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A Characterization of the Essential Spectrum and Applications.

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Sunto. – *In questo articolo lo spettro essenziale di operatori lineari chiusi e densamente definiti è caratterizzato in una grande classe degli spazi, che possiedono la proprietà di Dunford-Pettis o che sono isomorfi ad uno degli spazi $L_p(\Omega)$ $p > 1$. È dato un test di verifica pratico che garantisce la sua stabilità, per gli operatori perturbati. Inoltre applichiamo i risultati ottenuti per studiare lo spettro essenziale dell'equazione unidimensionale di trasporto con gli stati di contorno generali. Per concludere, sono discusse le condizioni sufficienti in termini di contorno e di operatori di scontro che assicurano l'invarianza dello spettro essenziale dell'operatore di flusso continuo.*

Summary. – *In this article the essential spectrum of closed, densely defined linear operators is characterized on a large class of spaces, which possess the Dunford-Pettis property or which isomorphic to one of the spaces $L_p(\Omega)$ $p > 1$. A practical criterion guaranteeing its stability, for perturbed operators, is given. Further we apply the obtained results to investigate the essential spectrum of one-dimensional transport equation with general boundary conditions. Finally, sufficient conditions in terms of boundary and collision operators assuring the invariance of the essential spectrum of the streaming operator are discussed.*

1. – Introduction.

Let A be a closed, densely defined linear operator on a Banach space X, and let $\sigma(A)$ (resp. $\varrho(A)$) denote the spectrum (resp. the resolvent set) of A. We denote by $\mathcal{C}(X)$ (resp. $\mathcal{L}(X)$) the set of all closed, densely defined linear operators (resp. the set of all bounded linear operators) on X to itself and $\mathcal{K}(X)$ the ideal of compact operators of $\mathcal{L}(X)$.

DEFINITION 1.1. – *Let $A \in \mathcal{C}(X)$. We define the essential spectrum of the operator A by*

$$\sigma_{\text{ess}}(A) = \bigcap_{C \in \mathcal{K}(X)} \sigma(A + C). \qquad \blacksquare$$

It is well known that if A is a self-adjoint operator on a Hilbert space, the essential spectrum of A is the set of limit points of the spectrum of A (with eigenvalues counted according to their multiplicities), i.e., all points of the spectrum except isolated eigenvalues of finite multiplicity (see, for example, [43, 44]).

There are many ways to define the essential spectrum of a closed, densely defined linear operator on a Banach space. Hence several definitions of the essential spectrum may be found in the literature see, for example, [11, 38] or the comments in [36], Chapter 11, p. 283, which coincide for self-adjoint operators on Hilbert spaces. Throughout this paper we are concerned with the so-called Weyl spectrum.

In 1996 and 1998, motivated by a problem concerning the spectrum of the transport operator posed in [19], Latrach and Jeribi [24, 27] obtained the following result:

THEOREM 1.1 ([27], Theorem 3.2). – *Let* (Ω, Σ, μ) *be an arbitrary positive measure space. If* A *is a closed densely defined linear operator on* $L_p(\Omega)$ $(1 \leqslant p < \infty)$ *then*

$$\sigma_{\mathrm{ess}}(A) = \bigcap_{S \in \mathcal{S}(L_p(\Omega))} \sigma(A+S)$$

where $\mathcal{S}(L_p(\Omega))$ *stands for the ideal of strictly singular operators on* $L_p(\Omega)$. ■

Recently, in 1999 Latrach [23] gives an extension of the Theorem 1.1 to general Banach spaces which possess the Dunford-Pettis property in terms of weakly compact operators and obtained the following results:

THEOREM 1.2 ([23], Theorem 3.2). – *Let* $A \in \mathcal{C}(X)$. *If* X *has the Dunford-Pettis property, then*

$$\sigma_{\mathrm{ess}}(A) = \bigcap_{F \in \mathcal{F}(X)} \sigma(A+F)$$

where $\mathcal{F}(X)$ *denote the family of weakly compact operators on* X. ■

Let $A \in \mathcal{C}(X)$, we suppose that the esssential spectrum of A is known. Let's perturbe the operator A with the bounded operator K i.e., $A + K$. What wiIl the essential spectrum of the operator $A + K$ be? If K is a compact operator on Banach spaces then $\sigma_{\mathrm{ess}}(A+K) = \sigma_{\mathrm{ess}}(A)$ (see Definition 1.1). If K is a strictly singular on L_p-spaces then $\sigma_{\mathrm{ess}}(A+K) = \sigma_{\mathrm{ess}}(A)$ (see Theorem 1.1). If K is a weakly compact on Banach spaces which possess the Dunford-Pettis property then $\sigma_{\mathrm{ess}}(A+K) = \sigma_{\mathrm{ess}}(A)$ (see Theorem 1.2). But in practice, the perturbed operator K is neither strictly singular nor weakly compact. So, it is natural to ask what are the conditions that we must impose on K such that $\sigma_{\mathrm{ess}}(A+K) =$

$\sigma_{\mathrm{ess}}(A)$. For this, we define the notion of the weak spectrum (which we denote by $\sigma_w^F(.)$ or $\sigma_w^S(.)$) by means of the operators $\mathcal{G}_A^F(X)$ or $\mathcal{G}_A^S(X)$, containing strictly the following sets $\mathcal{F}(X)$ and $\mathcal{S}(L_p(\Omega))$ (see Section 3) and we show the equality (in the sense of the inclusion) of the sets $\sigma_{\mathrm{ess}}(.)$ and $\sigma_w^F(.)$ or $\sigma_w^S(.)$. This gives a positive answers to an open question posed in [19].

