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Partial Hölder Continuity Results for Solutions of non Linear non Variational Elliptic Systems with Limit Controlled Growth.

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Sunto. – *Sia Ω un aperto limitato di R^n, $n > 4$, di classe C^2. Sia $u \in H^2(\Omega)$ una soluzione del sistema ellittico non lineare non variazionale*

$$a(x, u, Du, H(u)) = b(x, u, Du)$$

dove $a(x, u, \mu, \xi)$ e $b(x, u, \mu)$ sono vettori in R^N, $N \geq 1$, misurabili in x, continui in (u, μ, ξ) e (u, μ) rispettivamente. Si dimostra che se $b(x, u, \mu)$ ha andamenti controllati limite, se $a(x, u, \mu, \xi)$ è di classe C^1 in ξ e soddisfa la condizione (A) di Campanato e, unitamente a $\frac{\partial a}{\partial \xi}$, alcune condizioni di continuità, allora il vettore Du è parzialmente hölderiano per ogni esponente $\alpha < 1 - \frac{n}{p}$.

Summary. – *Let Ω be a bounded open subset of R^n, $n > 4$, of class C^2. Let $u \in H^2(\Omega)$ a solution of elliptic non linear non variational system*

$$a(x, u, Du, H(u)) = b(x, u, Du)$$

where $a(x, u, \mu, \xi)$ and $b(x, u, \mu)$ are vectors in R^N, $N \geq 1$, measurable in x, continuous in (u, μ, ξ) and (u, μ) respectively. Here, we demonstrate that if $b(x, u, \mu)$ has limit controlled growth, if $a(x, u, \mu, \xi)$ is of class C^1 in ξ and satisfies the Campanato condition (A) and, together with $\frac{\partial a}{\partial \xi}$, certain continuity assumptions, then the vector Du is partially Hölder continuous for every exponent $\alpha < 1 - \frac{n}{p}$.

1. – Introduction.

In this work we study the partial Hölder continuity for solutions of second order non linear non variational elliptic systems of type

$$a(x, u, Du, H(u)) = b(x, u, Du) \tag{1.1}$$

with limit controlled growth. This result is similar to the one demonstrated in the case of strictly controlled growth by L. Fattorusso - G. Idone [1].

Let Ω be a bounded open subset of R^n, $n > 4$, of class C^2 and $u \in H^2(\Omega)$ a solution of elliptic non linear non variational system (1.1) where $a(x, u, \mu, \xi)$ and $b(x, u, \mu)$ are vectors of R^N, $N \geq 1$, measurable in x, continuous in (u, μ, ξ) and (u, μ) respectively, satisfying the conditions:

(1.2) $a(x, u, \mu, 0) = 0$

(1.3) $a(x, u, \mu, \xi)$ is of class C^1 in ξ with derivatives $\dfrac{\partial a}{\partial \xi_{ij}}$ (1) uniformly continuous and bounded in $\Omega \times R^N \times R^{nN} \times R^{n^2 N}$,

(1.4) there exists a constant c such that, $\forall u \in R^N$, $\forall \mu \in R^{nN}$ and for almost every $x \in \Omega$ we have

$$\|b(x, u, \mu)\| \leq c\{f(x) + \|u\|^\alpha + \|\mu\|^\beta\}$$

with $f \in L^2(\Omega)$ and with $\alpha = \dfrac{n}{n-4}$ and $\beta = \dfrac{n}{n-2}$

(A) there exist three positive constants $\bar{\alpha}$, $\bar{\gamma}$, and $\bar{\delta}$, with $\bar{\gamma} + \bar{\delta} < 1$, such that, $\forall u \in R^N$, $\forall \mu \in R^{nN}$, $\forall \tau, \eta \in R^{n^2 N}$ and for almost every $x \in \Omega$ we have

$$\left\| \sum_{i=1}^{n} \tau_{ii} - \bar{\alpha}[a(x, u, \mu, \tau + \eta) - a(x, u, \mu, \eta)] \right\|^2 \leq \bar{\gamma}\|\tau\|^2 + \bar{\delta} \left\| \sum_{i=1}^{n} \tau_{ii} \right\|^2$$

