)$, $S(G)$, $H_p(G)$, $H(G)$, $M(G)$ and $C(G)$.

We recall the following lemma (see [3])

LEMMA 1. – *Let G be a finite group and let* $\mathcal{X}$ *be a quotient-closed class of finite groups. If* $\Phi(G) \subsetneq \mathcal{X}(G)$ *then:*

i) $G = \mathcal{X}(G)\,M$ *where M is a maximal subgroup belonging to* $\mathcal{X}$.

ii) If G is soluble, then $G = QN$, *where Q is a normal q-subgroup of G and N is a maximal subgroup of G belonging to* $\mathcal{X}$.

Finally we denote by $h(G)$ the nilpotent length (Fitting height) of G and by $l_p(G)$ the p-length of G. For the definition see for instance [8]

Section 1.

In this section we deal with $\Sigma(G)$, $S(G)$, $H_p(G)$, $H(G)$. In [1] Shidov proves that in insoluble groups $H(G)$ is nilpotent. Indeed it is immediate that it coincides with $\Phi(G)$ (see next Proposition 1). In [2] we have shown that $H_p(G) = \Phi(G)$ in a non p-soluble group, if p is a odd prime. However there exist insoluble groups such that $\Phi(G) \neq \Sigma(G)$. (see [3]). Also there exist insoluble groups in which $H_2(G)$ or $S(G)$ don't coincide with $\Phi(G)$. An example is $PGL(2, 9)$, where $H_2(G) = S(G) = PSL(2, 9)$. Observe that a double uncoincidence implies the solubility of G.

We begin with the easy:

PROPOSITION 1. – *Let G be a finite group. Then*

i) $H(G) \neq \Phi(G)$ implies that G is soluble and $h(G) \leqslant 2$

ii) $H_p(G) \neq \Phi(G)$ implies that G is p-soluble and $l_p(G) \leqslant 2$ if p is a odd prime.

PROOF. – i) By [1] (Shidov) G is soluble. So by Lemma 1 ii) $G = QN$ where Q is a q-group (q a prime) and N is nilpotent. It follows that $h(G) \leqslant 2$.

ii) By [2] (Gilotti-Tiberio) G is p-soluble. By [2] (Theorem 2) either $H_p(G)$ is p-nilpotent or $G = O_p(H_p(G))M$, M is a p-nilpotent group. In both cases we easily get $l_p(G) \leqslant 2$. ■

As we have already observed Proposition 1 i) does not hold for $\Sigma(G)$, $H_2(G)$ or $S(G)$, and Proposition 1 ii) does not hold for $p = 2$. But we can easily get the following two propositions:

PROPOSITION 2. – *Let G be a finite group such that $\Phi(G) \subsetneq \Sigma(G)) \subsetneq \mathcal{X}(G)$ where $\mathcal{X}$ is either $\mathcal{S}$ or $\mathcal{H}_2$. Then G is soluble and $h(G) \leqslant 3$.*

PROOF. – If the maximal subgroups of G belong to $\mathcal{X}$ then $\mathcal{X}(G) = G$. There are two cases: either $G \in \mathcal{X}$ or G is minimal non $-\ \mathcal{X}$. In the first case G is soluble and so by Lemma 1 ii) $G = QM$ where Q is a normal q-subgroup (q a prime) and M is supersoluble. So in this case $G/F(G)$ is supersoluble and $h(G) \leqslant 3$. If G is minimal non $-\ \mathcal{X}$, we have $G = \Sigma(G)M$ with $M \in \mathcal{X}$. Since $\mathcal{X} \subseteq \mathcal{S}$, M is soluble and since the proper subgroups of G are in $\mathcal{X}$, $\Sigma(G) \in \mathcal{X}$ and so G is soluble. If $\mathcal{X} = \mathcal{S}$ this is a contradiction. If $\mathcal{X} = \mathcal{H}_2$, by [10] 10.3.3 (Ito) G minimal non $-\ \mathcal{H}_2$ implies G minimal non $-\ \mathcal{H}$ and so $\Sigma(G) = G$, again a contradiction. If $\mathcal{X}(G) \neq G$, there exists at least one maximal subgroup M of G that does not belong to $\mathcal{X}$. On the other hand, since $\mathcal{X}(G) \neq \Sigma(G)$ there exist a maximal subgroup N of G which is not supersoluble, but belongs to $\mathcal{X}$. It follows $\Sigma(G) \subseteq N$ so $\Sigma(G) \in \mathcal{X}$. Since $G = \Sigma(G)L$, where L is supersoluble, G is soluble. This is a contradiction if $\mathcal{X} = \mathcal{S}$. If $\mathcal{X} = \mathcal{H}_2$, then $G = QL$, where Q is a q-subgroup (q a prime) and L is supersoluble, by Lemma 1 ii). So again G is soluble and $h(G) \leq 3$. ■