The purpose of the first part of this paper is to point out how, by means of the concept of regular operators (cf. [7, 17, 29]) and the technique developed in [24], Section 2, it is possible to improve the definition of the essential spectrum, in the same way as in Theorems 1.1 and 1.2, on Banach spaces X satisfying

$$(H1)\ \begin{cases} X \text{ has the Dunford-Pettis property or } X \\ \text{is isomorphic to one of the spaces } L_p(\Omega, \Sigma, d\mu) \\ p > 1 \text{ where } (\Omega, \Sigma, d\mu) \text{ is a positive mesure space.} \end{cases}$$

In the second part of the paper we apply the results described above to investigate the essential spectrum of the following integro-differential operator

$$A_H\psi(x, \xi) = -\xi\frac{\partial\psi}{\partial x}(x, \xi) - \sigma(\xi)\,\psi(x, \xi) + \int_{-1}^{1}\kappa(x, \xi, \xi')\,\psi(x, \xi')\,d\xi' = T_H\psi + K\psi$$

with general boundary conditions where $x \in [-a, a]$, $a > 0$, and $\xi \in [-1, 1]$. This operator describes the transport of particles (neutrons, photons, molecules of gas, etc.) in a plane parallel domain with a width of $2a$ mean free paths. The function $\psi(x, \xi)$ represents the number (or probability) density of gas particles having the position x and the direction cosine of propagation ξ. (The variable ξ may be thought of as the cosine of the angle between the velocity of particles and the x-direction). The functions $\sigma(.)$ and $\kappa(., ., .)$ are called, respectively, the collision frequency and the scattering kernel. The boundary conditions are modeled by

$$\psi_{|\Gamma_-} = H\psi_{|\Gamma_+}$$

where Γ_- (resp. Γ_+) is the incoming (resp. outgoing) part of the phase space boundary, $\psi_{|\Gamma_-}$ (resp. $\psi_{|\Gamma_+}$) is the restriction of ψ to Γ_- (resp. Γ_+) and H is a linear bounded operator from a suitable function space on Γ_+ to a similar one on Γ_-.

In the classical neutron transport theory ($H = 0$), it is well known that

$$(1.1)\qquad \sigma_{\mathrm{ess}}(T_0 + K) = \left\{\lambda \in \boldsymbol{C} \text{ such that } \operatorname{Re}\lambda \leqslant -\liminf_{|\xi| \to 0}\sigma(\xi)\right\} \quad \text{if} \quad K = 0\,.$$

If $K \neq 0$ and if some power of $(\lambda - T_0)^{-1}K$ is compact then it is well known that $\sigma(T_0+K) \cap \left\{\lambda \in \mathbf{C} \text{ such that } \operatorname{Re}\lambda > -\liminf_{|\xi| \to 0} \sigma(\xi)\right\}$ consists of, at most, isolated eigenvalues with finite algebraic multiplicities (see, for instance, [20] or [32]). On the other hand, under the above assumptions, the half plane $\left\{\lambda \in \mathbf{C} \text{ such that } \operatorname{Re}\lambda \leqslant -\liminf_{|\xi| \to 0} \sigma(\xi)\right\}$ may *contain, a priori, some holes in the resolvent set of* $T_0 + K$. So that (1.1) is not, a priori, true if $K \neq 0$ and some power of $(\lambda - T_0)^{-1}K$ is compact. These remarks remain valid if instead of K we consider the boundary operator H or both K and H. By taking advantage of the results of Section 2 and the compactness results obtained in Section 3, we are going to prove that (1.1) is, actually, true for general classes of boundary and collision operators H and K. More precisely, we give sufficient conditions on the collision operators K under which $\sigma_{\text{ess}}(T_H + K) = \sigma_{\text{ess}}(T_H)$ regardless of the boundary operator H. Furthermore, a broad class of boundary operators H (containing, in particular, those investigated in [12], [13], [14], [19], [24] and [27]) for which $\sigma_{\text{ess}}(T_H) = \sigma_{\text{ess}}(T_0)$ is considered.

Note that even though the spectral theory of transport operators is a classical theme in transport theory, generally, the analysis focuses on the point spectrum of these operators (see, for instance, [9], [21], [22], [31], [40], [41] or [33]). In fact, the knowledge of the (peripheral) point spectrum permits to obtain a simple description of the time asymptotic behaviour ($t \to \infty$) of the solution of the associated Cauchy problem (cf. [9], [41], [20] or [32]).

We organize the paper in the following way: The next section is devoted to the essential spectrum of closed densely defined linear operators on Banach spaces which possess the Dunford-Pettis property or which isomorphic to one of the spaces $L_p(\Omega)$ $p > 1$. The main result of this section is Theorem 2.1. In Section 3 we apply the results obtained in the second section to investigate the essential spectrum of the one-dimensional transport operator with general boundary conditions. Sufficient conditions, bearing on boundary and collision operators, assuring the invariance of the essential spectrum of the streaming operator T_0 are given. We discuss briefly by discussing the essential spectrum of transport operator with vacuum boundary conditions in arbitrary dimension and the essential spectrum of transport operators arising in growing cell populations.

2. – The main result.

The purpose of this section is to discuss essential spectrum of non-selfadjoint closed, densely defined linear operator on Banach spaces which possess the Dunford-Pettis property or which isomorphic to one of the spaces $L_p(\Omega)$ $p > 1$.

For the sake of completeness we first recall the following notions which will be used in the sequel. Let A be a closed, densely defined linear operator on a Banach space X. A is a Fredholm operator if the null space $N(A)$ of A is finite dimensional and the range $R(A)$ of A is closed and finite codimensional in X. The Fredholm index of A is the number $i(A) = \dim N(A) - \operatorname{codim} R(A)$. The Fredholm domain of A, Φ_A, is given by

$$\Phi_A := \{\lambda \in \boldsymbol{C} \text{ such that } \lambda - A \text{ is a Fredholm operator on } X\}.$$

DEFINITION 2.1. – *An operator $A \in \mathcal{L}(X)$ is called strictly singular if A is not an isomorphism when restricted to any infinite-dimensional subspace of X.* ■

The concept of strictly singular operators was introduced in the pioneering paper by Kato [17] as a generalization of the notion of compact operators. The class of strictly singular operators has been extensively studied in the late 60's (see, for example, [6, 7], [30, 34] and references therein). For our own use, let us recall the following three facts. The set of all strictly singular operators on X, $\mathcal{S}(X)$, forms a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$, if X is a Hilbert space then $\mathcal{K}(X) = \mathcal{S}(X)$ and the class of weakly compact operators on L_1-spaces is nothing else but the family of strictly singular operators on L_1-spaces (see [34], Theorem 1).

DEFINITION 2.2. – *An operator $A \in \mathcal{L}(X)$ is said to be weakly compact if $A(B)$ is relatively weakly compact for every bounded subset $B \subset X$.* ■

The family of weakly compact operators on X, $\mathcal{W}(X)$, is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$ (cf. [5, 7]). Note also that if $X = L_1(\Omega, \Sigma, d\mu)$, where $(\Omega, \Sigma, d\mu)$ is a positive measure space or $X = C(K)$ with K is a compact Hausdorff space then $\mathcal{W}(X) = \mathcal{S}(X)$ (cf. [34]).