(B) there exists a non negative function $\omega(t)$, defined for $t \geq 0$, continuous, bounded, concave, non decreasing with $\omega(0) = 0$ such that $\forall x, y \in \Omega$, $\forall u, v \in R^N$, $\forall \mu, \bar{\mu} \in R^{nN}$ and $\forall \xi, \tau \in R^{n^2 N}$

$$\|a(x, u, \mu, \xi) - a(y, u, \bar{\mu}, \xi)\| \leq \omega\big(d^2(x, y) + \|u - v\|^2 + \|\mu - \bar{\mu}\|^2\big) \cdot \|\xi\|,$$

$$\left\| \frac{\partial a(x, u, \mu, \xi)}{\partial \xi} - \frac{\partial a(x, u, \mu, \tau)}{\partial \xi} \right\| \leq \omega(\|\xi - \tau\|^2) \ (2)$$

Partial Hölder continuity results for solutions of the system (1.1) had been obtained in [1] in the strictly controlled growth case. For the case $n \leq 4$ we refer to [2] section n. 3.

In this work we obtain the following partial Hölder continuity result for solutions of (1.1) in the limit controlled growth case:

THEOREM 1.1. – *If* $u \in H^2(\Omega, R^N)$ *is a solution to the system* (1.1) *and if the assumptions* (1.3) (1.4) *with* $f \in L^p(\Omega)$, $p > n$ *hold, then there exists a set*

(1) $\dfrac{\partial a(x, u, \mu, \xi)}{\partial \xi_{ij}} = \left\{ \dfrac{\partial a^h(x, u, \mu, \xi)}{\partial \xi^k_{ij}} \right\}$ $\quad h, k = 1, \ldots, N.$

(2) $\dfrac{\partial a(x, u, \mu, \eta)}{\partial \xi} = \left\{ \dfrac{\partial a(x, u, \mu, \eta)}{\partial \xi_{ij}} \right\}$ $\quad i, j = 1, \ldots, n.$

$\mathcal{B}_0$, closed in Ω (3), *with*

$$\mathcal{B}_2 \subset \mathcal{B}_0 \subset \mathcal{B}_1 \cup \mathcal{B}_2$$

such that

$$Du \in C^{0,\alpha}(\Omega \setminus \mathcal{B}_0, R^{nN}), \qquad \forall \alpha < 1 - \frac{n}{p}.$$

In order to show the theorem mentioned, above, we need the following L^p local regularity theorem for solutions of (1.1). It serves an auxiliary role, but it has indeed an interest in its own right.

THEOREM 1.2. – *If $u \in H^2(\Omega, R^N)$ is a solution to the system* (1.1) *and if* (1.2), (1.4) *with $f \in L^p(\Omega)$, $p > 2$, and* (A) *hold, then $u \in H^{2,p}_{\mathrm{loc}}(\Omega, R^N)$ and there exists $\sigma_0(u)$ such that $\forall B(x^0, \sigma) \subset B(x^0, 2\sigma) \subset\subset \Omega$, with $\sigma < \sigma_0$ one has:*

$$(1.5) \qquad \fint_{B(x^0,\sigma)} (\|u\|^{\frac{2n}{n-4}} + \|Du\|^{2^*} + \|H(u)\|^2)^{p/2}\, dx \leqslant$$

$$\leqslant k\left\{\left[\fint_{B(x^0,2\sigma)} (\|u\|^{\frac{2n}{n-4}} + \|Du\|^{2^*} + \|H(u)\|^2) dx\right]^{p/2} + \fint_{B(x^0,2\sigma)} |f|^p\, dx\right\}$$

where k does not depend on σ.

2. – Local L^p-regularity of the matrix $H(u)$.

Proof of the theorem 1.2.

Let $u \in H^2(\Omega, R^N)$ be a solution in Ω to the system (1.1), being $a(x, u, \mu, \xi)$ and $b(x, u, \mu)$ vectors of R^N satisfying assumptions (A), (1.2) and (1.4) with $f \in L^p$, $p > 2$.

From the estimate (3.2) of lemma (3.1) of [2],

$$(2.1) \qquad \int_{B(x^0,\sigma)} \|H(u)\|^2\, dx \leqslant c\sigma^{-2} \int_{B(x^0,2\sigma)} \|Du - (Du)_{2\sigma}\|^2\, dx + $$

$$+ c \int_{B(x^0,2\sigma)} \|b(x, u, Du)\|^2\, dx$$

(3) In particular $m(\mathcal{B}_0) = 0$.

and by the theorem of Poincaré, we have

$$(2.2)\qquad \int_{B(x^0,\,2\sigma)} \|Du-(Du)_{2\sigma}\|^2\,dx \leqslant c\left(\int_{B(x^0,\,2\sigma)} \|H(u)^{\frac{2n}{n+2}}\|^2\,dx\right)^{\frac{n+2}{n}}.$$