PROPOSITION 3. – *Let G be a finite group such that*

$$\Phi(G) \subsetneq H_2(G) \subsetneq \mathcal{S}(G)$$

Then G is soluble and $l_2(G) \leq 2$.

PROOF. – If $G = S(G)$ then either G is soluble or G is minimal insoluble. In the first case by lemma 1 ii) $G = QN$, where Q is a normal q-subgroup and N is 2-nilpotent. It follows $l_2(G) \leq 2$. In the second case, since $G = H_2(G)\, N$, where N is 2-nilpotent and since $H_2(G)$ is a proper subgroup of G, we deduce that G is soluble, a contradiction.

It follows that we may assume $S(G) \neq G$. Since $S(G) \neq H_2(G)$, there exist maximal subgroups of G that are soluble but not 2-nilpotent. It follows then that $H_2(G)$, being contained in them, is soluble. Since $G = H_2(G)N$, where N is 2-nilpotent, this implies G soluble, which is a contradiction. ■

PROPOSITION 4. – *Let G be a finite group such that*

$$G \supsetneq H_2(G) \supsetneq \Sigma(G) \supsetneq \Phi(G)$$

then $\Sigma(G)$ is nilpotent and $G^{\mathcal{H}_2} = G^{\Sigma}$. Further $\Sigma(G) = G^{\Sigma}\Phi(G)$.

PROOF. – With an argument used several times we easily get that $\Sigma(G)$ is 2-nilpotent, so $\Sigma(G) = KQ_2$ where K is the Hall $2'$-subgroup of $\Sigma(G)$ normal in G and Q_2 is a Sylow 2-subgroup of $\Sigma(G)$. If $K \not\subseteq \Phi(G)$ then $G = KN$, where N is a maximal subgroup not containing K, and for this reason, supersoluble and therefore 2-nilpotent. Since $G = KN$ and K is a $2'$-subgroup, we have that G is 2-nilpotent in contradiction with the assumption that $G \subsetneq H_2(G)$. So $K \subset \Phi(G)$ and $\Sigma(G) = \Phi(G)Q_2$. By Frattini's argument, $G = \Sigma(G)N_G(Q_2) = \Phi(G)N_G(Q_2) = N_G(Q_2)$ so Q_2 is normal in G. It follows $\Sigma(G)$ nilpotent. By Theorem 4 in [3], $\Sigma(G) = G^{\Sigma}\Phi(G)$. Since $\Sigma \subset \mathcal{H}_2$, $G^{\mathcal{H}_2} \subseteq G^{\Sigma}$.

Since $G^{\mathcal{H}_2} \not\subseteq \Phi(G)$, there exists a maximal subgroup M such that $G^{\mathcal{H}_2} \not\subseteq M$. It follows $G = G^{\mathcal{H}_2}M$ and so $G = G^{\Sigma}M$.

Since $\Sigma(G) = G^{\Sigma}\Phi(G)$, $G = \Sigma(G)\, M$ so M is supersoluble. It follows that $G/G^{\mathcal{H}_2}$ is supersoluble, and so $G^{\Sigma} \subseteq G^{\mathcal{H}_2}$. Then $G^{\Sigma} = G^{\mathcal{H}_2}$. ■

The following two theorems extend to $S(G)$ in an insoluble group the results obtained for $\Sigma(G)$ in a soluble group (cf. [3] Theorem 4).