DEFINITION 2.3. – *A Banach space X is said to have the Dunford-Pettis property (for short property DP) if for each Banach space Y every weakly compact operator $T : X \to Y$ takes weakly compact sets in X into norm compact sets of Y.* ■

The Dunford-Pettis property as defined above was explicity defined by Grothendieck [10] who undertook an extensive study of this and related properties. It is well known that any L_1 space has the property DP [4]. Also, if Ω is a compact Hausdorff space $C(\Omega)$ has the property DP [10]. For further examples we refer to [3] or [5], p. 494, 497, 508, and 511. Note that the property DP is not preserved under conjugation. However, if X is a Banach space whose dual has the property DP then X has the property DP (see, e.g., [10]). For more information we refer to the paper by Diestel [3] which contains a survey and exposition of the Dunford-Pettis property and related topics.

The following elementary lemma is needed later.

LEMMA 2.1. – *(i)* If *X has the property DP, then*

$$\mathcal{W}(X)\,\mathcal{W}(X) \subset \mathcal{K}(X).$$

(ii) Let $(\Omega, \Sigma, d\mu)$ *be a positive mesure space and* $p > 1$. *If X is isomorphic to one of the spaces* $L_p(\Omega, \Sigma, d\mu)$, *then*

$$\mathcal{S}(X)\ \mathcal{S}(X) \subset \mathcal{K}(X). \quad \blacksquare$$

PROOF. – (i) Let $T_1, T_2 \in \mathcal{W}(X)$. If U is a bounded subset of X, then $T_1(U)$ is relatively weakly compact. Accordingly, since X has the property DP, $T_2(T_1(U))$ is a relatively compact subset of X. That is, $T_2 T_1 \in \mathcal{K}(X)$.

Assertion (ii) follows from [30], Theorem 1.b and the proof of Lemma is finished. Q.E.D.

The main result of this section is the following:

THEOREM 2.1. – *Let X be a Banach space satisfying the hypothesis* $(H1)$.

i) Let $A \in \mathcal{C}(X)$.

If X has the Dunford-Pettis property then $\sigma_{\text{ess}}(A) = \sigma_w^F(A)$, *where* $\sigma_w^F(A) = \bigcap_{C \in \mathcal{G}_A^F(X)} \sigma(A+C)$ *and* $\mathcal{G}_A^F(X) = \{K \in \mathcal{L}(X)$ *such that* $(\lambda - A)^{-1}K \in \mathcal{W}(X)$ *for some* $\lambda \in \varrho(A)\}$.

If X is isomorphic to one of the spaces $L_p(\Omega)$ $p > 1$ *then* $\sigma_{\text{ess}}(A) = \sigma_w^S(A)$, *where* $\sigma_w^S(A) = \bigcap_{C \in \mathcal{G}_A^S(X)} \sigma(A+C)$ *and* $\mathcal{G}_A^S(X) = \{K \in \mathcal{L}(X)$ *such that* $(\lambda - A)^{-1}K \in \mathcal{S}(X)$ *for some* $\lambda \in \varrho(A)\}$.

ii) Let A and $B \in \mathcal{C}(X)$.

If X has the Dunford-Pettis property and if for some $\lambda \in \varrho(A) \cap \varrho(B)$ *we have* $(\lambda - A)^{-1} - (\lambda - B)^{-1} \in \mathcal{W}(X)$, *then*

$$\sigma_w^F(A) = \sigma_w^F(B).$$

If X is isomorphic to one of the spaces $L_p(\Omega)$ $p > 1$ *and if for some* $\lambda \in \varrho(A) \cap \varrho(B)$ *we have* $(\lambda - A)^{-1} - (\lambda - B)^{-1} \in \mathcal{S}(X)$, *then*

$$\sigma_w^S(A) = \sigma_w^S(B). \quad \blacksquare$$

REMARK 2.1. – $a)$ Let us notice that following Pelczynski [34], Theorem 1, the class of weakly compact operators on L_1-spaces is nothing else but the family of strictly singular operators on L_1-spaces. So, Theorem 2.1 may be regarded as an extension of [24], Theorems 3.1 and 3.2, to L_p-spaces for $1 < p < \infty$, and an generalization of [24], Theorems 3.1 and 3.2, [27], Theorems 3.2 and 3.3, and [23], Theorem 3.2 and gives an unified definition of the essential spectrum on Banach spaces which possess the Dunford-Pettis property or which isomorphic to $L_p(\Omega)$ $p > 1$.

b) Due to $\mathcal{K}(X) \subsetneq \mathcal{G}_A^F(X)$ and $\mathcal{K}(X) \subsetneq \mathcal{G}_A^S(X)$ (see Section 3), the first part of Theorem 2.1 shows that the definition of the essential spectrum on these spaces by means of compact operators is restrictive.

c) Observe that, in the definition of the sets $\mathcal{G}_A^F(X)$ and $\mathcal{G}_A^S(X)$, if an operator satisfies the required condition for a fixed $\lambda \in \varrho(A)$, then it satisfies it for every $\lambda \in \varrho(A)$.

d) The statement *ii*) of Theorem 2.1 provides a practical criterion for the stability of $\sigma_{\text{ess}}(.)$ for perturbed linear operators and generalizes [37], Theorem 4.7, p. 17. ■

PROOF OF THEOREM 2.1. – *i*) We first claim that $\sigma_{\text{ess}}(A) \subset \sigma_w^F(A)$ if X has the Dunford-Pettis property and $\sigma_{\text{ess}}(A) \subset \sigma_w^S(A)$ if X is isomorphic to one of the spaces $L_p(\Omega)$ $p > 1$. Indeed, if $\lambda \notin \sigma_w^F(A)$ (resp. $\lambda \notin \sigma_w^S(A)$) then there exists $K \in \mathcal{G}_A^F(X)$ (resp. $K \in \mathcal{G}_A^S(X)$) such that $\lambda \in \varrho(A+K)$, hence $\lambda \in \Phi_{(A+K)}$ and $i(\lambda - A - K) = 0$.