From the assumption (1.4) one has:

$$(2.3)\qquad \int_{B(x^0,\,2\sigma)} \|b(x,\,u,\,Du)\|^2\,dx \leqslant c\int_{B(x^0,\,2\sigma)} |f(x)|^2\,dx + \int_{B(x^0,\,2\sigma)} \|u\|^{\frac{2n}{n-4}}\,dx + $$

$$+\int_{B(x^0,\,2\sigma)} \|Du\|^{\frac{2n}{n-2}}\,dx\,.$$

Now, observing that

$$\int_{B(x^0,\,2\sigma)} \|Du\|^{2^*}\,dx \leqslant c\int_{B(x^0,\,2\sigma)} \|Du-(Du)_{2\sigma}\|^{2^*}\,dx + \int_{B(x^0,\,2\sigma)} \|(Du)_{2\sigma}\|^{2^*}\,dx \leqslant$$

$$\leqslant c\int_{B(x^0,\,2\sigma)} \|Du-(Du)_{2\sigma}\|^{2^*}\,dx + \sigma^{n(1-2^*)}\left(\int_{B(x^0,\,2\sigma)} \|Du\|\right)^{2^*} dx \leqslant$$

$$\leqslant c\int_{B(x^0,\,2\sigma)} \|Du-(Du)_{2\sigma}\|^{2^*}\,dx + \sigma^{-2}\left[\int_{B(x^0,\,2\sigma)} \|Du\|^{2^*\frac{n}{n+2}}\,dx\right]^{\frac{n+2}{n}}$$

and taking into account the (3.19) of page 21 of [3], we have

$$\int_{B(x^0,\,2\sigma)} \|Du\|^{2^*}\,dx \leqslant \left(\int_{B(x^0,\,2\sigma)} \|H(u)\|\right)^{2^*/2} dx + \sigma^{-2}\left[\int_{B(x^0,\,2\sigma)} \|Du\|^{2^*\frac{n}{n+2}}\,dx\right]^{\frac{n+2}{n}}.$$

Now from the absolute-continuity of the integral, we have that, $\forall\lambda$ there exists $\sigma(\mu,\,\lambda)$ such that if $\sigma<\sigma_1$

$$(2.4)\qquad \int_{B(x^0,2\sigma)} \|Du\|^{2^*}dx \leqslant \frac{\lambda}{3}\left(\int_{B(x^0,2\sigma)} \|H(u)\|^2 dx\right) + \sigma^{-2}\left[\int_{B(x^0,2\sigma)} \|Du\|^{2^*\frac{n}{n+2}}dx\right]^{\frac{n+2}{n}}.$$

Moreover being

$$\int_{B(x^0,\,2\sigma)} \|u\|^{\frac{2n}{n-4}}\,dx \leqslant c\sigma^n\|P\|^{\frac{2n}{n-4}} + \int_{B(x^0,\,2\sigma)} \|u-P\|^{\frac{2n}{n-4}}\,dx$$

where $P \equiv (P_1, P_2, \ldots, P_N)$ is the polynomial vector of degree $\leqslant 1$ such that

$$\int_{B(x^0,\, \sigma)} D^\alpha (u - P)\, dx = 0 \qquad \forall \alpha,\ |\alpha| \leqslant 1$$

we have:

$$\sigma^n \|P\|^{\frac{2n}{n-4}} \leqslant \sigma^{-2} \left[\int_{B(x^0,\, 2\sigma)} \|u\|^{\frac{2n}{n-4} \frac{n}{n+2}}\, dx \right]^{\frac{n+2}{n}} + c(u) \int_{B(x^0,\, 2\sigma)} \|Du\|^{2^*} dx$$

and

$$\int_{B(x^0,\, 2\sigma)} \|u - P\|^{\frac{2n}{n-4}}\, dx \leqslant c \left(\int_{B(x^0,\, 2\sigma)} \|H(u)\|^2\, dx \right)^{\frac{n}{n-4}}.$$

Hence

$$(2.5) \qquad \int_{B(x^0,\, 2\sigma)} \|u\|^{\frac{2n}{n-4}}\, dx \leqslant c\sigma^{-2} \left[\int_{B(x^0,\, 2\sigma)} \|u\|^{\frac{2n}{n-4} \frac{n}{n+2}}\, dx \right]^{\frac{n+2}{n}} + \\ + c(u) \int_{B(x^0,\, 2\sigma)} \|Du\|^{2^*} dx + c \left(\int_{B(x^0,\, 2\sigma)} \|H(u)\|^2\, dx \right)^{\frac{n}{n-4}}.$$