Recall that $F^*(G)$ denotes the generalized Fitting subgroup of G and

$E(G)$ is the maximal normal semisimple subgroup of G (for the definitions see [8] chapter 6, paragraph 6). It holds $F^*(G) = E(G)\,F(G)$.

THEOREM 1. – *Let G be a finite insoluble group such that*

$$\Phi(G) \subsetneq S(G) \subseteq F^*(G).$$

Then

$$S(G) = E(G)\,\Phi(G) = G^S\,\Phi(G).$$

PROOF. – Claim a): $S(G) \cap F(G) = \Phi(G)$.

Obviously $\Phi(G) \subseteq S(G) \cap F(G)$. If $\Phi(G) \subsetneq S(G) \cap F(G)$, we could find a maximal subgroup M such that $G = (S(G) \cap F(G))M$. So $G = S(G)M$, which implies M soluble. Since $S(G) \cap F(G)$ is nilpotent, we get G soluble, in contradiction with the assumption. So claim a) is proved.

Claim b) $E(G) \subseteq S(G)$.

Since $G/S(G)$ is soluble $(S(G)E(G))/S(G) \simeq E(G)/(S(G) \cap E(G))$ is soluble. Since $E(G)$ has no soluble proper quotients, we have $E(G) = S(G) \cap E(G)$ so $S(G) \supseteq E(G)$. Claim b) is proved.

Now we prove that $S(G) = E(G)\,\Phi(G)$. Since $S(G) \subseteq F^*(G)$, by using claim a), claim b) and Dedekind modular law we have:

$$S(G) \cap F^*(G) = S(G) \cap (F(G)\,E(G)) = E(G)(S(G) \cap F(G)) = E(G)\,\Phi(G).$$

It remains to prove that $G^S\,\Phi(G) = S(G)$. We obviously have $G^S \subseteq S(G)$ and so $G^S\,\Phi(G) \subseteq S(G)$.

Since $G/G^S\,\Phi(G)$ is soluble, $S(G)/G^S\,\Phi(G)$ is also soluble and so

$$(E(G)\,\Phi(G))/G^S\,\Phi(G)$$

is soluble. But

$$\frac{E(G)\,\Phi(G)}{G^S\,\Phi(G)} \simeq \frac{E(G)\,\Phi(G)}{\Phi(G)} \Big/ \frac{G^S\,\Phi(G)}{\Phi(G)}$$

so it is isomorphic to a soluble quotient of

$$(E(G)\,\Phi(G))/\Phi(G) \simeq E(G)/(E(G) \cap \Phi(G)).$$

But this last group does not have any proper soluble quotient. So $E(G)\,\Phi(G) = G^S\,\Phi(G)$ as we wanted. ■

The following theorem is a sort of converse of the previous theorem:

THEOREM 2. – *Let G be a finite (insoluble) group such that*

$$S(G) = G^S\,\Phi(G).$$

then

$$\frac{S(G)}{\Phi(G)} \subseteq F^*\left(\frac{G}{\Phi(G)}\right).$$

PROOF. – Since $S(G) = G^S \Phi(G)$ we have $G/S(G)$ soluble and G non soluble. Since $S(G) \supsetneq \Phi(G)$, $G = S(G)\, M$ where M is a maximal soluble subgroup of G. We distinguish two cases:

$$\text{a)}\ S(G) \cap F^*(G) \nsubseteq M \qquad \text{b)}\ S(G) \cap F^*(G) \subseteq M\,.$$

In case a) $G = (S(G) \cap F^*(G))\, M$ and $G/(S(G) \cap F^*(G))$ is soluble. It follows that $G^S \subseteq S(G) \cap F^*(G)$ and so $S(G) = G^S \Phi(G) \subseteq S(G) \cap F^*(G)$. So $S(G) \subseteq F^*(G)$. Hence

$$\frac{S(G)}{\Phi(G)} \subseteq \frac{F^*(G)}{\Phi(G)} \subseteq F^*\left(\frac{G}{\Phi(G)}\right)$$

So assume that we are in case b) $S(G) \cap F^*(G) \subseteq M$. So $S(G) \cap F^*(G)$ is soluble. It follows that $S(G) \cap F^*(G) \subseteq F(G)$.