Let $\mu \in \varrho(A)$, we have

$$(2.1) \qquad (\lambda - A - K)^{-1} K = [I + (\lambda - A - K)^{-1}(\mu - \lambda + K)](\mu - A)^{-1} K .$$

If X has the Dunford-Pettis property, then using (2.1), and the fact that $\mathcal{T}(X)$ is a two-sided ideal of $\mathcal{L}(X)$, we infer that $(\lambda - A - K)^{-1} K \in \mathcal{T}(X)$ and consequently $((\lambda - A - K)^{-1} K)^2 \in \mathcal{K}(X)$ (see Lemma 2.1 (*i*)).

If X is isomorphic to one of the spaces $L_p(\Omega)$ $p > 1$, then using (2.1), and the fact that $\mathcal{S}(X)$ is a two-sided ideal of $\mathcal{L}(X)$, we infer that $(\lambda - A - K)^{-1} K \in \mathcal{S}(X)$ and consequently $((\lambda - A - K)^{-1} K)^2 \in \mathcal{K}(X)$ (see Lemma 2.1 (*ii*)).

Applying [24] Theorem 2.1, we infer that $(I + (\lambda - A - K)^{-1} K)$ is a Fredholm operator and $i(I + (\lambda - A - K)^{-1} K) = 0$. Using the equality $\lambda - A = (\lambda - A - K)(I + (\lambda - A - K)^{-1} K)$ together with Atkinson's theorem ([29], Proposition 2.c.7.(ii), p. 77) one gets $\lambda \in \Phi_A$ and $i(\lambda - A) = 0$. Finally, the use of [37], Theorem 4.5, p. 15, shows that $\lambda \notin \sigma_{\text{ess}}(A)$ which proves the claim.

On the other hand, since $\mathcal{K}(X) \subset \mathcal{G}_A^F(X)$ (resp. $\mathcal{K}(X) \subset \mathcal{G}_A^S(X)$) we infer that $\sigma_w^F(A) \subset \sigma_{\text{ess}}(A)$ (resp. $\sigma_w^S(A) \subset \sigma_{\text{ess}}(A)$) which completes the proof of *i*).

ii) Without loss of generality, we may suppose that $\lambda = 0$. If X has the Dunford-Pettis property, then the operator $A^{-1} - B^{-1} \in \mathcal{T}(X)$. It follows both from Lemma 2.1 (*i*) and [42], Theorem p. 287, that $\Phi_{A^{-1}} = \Phi_{B^{-1}}$ and $i(\mu - A^{-1}) = i(\mu - B^{-1})$ for all $\mu \in \Phi_{A^{-1}}$.

If X is isomorphic to one of the spaces $L_p(\Omega)$ $p>1$, then the operator $A^{-1}-B^{-1}\in\mathcal{S}(X)$. It follows both from Lemma 2.1 (*ii*) and [42], Theorem p. 287, that $\Phi_{A^{-1}}=\Phi_{B^{-1}}$ and $i(\mu-A^{-1})=i(\mu-B^{-1})$ for all $\mu\in\Phi_{A^{-1}}$.

We next infer from [37], Lemma 4.6, p. 16, that $\Phi_A=\Phi_B$ and $i(\nu-A)=i(\nu-B)$ for all $\nu\in\Phi_A$. Now, the use of [37], Theorem 4.5, p. 15, concludes the proof of *ii*). Q.E.D.

By Theorem 2.1 and [19], Lemma 4.1, we have:

COROLLARY 2.1. – *Let X be a Banach space satisfying the hypothesis $(H1)$ and let $A\in\mathcal{C}(X)$. If X has the Dunford-Pettis property, then*

$$\sigma C(A)\subset\sigma_w^F(A)\quad\textit{and}\quad\sigma R(A)\subset\sigma_w^F(A)$$

where $\sigma C(A)$ (resp. $\sigma R(A)$) denotes the continuous spectrum (resp. the residual spectrum) of A.

If X is isomorphic to one of the spaces $L_p(\Omega)$ $p>1$, then

$$\sigma C(A)\subset\sigma_w^S(A)\quad\textit{and}\quad\sigma R(A)\subset\sigma_w^S(A).\qquad\blacksquare$$

The following result provides a characterization of the weak spectrum on a Banach space X satisfying the hypothesis $(H1)$.

COROLLARY 2.2. – *Let X be a Banach space satisfying the hypothesis $(H1)$ and let $A\in\mathcal{C}(X)$. If X has the Dunford-Pettis property, then*

$$\lambda\notin\sigma_w^F(A)\quad\textit{if and only if}\quad\lambda\in\Phi_A\ \textit{and}\ i(\lambda-A)=0\,.$$

If X is isomorphic to one of the spaces $L_p(\Omega)$ $p>1$, then

$$\lambda\notin\sigma_w^S(A)\quad\textit{if and only if}\quad\lambda\in\Phi_A\ \textit{and}\ i(\lambda-A)=0\,.\qquad\blacksquare$$

PROOF. – This corollary immediately follows from Theorem 2.1 (i) and [37], Theorem 4.5, p. 15. Q.E.D.

3. – Application to transport equations.

In this section we shall apply the results of Theorem 2.1 to the one-dimensional transport equation on L_p-spaces with $p\in[1,\infty)$. Indeed, we prove the invariance of the essential spectrum of T_0, under boundary perturbations, for a wide class of boundary operators H, that is, $\sigma_{\text{ess}}(T_H)=\sigma_{\text{ess}}(T_0)$. Moreover, a general class of collision operators for which $\sigma_{\text{ess}}(T_H+K)=\sigma_{\text{ess}}(T_0)$ is also given. The main tools of proof are Theorem 2.1 and the compactness results obtained in this section.

Let

$$X_p = L_p[(-a, a) \times (-1, 1); dx\, d\xi], \ (a > 0, \ 1 \leq p < \infty)$$

and

$$\begin{aligned} X_p^o &:= L_p[\{-a\} \times (-1, 0); |\xi| d\xi] \times L_p[\{a\} \times (0, 1); |\xi| d\xi] \\ &:= X_{1,p}^o \times X_{2,p}^o \end{aligned}$$

equipped with the norm

$$\|\psi^o; X_p^o\| = [\|\psi_1^o; X_{1,p}^o\|^p + \|\psi_2^o; X_{2,p}^o\|^p]^{1/p} = \left[\int_{-1}^{0} |\psi(-a, \xi)|^p |\xi| d\xi + \int_{0}^{1} |\psi(a, \xi)|^p |\xi| d\xi\right]^{1/p}.$$