From (2.1), taking into account (2.2) (2.3) (2.4) and (2.5) it follows

$$(2.6) \qquad \int_{B(x^0,\, 2\sigma)} \|H(u)\|^2\, dx \leqslant \\ \leqslant c\sigma^{-2} \left[\int_{B(x^0,\, 2\sigma)} \left(\|u\|^{\frac{2n}{n-4}} + \|Du\|^{2^*} + \|H(u)\|^2\right)^{\frac{n}{n+2}} dx \right]^{\frac{n+2}{n}} + \\ + \frac{\lambda}{3} \left(\int_{B(x^0,\, 2\sigma)} \|H(u)\|^2\, dx \right) + \int_{B(x^0,\, 2\sigma)} |f(x)|^2\, dx + c(u) \int_{B(x^0,\, 2\sigma)} \|Du\|^{2^*} dx + \\ + c \left(\int_{B(x^0,\, 2\sigma)} \|H(u)\|^2\, dx \right)^{\frac{n}{n-4}}.$$

Now, from the absolute continuity of the integral, we have that $\forall \lambda > 0$

$\exists \sigma_2(u, \lambda)$ such that, if $\sigma < \sigma_2$,

$$c\left(\int_{B(x^0, 2\sigma)} \|H(u)\|^2 dx\right)^{\frac{n}{n-4}} < \frac{\lambda}{3}.$$

From this and from (2.4) (2.6)

$$(2.7) \qquad \int_{B(x^0, 2\sigma)} \|H(u)\|^2 dx \leqslant$$

$$\leqslant c\left\{\sigma^{-2}\left[\int_{B(x^0, 2\sigma)} (\|u\|^{\frac{2}{n-4}} + \|Du\|^{2^*} + \|H(u)\|^2)^{\frac{n}{n+2}} dx\right]^{\frac{n+2}{n}} + \right.$$

$$\left. + \lambda \int_{B(x^0, 2\sigma)} \|H(u)\|^2 dx + \int_{B(x^0, 2\sigma)} |f(x)|^2 dx\right\}.$$

Adding member to member (2.4), (2.5) and (2.7)

$$(2.8) \qquad \int_{B(x^0,2\sigma)} \|u\|^{\frac{2n}{n-4}} dx + \int_{B(x^0,2\sigma)} \|Du\|^{2^*} dx + \int_{B(x^0,2\sigma)} \|H(u)\|^2 dx \leqslant$$

$$\leqslant c\left\{\sigma^{-2}\left[\int_{B(x^0, 2\sigma)} (\|u\|^{\frac{2n}{n-4}} + \|Du\|^{2^*} + \|H(u)\|^2)^{\frac{n}{n+2}} dx\right]^{\frac{n+2}{n}} + \right.$$

$$\left. + \lambda\left(\int_{B(x^0, 2\sigma)} \|H(u)\|^2 dx\right) + \int_{B(x^0, 2\sigma)} |f(x)|^2 dx\right\}.$$

Now, setting

$$U(x) = \{\|u\|^{\frac{2n}{n-4}} + \|Du\|^{2^*} + \|H(u)\|^2 dx\}^{\frac{n}{n+2}}$$

$$G(x) = \{|f(x)|^2\}^{\frac{n}{n+2}}$$

the inequality (2.8), if $\sigma < \sigma_0(u, \lambda)$, can be written in the following form:

$$⨍_{B(x^0, \sigma)} U^{\frac{n}{n+2}} dx \leqslant c\left\{\left(⨍_{B(x^0, 2\sigma)} U dx\right)^{\frac{n}{n+2}} + \lambda ⨍_{B(x^0, 2\sigma)} U^{\frac{n}{n+2}} dx + ⨍_{B(x^0, 2\sigma)} G^{\frac{n}{n+2}} dx\right\}.$$

From this, using a lemma of Gehring-Giaquinta-Modica (see lemma 4.1 page 125 of [3]), written for $r = \dfrac{n+2}{n}$, $s = p\dfrac{n+2}{n}$ where $p > 2$, we deduce that there exists $\lambda_0(r, s)$ such that $\forall \lambda < \lambda_0$ $\exists \varepsilon > 0$ for which $U \in L^1_{\text{loc}}(\Omega)$, $\forall t \in$

$[r, r+\varepsilon)$ and

$$\left(\fint_{B(x^0,\sigma)} U^t\,dx \right)^{\frac{1}{t}} \leqslant k \left\{ \left(\fint_{B(x^0,2\sigma)} U^r\,dx \right)^{\frac{1}{r}} + \left(\fint_{B(x^0,2\sigma)} G^t\,dx \right)^{\frac{1}{t}} \right\}. \tag{2.9}$$

Setting in (2.9) $t = p\dfrac{n+2}{2n}$ it follows (1.5).