On the other hand

$$\frac{F^*(G)}{F^*(G) \cap S(G)} \simeq \frac{S(G)\, F^*(G)}{S(G)} \leqslant \frac{G}{S(G)}$$

so it is soluble. It follows that $F^*(G)$ is soluble, so $F^*(G) = F(G)$. Obviously $\Phi(G) \subseteq S(G) \cap F^*(G)$. If $\Phi(G) \subsetneq S(G) \cap F(G)$, with the same reasoning as in Theorem 1, we would obtain G soluble. So $\Phi(G) = S(G) \cap F(G)$.

Now we proceed by induction on the order of G. If $\Phi(G) \neq 1$, let us denote $G/\Phi(G) = \overline{G}$. Then $\overline{G}^S = (G^S \Phi(G))/\Phi(G)$ (cf. [6] p. 272) and $S(\overline{G}) = S(G)/\Phi(G)$. So $\overline{G}^S = S(\overline{G})$ (remember that in this case $\Phi(\overline{G}) = 1$).

So $\overline{G}$ verifies the same hypothesis as G. By induction we get

$$\frac{S(\overline{G})}{\Phi(\overline{G})} \subseteq F^*(\overline{G}/\Phi(G))\,,$$

i.e.

$$S(G/\Phi(G)) \subseteq F^*(G/\Phi(G))$$

as we wanted.

So we may assume $\Phi(G) = 1$. It follows then:

$$S(G) \cap F^*(G) = S(G) \cap F(G) = \Phi(G) = 1\,.$$

We then obtain $[S(G), F(G)] = 1$ and so $S(G) \subseteq C_G(F(G)) \subseteq F^*(G)$ as we wanted. ■

To finish this section we observe that while Theorems 1 and 2 of [2] do not hold for $p = 2$, Theorem 3 of [2] is valid even for $p = 2$. The proof can be done in the same way as in [2], by using Lemma 1 ii) of this paper instead of Theorem 2 of [2].

In addition, an example, similar to Example 1 of [2], can be provided of a finite soluble group G in which $H_2(G)$ is neither 2-nilpotent nor it has a normal Sylow 2-subgroup.

EXAMPLE. – Let

$$M = \langle a, b, c \mid a^3 = b^3 = c^8 = 1, [a, b] = 1, [a, c] = b, b^c = a \rangle$$

It is easy to see that M is a non supersoluble 2-nilpotent group. Since $O_2(M) = 1$, M possesses a faithful irreducible $GF(2)$-module V (see f.i. [6] p. 177).

Let $G = VM$. Obviously $(|G|, \bar{r}_2(G)) \neq 1$. (For the definition of the arithmetical p-rank $\bar{r}_p(G)$ see [11] VI 8.2 p. 712). We have $G = O_{2, 2', 2}(G)$ and $X = H_2(G)$ is a maximal subgroup of G of index 2, so it is not 2-nilpotent and it does not have normal Sylow 2-subgroup.

Also, since M is a maximal subgroup of G and M is non-supersoluble, $\Sigma(G) = H_3(G) = \Phi(G) = 1$.

Section 2.

In this section we deal with $C(G)$, the intersection of maximal non-cyclic subgroups of a finite group G and with $M(G)$, the intersection of non-abelian maximal subgroups of G.

The first three theorems characterize non-abelian groups G, in which $C(G) \neq \Phi(G)$ (the abelian case being obvious).