Moreover we introduce

$$\begin{aligned} X_p^i &:= L_p[\{-a\} \times (0, 1); |\xi| d\xi] \times L_p[\{a\} \times (-1, 0); |\xi| d\xi] \\ &:= X_{1,p}^i \times X_{2,p}^i \end{aligned}$$

and equipped with the norm

$$\|\psi^i; X_p^i\| = [\|\psi_1^i; X_{1,p}^i\|^p + \|\psi_2^i; X_{2,p}^i\|^p]^{1/p} = \left[\int_{0}^{1} |\psi(-a, \xi)|^p |\xi| d\xi + \int_{-1}^{0} |\psi(a, \xi)|^p |\xi| d\xi\right]^{1/p}.$$

We define the partial Sobolev space W_p by

$$W_p = \left\{\psi \in X_p \text{ such that } \xi \frac{\partial \psi}{\partial x} \in X_p\right\}.$$

It is well know that any function $\psi \in W_p$ has traces on $\{-a\}$ and $\{a\}$ in X_p^o and X_p^i (see, for instance [2] or [8]). They are denoted, respectively, by ψ^o and ψ^i, and represent the outgoing and the incoming fluxes («o» for outgoing and «i» for incoming).

We define the operator T_H by

$$\begin{cases} T_H\colon D(T_H) \subset X_p \to X_p \\ \qquad \psi \to T_H \psi(x, \xi) = -\xi \dfrac{\partial \psi}{\partial x}(x, \xi) - \sigma(\xi)\, \psi(x, \xi) \\ D(T_H) = \{\psi \in W_p \text{ such that } H\psi^o = \psi^i\}, \end{cases}$$

where $\sigma(.) \in L^{\infty}(-1, 1)$ and H is the boundary operator defined by

$$\begin{cases} H : X_p^o \to X_p^i \\ H \in \mathcal{L}(X_p^o, X_p^i). \end{cases}$$

Note that the spectrum of the operator T_0 (i.e., $H = 0$) was analyzed in [19]. In particular we have

$$\sigma(T_0) = \sigma C(T_0) = \{\lambda \in \boldsymbol{C} \text{ such that } \operatorname{Re}\lambda \leqslant -\lambda^*\} \tag{3.1}$$

where $\sigma C(T_0)$ denotes the continuous spectrum of T_0 and $\lambda^* := \liminf_{|\xi| \to 0} \sigma(\xi)$.

REMARK 3.1. – As a consequence of (3.1) and Corollary 2.1 is that

$$\sigma_{\text{ess}}(T_0) = \{\lambda \in \boldsymbol{C} \text{ such that } \operatorname{Re}\lambda \leqslant -\lambda^*\}. \quad \blacksquare$$

Let us now consider the resolvent equation for T_H

$$(\lambda - T_H)\,\psi = \varphi \tag{3.2}$$

where φ is a given element of X_p and the unknown ψ must be sought in $D(T_H)$. For $\operatorname{Re}\lambda + \lambda^* > 0$, the solution of (3.2) is formally given by

$$\psi(x, \xi) = \begin{cases} \psi(-a, \xi)\, e^{-\frac{(\lambda + \sigma(\xi))\,|a + x|}{|\xi|}} + \\ \qquad \dfrac{1}{|\xi|} \displaystyle\int_{-a}^{x} e^{-\frac{(\lambda + \sigma(\xi))\,|x - x'|}{|\xi|}} \varphi(x', \xi)\, dx' \quad 0 < \xi < 1, \\ \psi(a, \xi)\, e^{-\frac{(\lambda + \sigma(\xi))\,|a - x|}{|\xi|}} + \\ \qquad \dfrac{1}{|\xi|} \displaystyle\int_{x}^{a} e^{-\frac{(\lambda + \sigma(\xi))\,|x - x'|}{|\xi|}} \varphi(x', \xi)\, dx' \quad -1 < \xi < 0, \end{cases} \tag{3.3}$$

whereas $\psi(a, \xi)$ and $\psi(-a, \xi)$ are given by

$$\psi(a,\xi) = \psi(-a,\xi)\, e^{\frac{-2a(\lambda + \sigma(\xi))}{|\xi|}} + \frac{1}{|\xi|} \int_{-a}^{a} e^{-\frac{(\lambda + \sigma(\xi))|a - x|}{|\xi|}} \varphi(x,\xi)\, dx \quad 0 < \xi < 1, \tag{3.4}$$

$$\psi(-a, \xi) = \psi(a, \xi)\, e^{\frac{-2a(\lambda + \sigma(\xi))}{|\xi|}} + \frac{1}{|\xi|} \int_{-a}^{a} e^{-\frac{(\lambda + \sigma(\xi))\,|a + x|}{|\xi|}} \varphi(x, \xi)\, dx \quad -1 < \xi < 0. \tag{3.5}$$

In the sequel we shall need the following operators:

$$\begin{cases} M_\lambda : X_p^i \to X_p^o, \ M_\lambda u := (M_\lambda^+ u, M_\lambda^- u) \text{ with} \\ (M_\lambda^+ u)(-a, \xi) := u(-a, \xi)\, e^{\frac{-2a}{|\xi|}(\lambda + \sigma(\xi))}, \quad 0 < \xi < 1, \\ (M_\lambda^- u)(a, \xi) := u(a, \xi)\, e^{\frac{-2a}{|\xi|}(\lambda + \sigma(\xi))}, \quad -1 < \xi < 0 \end{cases}$$

$$\begin{cases} B_\lambda : X_p^i \to X_p, \ B_\lambda u := \chi_{(-1,0)}(\xi)\, B_\lambda^- u + \chi_{(0,1)}(\xi)\, B_\lambda^+ u \text{ with} \\ (B_\lambda^+ u)(-a, \xi) := u(-a, \xi)\, e^{\frac{-1}{|\xi|}(\lambda + \sigma(\xi))|a+x|}, \quad 0 < \xi < 1, \\ (B_\lambda^- u)(a, \xi) := u(a, \xi)\, e^{\frac{-1}{|\xi|}(\lambda + \sigma(\xi))|a-x|}, \quad -1 < \xi < 0 \end{cases}$$

$$\begin{cases} G_\lambda : X_p \to X_p^o, \ G_\lambda \varphi := (G_\lambda^+ \varphi, G_\lambda^- \varphi) \text{ with} \\ G_\lambda^+ \varphi := \dfrac{1}{|\xi|} \displaystyle\int_{-a}^{a} e^{\frac{-1}{|\xi|}(\lambda + \sigma(\xi))|a-x|} \varphi(x, \xi)\, dx, \quad 0 < \xi < 1, \\ G_\lambda^- \varphi := \dfrac{1}{|\xi|} \displaystyle\int_{-a}^{a} e^{\frac{-1}{|\xi|}(\lambda + \sigma(\xi))|a+x|} \varphi(x, \xi)\, dx, \quad -1 < \xi < 0 \end{cases}$$