3. – Partial Hölder continuity of the vector Du.

With the same technique used in [1], we have the following lemmas:

LEMMA 3.1. – *If* $u \in H^2(\Omega, R^N)$ *is a solution to the system* (1.1) *and if the assumptions* (1.3) (1.4) *with* $p > 2$ *hold, then,* $\forall B(x^0, \sigma) \subset\subset \Omega$, *with* $\sigma < 2$, $\forall \tau \in (0, 1)$ *and* $\forall \varepsilon \in \left(0, (n-2)\left(1 - \frac{2}{p}\right)\right]$, *it results:*

$$\Phi(u, x^0, \tau\sigma) \leqslant A\Phi(u, x^0, \sigma) \left\{ \tau^\lambda + \sigma^{2(1-\frac{2}{p})} + [\omega(c\sigma^{2-n}\Phi(u, x^0, \sigma)]^{1-\frac{2}{p}} + \right.$$
$$\left. + \left[\omega \left(\fint_{B(x^0,\sigma)} \|H(u) - (H(u))_\sigma\|^2\,d\sigma \right) \right]^{1-\frac{2}{p}} \right\} \tag{3.1}$$

where $\lambda = n\left(1 - \frac{2}{p}\right) - \varepsilon$ *and*

$$\Phi(u, x^0, \sigma) = \sigma^\xi + \int_{B(x^0,\sigma)} \left[\|u\|^{\frac{2n}{n-4}} + \|Du\|^{2^*} + \|H(u)\|^2 \right] dx \tag{3.2}$$

with $\xi = n\left(1 - \frac{2}{p}\right)$.

Let us set:

$$\mathcal{B}_1 = \left\{ x^0 \in \Omega : \lim_{\sigma \to 0}{}'' \fint_{B(x^0,\sigma)} \|H(u) - (H(u))_\sigma\|^2\,dy > 0 \right\}$$

$$\mathcal{B}_2 = \left\{ x^0 \in \Omega : \lim_{\sigma \to 0}{}' \sigma^{2-n}\Phi(u, x^0, \sigma) > 0 \right\}.$$

By a known property of the Lebesgue integral we have

$$\text{mis}\,\mathcal{B}_1 = 0$$

and taking into account a theorem of Giusti ([3] p. 142) we obtain

$$\mathcal{H}_{n-2}(\mathcal{B}_2) = 0$$

where $\mathcal{H}_\gamma$ is the γ-dimensional Hausdorff measure.

Hence the set $\mathcal{B}_1 \cup \mathcal{B}_2$ has the measure zero.

Now, reasoning exactly as in theorem 5.1 of [4], it is easy to prove:

LEMMA 3.2. – *If* $u \in H^2(\Omega, R^N)$ *is a solution to the system* (1.1), *if the assumptions* (1.3) (1.4) *hold*, then, for every fixed $\varepsilon \in \left(0, 1 - \frac{n}{p}\right)$, *it is possible to associate to every* $x^0 \in \Omega \setminus \mathcal{B}_1 \cup \mathcal{B}_2$ *a ball* $B(x^0, R_{x^0}) \subset \Omega \setminus \mathcal{B}_2$ *and a positive number* σ_ε *such that,* $\forall t \in (0, 1)$ *and* $\forall y \in B(x^0, R_{x^0})$

$$\Phi(u, y, t\sigma_\varepsilon) \leqslant (1 + A) t^{n(1 - \frac{2}{p}) - 2\varepsilon} \Phi(u, y, \sigma_\varepsilon)$$

and hence (*see* [4])

$$H(u) \in L^{2, n(1 - \frac{2}{p}) - 2\varepsilon}(B(x^0, R_{x^0}), R^{n^2 N})$$

$$Du \in \mathcal{L}^{2, n(1 - \frac{2}{p}) - 2\varepsilon + 2}(B(x^0, R_{x^0}), R^{nN}) \, .$$

From Lemma (3.2) the theorem 1.1 easily follows.

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