We begin with p-groups, p a prime, with the following easy theorem:

THEOREM 3. – *Let G be a non-abelian group of order p^n, p a prime. Then $C(G) \neq \Phi(G)$ if and only if G is isomorphic to one of the following (classes of) groups:*

a) $G = \langle a, b \mid a^{p^{n-1}} = b^p = 1, a^b = a^{1+p^{n-2}} \rangle$ *where $n \geq 3$ if $p > 2$ and $n > 3$ if $p = 2$*

b) $G \simeq Q$ *the quaternion group of order* 8.

PROOF. – Let $C(G) \neq \Phi(G)$. Suppose first that $p > 2$, obviously $n > 2$. By [11] III. 8.4 $C(G) \neq G$. It follows that there exist a maximal cyclic subgroup A of order p^{n-1} and $G = C(G)A$. By [12] (Theorem 4.4 p.193) $G \simeq M_n(p)$, i.e. to the group described in a). Conversely, with very easy calculation we can prove that $\langle a^p, b \rangle$ is the unique non-cyclic maximal subgroup of G and that every other maximal subgroup is cyclic.

So it holds $C(G) \neq \Phi(G)$.

Now suppose $p = 2$, and $C(G) \neq \Phi(G)$. By [11] III. 8.4 either $C(G) \neq G$ or G is the quaternion group of order 8. So either exist a maximal cyclic subgroup of order 2^{n-1} and $n \geqslant 3$ or G is the quaternion group of order 8.

In the first case, by [12] (Theorem 4.4), if $n = 3$ G is either the quaternion group Q or the dihedral group D of order 8. But D cannot occur since if $G \simeq D$, $C(G) = \Phi(G)$ as it is easily seen. If $n > 3$ then G is isomorphic to $M_n(2)$, D_n (dihedral group D of order 2^n), Q_n (generalized quaternion group of order 2^n) or S_n (the semidihedral group of order 2^n). But only $M_n(2)$ can occur, since for $n > 3$ in all other case $C(G) = \Phi(G)$, as it is easily seen. So G is either Q or, if $n > 3$, $M_n(2)$.

Conversely both of these groups verify the condition $C(G) \neq \Phi(G)$, since they have a unique maximal non cyclic subgroup. ■

Next two theorems concern groups with composite order. Obviously if $C(G) = G$, i.e., if G is cyclic or minimal non cyclic, the condition $C(G) \neq \Phi(G)$ is automatically satisfied. So we are interested in the case $G \neq C(G)$. So under the assumption $\Phi(G) \subsetneq C(G) \subsetneq G$,by lemma 1)i, we have $G = C(G)N$ where N is a cyclic maximal subgroup of G.

We study separately the cases: N normal in G and no such N normal in G exists.

THEOREM 4. – *Let G be a non abelian group. Then* $G = C(G)N$, $C(G) \neq G$, *N cyclic maximal normal subgroup of G if and only if G is isomorphic to one of the following groups:*

A) $G = \langle x, y \mid x^m = 1 = y^{p^n}, y^{-1}xy = x^r \rangle$ *where* $(m, p) = 1$, $r^p \equiv$ 1)mod p) *and* $(r-1, m) \neq 1$.

B) G is nilpotent, $G = K \times P$ *where K, the* p'*-Hall subgroup, is cyclic and where P, the Sylow p-subgroup, is a p-group described in Theorem* 3 *a), i.e.*

$$P = \langle y, z \mid y^p = z^{p^{n-1}} = 1, \quad y^{-1}zy = z^{1+p^{n-2}} \rangle$$

PROOF. – Suppose $G \neq C(G)$, $G = C(G)N$, where N is a cyclic, normal, maximal subgroup of G. Since N is maximal G/N does not have any proper subgroup, so $[G:N] = p$ for a prime p.

Distinguish two cases: i) all Sylow subgroups of G are cyclic; ii) there exists at least one Sylow subgroup of G, which is not cyclic.