and

$$\begin{cases} C_\lambda : X_p \to X_p, \ C_\lambda \varphi := \chi_{(-1,0)}(\xi)\, C_\lambda^- \varphi + \chi_{(0,1)}(\xi)\, C_\lambda^+ \varphi \text{ with} \\ C_\lambda^+ \varphi := \dfrac{1}{|\xi|} \displaystyle\int_{-a}^{x} e^{\frac{-1}{|\xi|}(\lambda + \sigma(\xi))|x-x'|} \varphi(x', \xi)\, dx', \quad 0 < \xi < 1, \\ C_\lambda^- \varphi := \dfrac{1}{|\xi|} \displaystyle\int_{x}^{a} e^{\frac{-1}{|\xi|}(\lambda + \sigma(\xi))|x-x'|} \varphi(x', \xi)\, dx', \quad -1 < \xi < 0 \end{cases}$$

where $\chi_{(-1,0)}(.)$ and $\chi_{(0,1)}(.)$ denote, respectively, the characteristic functions of the intervals $(-1, 0)$ and $(0, 1)$. The operators M_λ, B_λ, G_λ and C_λ are bounded on their respective spaces. In fact, their norms are bounded above, respectively, by $e^{-2a(Re\lambda + \lambda^*)}$, $[p(Re\lambda + \lambda^*)]^{-\frac{1}{p}}$, $(\mathrm{Re}\,\lambda + \lambda^*)^{-\frac{1}{q}}$ and $(\mathrm{Re}\,\lambda + \lambda^*)^{-1}$ where q denotes the conjugate of p. For the details we refer to [18].

Now we may write Eqs. (3.4) and (3.5) abstractly in the space X_p^o in the operator form

$$\psi^o = M_\lambda H \psi^o + G_\lambda \varphi\,. \tag{3.6}$$

We define the real λ_0 by

$$\lambda_0 := \begin{cases} -\lambda^* & \text{if } \|H\| \leqslant 1 \\ -\lambda^* + \dfrac{1}{2a} \log(\|H\|) & \text{if } \|H\| > 1 . \end{cases}$$

It follows from the norm estimate of M_λ that, for $\mathrm{Re}\,\lambda > \lambda_0$, $\|M_\lambda H\| < 1$ and consequently

$$\psi^o = \sum_{n \geqslant 0} (M_\lambda H)^n G_\lambda \varphi . \tag{3.7}$$

On the other hand, Eq. (3.3) can be rewritten in form

$$\psi = B_\lambda H \psi^o + C_\lambda \varphi .$$

Substituting (3.7) into the above equation we get

$$\psi = \sum_{n \geqslant 0} B_\lambda H (M_\lambda H)^n G_\lambda \varphi + C_\lambda \varphi .$$

Hence, $\{\lambda \in \boldsymbol{C}$ such that $\mathrm{Re}\,\lambda > \lambda_0\} \subset \varrho(T_H)$ and for $\mathrm{Re}\,\lambda > \lambda_0$ we have

$$(\lambda - T_H)^{-1} = \sum_{n \geqslant 0} B_\lambda H (M_\lambda H)^n G_\lambda + C_\lambda. \tag{3.8}$$

THEOREM 3.1. – *Suppose that the boundary operator H is strictly singular, then*

$$\sigma_{\mathrm{ess}}(T_H) = \sigma_{\mathrm{ess}}(T_0) . \qquad \blacksquare$$

PROOF. – Let us first note that the operator C_λ is nothing else but $(\lambda - T_0)^{-1}$. Therefore, if $\mathrm{Re}\,\lambda > \lambda_0$, then $\lambda \in \varrho(T_H) \cap \varrho(T_0)$ and

$$(\lambda - T_H)^{-1} - (\lambda - T_0)^{-1} = Q_\lambda, \tag{3.9}$$

where

$$Q_\lambda = \sum_{n \geqslant 0} B_\lambda H (M_\lambda H)^n G_\lambda .$$

Since H is strictly singular, we infer from [29], Proposition 2.c.5.(ii), p. 76, that Q_λ is strictly singular too. Now the use of (3.9) together with Theorem 2.1, gives the desired result. Q.E.D.

REMARK 3.2. – The series defining Q_λ converges in the operator norm. $\blacksquare$

REMARK 3.3. – It should be observed that the result of Theorem 3.1 is not optimal. Indeed, let $p = 2$ and consider the following boundary operator

$$\begin{cases} \widetilde{H} : X_2^o \to X_2^i \\ u \to \widetilde{H} u \\ \widetilde{H} u := \begin{pmatrix} H_{11} & 0 \\ H_{21} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \end{cases}$$

where

$$\begin{cases} H_{11} : X_{1,2}^o \to X_{1,2}^i \\ u(-a, \xi) \to u(-a, -\xi) \end{cases}$$

with H_{21} is an arbitrary operator.

Hence, in spite of the fact that $\widetilde{H}$ is not strictly singular, the use of [19], Proposition 4.1, shows $\sigma_{\text{ess}}(T_{\widetilde{H}}) = \sigma_{\text{ess}}(T_0)$. ■

Next we consider the transport operator $A_H = T_H + K$ where K is the bounded operator given by

$$\begin{cases} K : X_p \to X_p \\ \psi \to \int_{-1}^{1} \kappa(x, \xi, \xi') \, \psi(x, \xi') \, d\xi' \end{cases}$$

where $\kappa(.,.,.)$ is a measurable function from $[-a, a] \times [-1, 1] \times [-1, 1]$ to $\boldsymbol{R}$.

Observe that the operator K acts only on the variable ξ', so x may be viewed merely as a parameter in $[-a, a]$. Hence we may consider K as a function $K : x \in [-a, a] \to K(x) \in Z$ where $Z := \mathcal{L}(L_p([-1, 1], d\xi))$.

In the following we will make the assumptions:

$$(H2) \begin{cases} K \text{ is measurable, i.e.,} \\ (3.10) \quad \{x \in [-a, a] \text{ such that } K(x) \in \mathcal{O}\} \text{ is measurable if } \mathcal{O} \subset Z \text{ is open,} \\ \text{there exists a compact subset } T \subset Z \text{ such that} \\ (3.11) \qquad K(x) \in T \quad \text{a.e.} \\ \text{and finally} \\ (3.12) \qquad K(x) \in \mathcal{K}(L_p([-1, 1], d\xi)) \quad a.e. \end{cases}$$

where $\mathcal{K}(L_p([-1, 1], d\xi))$ denotes the set of all compact operators on $L_p([-1, 1], d\xi)$.

Obviously, the hypothesis (3.11) implies that

$$K(.) \in L^{\infty}(]-a, a[, Z). \tag{3.13}$$

Let $\psi \in X_p$. It is easy to see that $(K\psi)(x, \xi) = K(x)\psi(x, \xi)$ and then, by (3.13), we have

$$\int_{-1}^{1} |(K\psi)(x, \xi)|^p d\xi \leq \|K(.)\|^p_{L^{\infty}(]-a, a[, Z)} \int_{-1}^{1} |\psi(x, \xi)|^p d\xi$$

and therefore

$$\int_{-a}^{a}\int_{-1}^{1} |(K\psi)(x, \xi)|^p d\xi dx \leq \|K(.)\|^p_{L^{\infty}(]-a, a[, Z)} \int_{-a}^{a}\int_{-1}^{1} |\psi(x, \xi)|^p d\xi\, dx .$$

Thus leads to the estimate

$$\|K\|_{\mathcal{L}(X_p)} \leq \|K(.)\|_{L^{\infty}(]-a, a[, Z)}. \tag{3.14}$$

THEOREM 3.2. – *Let $p \in [1, \infty)$ and suppose that the collision operator K satisfies the hypothesis $(H2)$ on X_p. Then*

$$\sigma_{\text{ess}}(A_H) = \sigma_{\text{ess}}(T_H).$$

Further, if the boundary operator H is strictly singular then

$$\sigma_{\text{ess}}(A_H) = \sigma_{\text{ess}}(T_0) = \{\lambda \in \boldsymbol{C} \text{ such that } \operatorname{Re}\lambda \leq -\lambda^*\}. \qquad \blacksquare$$

REMARK 3.4. – As we have already mentioned in Section 2, $\mathcal{F}(X_1) = \mathcal{S}(X_1)$. Accordingly, Theorem 3.2 is a natural extension to L_p-spaces $(1 < p < \infty)$ of [24], Corollary 4.1 and Theorem 4.3. Note also that, since $\mathcal{K}(X_p) \subset \mathcal{G}_{T_H}(X_p)$, Theorem 3.2 generalizes [19], Theorem 4.5 and Corollary 4.1. $\blacksquare$

To prove this theorem the following lemmas are required.

LEMMA 3.1. – *If K satisfies $(H2)$ then, for any $\lambda \in \boldsymbol{C}$ such that $\operatorname{Re}\lambda > -\lambda^*$, the operator $(\lambda - T_H)^{-1}K$ is compact on X_p for $1 < p < \infty$ and weakly compact on X_1.* $\blacksquare$

The following lemma is inspired and adapted from [32], Lemma 2.3.

LEMMA 3.2. – *Assume that K satisfies the hypothesis $(H2)$. Then K can by approximated, in the uniform topology, by a sequence $(K_n)_n$ of operators of* *the form*

$$\kappa_n(x, \xi, \xi') = \sum_{j=1}^{n} \eta_j(x)\, \theta_j(\xi)\, \beta_j(\xi')$$

where $\eta_j(.) \in L^\infty([-a, a], dx)$, $\theta_j(.) \in L_p([-1, 1], d\xi)$ *and* $\beta_j(.) \in L_q([-1, 1], d\xi)$ (*q denotes the conjugate of p*). ■

PROOF. – Let $\varepsilon > 0$. By the assumption (3.11) there exist $K_1, \ldots, K_m$ such that $(K_i)_i \subset T$ and $T \subset \bigcap_{1 \leq i \leq m} B(K_i, \varepsilon)$ where $B(K_i, \varepsilon)$ is the open ball, in $\mathcal{K}(L_p([-1, 1], d\xi))$, centred at K_i and with radius ε.

Let $A_1 = B(K_1, \varepsilon)$, $A_2 = B(K_2, \varepsilon) - A_1, \ldots, A_m = B(K_m, \varepsilon) - A_{m-1}$. Clearly, $A_i \cap A_j = \emptyset$ if $i \neq j$ and $T \subset \bigcap_{1 \leq i \leq m} A_i$.

Let $1 \leq i \leq m$ and denote by I_i the set

$$I_i = K^{-1}(A_i) = \{x \in]-a, a[\text{ such that } K(x) \in A_i\}.$$

Hence we have $I_i \cap I_j = \emptyset$ if $i \neq j$ and $]0, 1[= \bigcap_{i=1}^{m} I_i$.

Consider now the following step function from $]-a, a[$ to Z defined by

$$S(x) = \sum_{i=1}^{m} \chi_{I_i}(x)\, K_i$$

where $\chi_{I_i}(.)$ denotes the characteristic function of I_i. Obviously, $S(.)$ satisfies the hypothesis $(H2)$ i.e., (3.10), (3.11) and (3.12). Then using (3.13) we get $K - S \in L^\infty(]-a, a[, Z)$. Moreover, an easy calculation leads to

$$\|K - S\|_{L^\infty(]-a,\, a[,\, Z)} \leq \varepsilon .$$

Now, using (3.14) we obtain

$$\|K - S\|_{\mathcal{L}(X_p)} \leq \|K - S\|_{L^\infty(]-a,\, a[,\, Z)} \leq \varepsilon .$$

Hence, we infer that the operator K may be approximated (for the uniform topology) by operators of the form

$$U(x) = \sum_{i=1}^{m} \eta_i(x)\, K_i$$

where $\eta_j(.) \in L^\infty([-a, a], dx)$ and $K_i \in \mathcal{K}(L_p([-1, 1], d\xi))$. On the other hand, each compact operator K_i on $L_p([-1, 1], d\xi)$ is a limit (for the norm topology) of a sequence of finite rank operators because $L_p([-1, 1], d\xi)$ $(1 \leq p < \infty)$ admits a Schauder Basis. This ends the proof. Q.E.D.