Suppose we are in the case i). $N = K \times P_1$, where K is the Hall p'-subgroup of N (of G) and P_1 is the Sylow p-subgroup of N. Suppose that P is a Sylow p-subgroup of G containing P_1, then $P = \langle y \rangle$, $P_1 = \langle y^p \rangle$. If $K = \langle x \rangle$, $[y^p, x] = 1$. If $|K| = m$, we have:

$$G = KP = \langle x, y \mid x^m = y^{p^n} = 1, \, y^{-1}xy = x^r \rangle, \, (m, p) = 1, \; r^p \equiv 1 (\mathrm{mod}\, m).$$

We have $G' = \langle [x, y] \rangle = \langle x^{r-1} \rangle$. So if $(r-1, m) = 1$, we would have $G' = K \subset C(G)$. So K would be contained in every non-cyclic maximal subgroup of G. But $G/K \simeq P$ is cyclic, so N is the unique maximal subgroup containing K and this is a contradiction. So $(r-1, m) \neq 1$ as required in A).

Suppose now we are in the case ii). Since $[G : N] = p$ and N is cyclic, the only non-cyclic Sylow subgroups can be those relative to the prime p. Also, with the some notations introduced above, P is metacyclic with a cyclic maximal subgroup $N \cap P$. Let $\langle z \rangle = P \cap N$. Distinguish the cases $p \neq 2$ and $p = 2$.

If $p > 2$ by Theorem 14.9 [11], $P = \langle y, z \mid y^p = z^{p^{n-1}} = 1, y^{-1}zy = z^{1+p^{n-2}} \rangle$ (see the previous theorem 3) $C(P) = \langle y, z^p \rangle$ is a non-cyclic maximal subgroup of P. If we let K be the Hall p'-subgroup of G, $T = KC(P)$ is a maximal non cyclic-subgroup of G. As before let $K = \langle x \rangle$. If $[y, x] \neq 1$ we would also have $[yz, x] \neq 1$ since $[x, z] = 1$. So $M = \langle x, yz \rangle$ would be a non-cyclic maximal subgroup of G different from T. So

$$C(G) \subseteq T \cap M = K\langle y, z^p \rangle \cap K\langle yz \rangle = K(\langle y, z^p \rangle \cap \langle yz \rangle) = K\Phi(P) = K\langle z^p \rangle \subseteq N .$$

This is a contradiction with the assumption $C(G)\, N = G$. So $[y, x] = 1$ and G is nilpotent, and we get the case B).

Let now $p = 2$. The P can be dihedral, semidihedral, generalized quaternion group and for $n > 3$, $M_n(2)$.

In the first three cases there are in P at least two maximal non-cyclic subgroups P_1 and P_2 such that $P_1 \cap P_2 = \Phi(P) \subseteq N$. As before $C(G) \subseteq N$ a contradiction. So $P \simeq M_n(2)$, $n > 3$. By the same reasoning as in case $p > 2$, we get G nilpotent and so case B).

Conversely now suppose that G belongs to the class described in A) and let M be a non-cyclic maximal subgroup of G. M cannot have index p in G, in fact otherwise $M = \langle x, y^p \rangle$ would be cyclic. $[G : M] = s$, where s is a prime different from p, $M \cap \langle x \rangle = \langle x^s \rangle$. Without loss of generality $y \in M$ and since M is not cyclic, $[x^s, y] \neq 1$. But $[y, x^s] = x^{s(r-1)}$ so $[y, x^s] \in \langle x^s \rangle \cap \langle x^{r-1} \rangle$. If $(s, r-1) = 1$ we would get $[y, x^s] = 1$ a contradiction. So s divides $r-1$. So $G' = \langle x^{r-1} \rangle \subseteq M$ which is normal in G. It follows that $\langle y \rangle = P \subseteq M$ for each M non-cyclic maximal subgroup of G.So $P \subseteq C(G)$. It follows then, that, if we set $N = \langle x, y^p \rangle$, N is a cyclic, normal maximal subgroup of P and $C(G)N = PN = G$. Let now be G as in B). If M is a cyclic maximal subgroup of G such that $[G : M] = p$ we have $M \cap P$ non-cyclic so $M \cap P = \langle y, z^p \rangle$. So $M = K \times C(P)$.

If T is an other non-cyclic maximal subgroup of G different from M, T must contain P. It follows that $\Phi(K) \times C(P) \subseteq C(G)$. Since $y \in C(P)$, $y \in C(G)$ so if $N = \langle yz \rangle$, $G = C(G)N$ as we wanted. ■

THEOREM 5. – *Let G be a finite (non-abelian) group. Then $G = C(G)N$, where $C(G) \neq G$ and N is a cyclic non-normal maximal subgroup of G, if and* *only if* $Z(G)$ *is cyclic,* $G/Z(G)$ *is primitive* $G/Z(G) = (M/Z(G))(N/Z(G))$, *where* $M/Z(G)$ *is the unique minimal normal subgroup of* $G/Z(G)$ *of order* p^n, N *is a cyclic maximal subgroup of* G *and* $(p, |N/Z(G)|) = 1$.

PROOF. – Suppose first $G = C(G)N$, $C(G) \neq G$, N cyclic non-normal maximal subgroup.

By Proposition 1.3 of [13] $(G/Z(G)) = (M/Z(G))(N/Z(G))$ with the described properties. Observe that $Z(G) \subseteq N$ so $Z(G)$ is cyclic.

Conversely let $(G/Z(G)) = (M/Z(G))(N/Z(G))$ be primitive and N be a cyclic non normal maximal subgroup of G, and $M/Z(G)$ be the unique minimal normal subgroup of $G/Z(G)$. Let T be a maximal non cyclic subgroup of G. If $T \supseteq Z(G)$, since it is not conjugate to N, $T \supseteq M \supseteq G'$. If $T \not\supseteq Z(G)$, $TZ(G) = G$ so T is normal in G, $T \supseteq G'$. In any case $C(G) \supseteq G'$. It follows then that $C(G) \neq \Phi(G)$ since N is a maximal and non normal. So $G = C(G)\,N$. ∎

REMARK. – We have learnt from by Prof. V. Zambelli that a student of hers, Dott. Cristina Mataloni, has obtained in her degree dissertation, results similar to ours concerning $C(G)$.

In [4] non-nilpotent groups with $M(G) \neq \Phi(G)$ have been characterized. Now we want to complete the classification in the nilpotent case. Everything is based on the following lemma.

LEMMA 2. – *Let* G *be a* p*-group,* p *a prime, that has non-abelian maximal subgroups. Then* $M(G) = \Phi(G)$.

PROOF. – If all maximal subgroups of G are non-abelian, the lemma is trivial. So we may assume that there exist abelian maximal subgroups.

Distinguish two cases:

1) G has more than one abelian maximal subgroup.

2) G has a unique abelian maximal subgroup.

Let us begin with the case 1). We easily get that $|G/Z(G)| = p^2$ and that $Z(G)$ coincides with the intersection of all abelian maximal subgroups of G. Therefore $\Phi(G) \subseteq Z(G)$ and $\Phi(G) = M(G) \cap Z(G)$. If $\Phi(G) = Z(G)$, all maximal subgroups of G are abelian, in contradiction with our hypothesis. It follows that $\Phi(G) \subsetneq Z(G)$. If $M(G) \subseteq Z(G)$, we get $\Phi(G) = M(G)$ and the lemma is proved. So assume $Z(G) \subsetneq M(G)Z(G)$. If

$$|G/\Phi(G)| = p^n, \qquad |Z(G)/\Phi(G)| = p^{n-2}$$

so suppose that $\{a_1\Phi(G), a_2\Phi(G), \ldots, a_{n-2}\Phi(G)\}$ is a basis of $Z(G)/\Phi(G)$. There exists at least one element $a \in M(G)$ such that $a \notin Z(G)$. Let b be another element in such a way that

$$\{a_1\Phi(G), a_2\Phi(G), \ldots, a_{n-2}\Phi(G), a\Phi(G), b\Phi(G)\}$$

is a basis of $G/\Phi(G)$. Since G is non abelian, $[a, b] \neq 1$.
Also $\{a_1, a_2, \ldots, a, b\}$ is a generating system for G. Let j be an index, $j \in \{1, \ldots, n-2\}$ and set $a_j = x$.

The subgroup $M_1 = \langle a, b, a_1, a_2, \ldots, \widehat{x}, \ldots, a_{n-2}, \Phi(G)\rangle$ (where x is removed) is a maximal non abelian subgroup of G.

Also $M_2 = \langle ax, b, a_1, a_2, \ldots, \widehat{x}, \ldots, a_{n-2}, \Phi(G)\rangle$ (where x is removed) is non-abelian and it is also maximal in G since

$$\{ax\Phi(G), b\Phi(G), a_1\Phi(G), a_2\Phi(G), \ldots, x\Phi(G), \ldots, a_{n-2}\Phi(G)\}$$

is a basis of $G/\Phi(G)$. Since $a \in M_1$ and $a \notin M_2$ we have $a \notin M(G)$. So this is a contradiction.

So we may suppose we are in case 2).

Let A be the unique maximal abelian subgroup of G. If $M(G) \subseteq A$, $M(G) = \Phi(G)$ and the lemma is proved. So suppose $M(G) \nsubseteq A$ and $G = M(G)A = \langle b, A\rangle$ where $b \in M(G)$.

If $|G/\Phi(G)| = p^n$, then $G/\Phi(G)$ will have a basis of the following shape:

$$\{a_1\Phi(G), a_2\Phi(G), \ldots, a_{n-1}\Phi(G), b\Phi(G)\}$$

where, for $1 \leq i \leq n-1$, $a_i \in A$. Let j be an index, $j \in \{1, \ldots, n-1\}$ and set $a_j = x$. As in the first case of the proof, consider the following two subgroups:

$M_1 = \langle b, a_1, a_2, \ldots, \widehat{x}, \ldots, a_{n-1}, \Phi(G)\rangle$

$M_2 = \langle bx, a_1, a_2, \ldots, \widehat{x}, \ldots, a_{n-1}, \Phi(G)\rangle$ (where x is removed) M_1 and M_2 are non-abelian maximal subgroups of G and $b \notin M_2$.

It follows that $b \notin M(G)$, a contradiction. The lemma is proved. ■

An equivalent formulation of the previous lemma is:

COROLLARY 1. – *If G is a p-group, p a prime, such that $M(G) \neq \Phi(G)$, then $M(G) = G$, i.e. G is a minimal non-abelian group.*

THEOREM 6. – *Let G be a nilpotent group. Then $M(G) \neq \Phi(G)$ if and only if there exists a prime p dividing the order of G such that the Sylow p-subgroup of G is minimal non abelian, while all the other Sylow q-subgroups ($q \neq p$) of G are abelian.*

PROOF. – Let $M(G) \neq \Phi(G)$. Since, then, there exists an abelian maximal subgroup A, $[G : A] = p$, for every different q from p, the Sylow q-subgroup of G is abelian. Let P be the Sylow p-subgroup of G. If P has non-abelian maximal subgroups, by lemma 2, $\Phi(P) = M(P)$. But, if K is the Hall p'-subgroup of G, it is immediate that $K\Phi(P)$ coincides with the intersection of all maximal non-abelian subgroups of index p. On the other hand the maximal subgroups that contain P are all non-abelian and their intersection is $\Phi(K)P$. So $M(G) = K\Phi(P) \cap \Phi(K)P = \Phi(K) \times \Phi(P) = \Phi(G)$, a contradiction. So P is minimal non-abelian.

Conversely if $G = K \times P$, K abelian, P minimal non-abelian, every non-abelian maximal subgroup of G contains P, so
$P \subseteq M(G)$.

Thus $M(G) \neq \Phi(G)$.

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A. L. Gilotti: Dipartimento di Matematica
P.za di P.ta S. Donato, Università di Bologna, 40127 Bologna
E-mail: gilotti@dm.unibo.it

U. Tiberio: Dipartimento di Matematica Applicata G. Sansone
Via di S. Marta 3, 50139 Firenze
E-mail: utiberio@dma.unifi.it

Pervenuta in Redazione
il 4 maggio 2001