PROOF OF LEMMA 3.1. – Let λ be such that $Re\lambda > -\lambda^*$. In view of (3.8) we have

$$(\lambda - T_H)^{-1} K = \sum_{n \geq 0} B_\lambda H (M_\lambda H)^n G_\lambda K + C_\lambda K .$$

In order to conclude, it suffices to show that $\sum_{n \geq 0} B_\lambda H (M_\lambda H)^n G_\lambda K$ and $C_\lambda K$ are compact on X_p $(1 < p < \infty)$ and weakly compact on X_1.

We claim that $G_\lambda K$ and $C_\lambda K$ are compact on X_p for $1 < p < \infty$ and weakly compact on X_1.

By Lemma 3.2, it suffices to prove the result for an operator K whose kernel is in the form $\kappa(x, \xi, \xi') = \eta(x)\, \theta(\xi)\, \beta(\xi')$ where $\eta(.) \in L^\infty([-a, a], dx)$, $\theta(.) \in L_p([-1, 1], d\xi)$ and $\beta(.) \in L_q([-1, 1], d\xi)$.

Consider $\varphi \in X_p$,

$$\begin{cases} (G_\lambda^+ K\varphi)(\xi) = \displaystyle\int_{-1}^{1} \int_{-a}^{a} \frac{1}{|\xi|} \eta(x)\, \theta(\xi)\, e^{\frac{-1}{|\xi|}(\lambda + \sigma(\xi))|a-x|} \beta(\xi')\, \varphi(x, \xi')\, dx\, d\xi', \quad 0 < \xi < 1 \\ \qquad = J_\lambda U \varphi \end{cases}$$

where U and J_λ denote the following bounded operators

$$\begin{cases} U : X_p \to L_p([-a, a], dx) \\ \qquad \varphi \to \displaystyle\int_{-1}^{1} \beta(\xi)\, \varphi(x, \xi)\, d\xi \end{cases}$$

$$\begin{cases} J_\lambda : L_p([-a, a], dx) \to X_{1,p}^o \\ \qquad \psi \to \displaystyle\int_{-a}^{a} \frac{1}{|\xi|} \eta(x)\, \theta(\xi)\, e^{\frac{-1}{|\xi|}(\lambda + \sigma(\xi))|a-x|} \psi(x)\, dx . \end{cases}$$

It is now sufficient to show that J_λ is compact for $1 < p < \infty$ and weakly compact for $p = 1$.

In fact, the compactness follows from [15], Theorem 11.6, p. 275, if we show

$$\int_{-1}^{1} \left[\int_{-a}^{a} \left| \frac{1}{|\xi|} \eta(x)\, \theta(\xi)\, e^{\frac{-1}{|\xi|}(\lambda + \sigma(\xi))|a-x|} \right|^q dx \right]^{p/q} |\xi| d\xi < +\infty$$

(J_λ is then a Hille-Tamarkin operator). Indeed, let us first observe that we have

$$\int_{-a}^{a} \left| \frac{1}{|\xi|} \eta(x)\, \theta(\xi)\, e^{\frac{-1}{|\xi|}(\lambda + \sigma(\xi))\,|a-x|} \right|^q dx \leqslant \|\eta\|_\infty^q \frac{|\theta(\xi)|^q}{|\xi|^q} \int_{-a}^{a} e^{-q\frac{(\lambda+\sigma(\xi))\,|a-x|}{|\xi|}}\, dx$$

$$\leqslant \|\eta\|_\infty^q \frac{|\theta(\xi)|^q}{q(\operatorname{Re}\lambda + \lambda^*)\,|\xi|^{(q-1)}}$$

which leads to

$$\left[\int_{-a}^{a} \left| \frac{1}{|\xi|} \eta(x)\, \theta(\xi)\, e^{\frac{-1}{|\xi|}(\lambda + \sigma(\xi))\,|a-x|} \right|^q dx\right]^{p/q} \leqslant \|\eta\|_\infty^p \frac{|\theta(\xi)|^p}{[q(\operatorname{Re}\lambda + \lambda^*)]^{p-1}\,|\xi|^{-1}}.$$

Integrating in ξ from -1 to 1 we obtain

$$\int_{-1}^{1}\left[\int_{-a}^{a} \left| \frac{1}{|\xi|} \eta(x)\, \theta(\xi)\, e^{\frac{-1}{|\xi|}(\lambda + \sigma(\xi))\,|a-x|} \right|^q dx\right]^{p/q} |\xi|\, d\xi \leqslant \int_0^1 \|\eta\|_\infty^p \frac{|\theta(\xi)|^p}{[q(\operatorname{Re}\lambda + \lambda^*)]^{p/q}}\, d\xi$$

$$\leqslant \|\eta\|_\infty^p \frac{\|\theta\|^p}{[q(\operatorname{Re}\lambda + \lambda^*)]^{p/q}} < \infty.$$

For the case $p = 1$, it is easy to see that the operator J_λ satisfies the following estimates:

$$\|J_\lambda\| \leqslant \|\eta\|_\infty \|\theta\|_{L_1}.$$

The last inequality shows that J_λ depends continuously (in the uniform topology) on $\theta(.) \in L_1([-1, 1], d\xi)$. But the set of bounded functions which vanich in a neighborhood of $\xi = 0$ is dense in $L_1([-1, 1], d\xi)$, so J_λ is a limit, in the uniform topology, of integral operators with bounded kernels. Hence J_λ is a weakly compact operator on $L_1([-a, a], dx)$ (see [5], Corollary 11, p. 294).

A similar reasoning allows us to reach the same result for the operator $G_\lambda^- K$ and $C_\lambda K$. This concludes the proofs of the claim and lemma. Q.E.D.

Now we are in a position to prove Theorem 3.2.

PROOF OF THEOREM 3.2. – The hypothesis on K together with Lemma 3.1 implies that $K \in \mathcal{G}_{T_H}(X_p)$. Now the result follows from Theorems 2.1 (*i*), 3.1 and Remark 3.1. Q.E.D.

We consider the essential spectrum of the multidimensional neutron transport equation. To this purpose, consider the neutron transport operator

$$A_0 \psi(x, v) =$$

$$-v \frac{\partial \psi}{\partial x}(x, v) - \sigma(v)\, \psi(x, v) + \int_V \kappa(x, v, v')\, \psi(x, v')\, dv' = T_0 \psi + K\psi$$

where T_0 is the streaming operator and K denotes the integral part of A_0 (the collision operator), $(x, v) \in D \times V$, where the configuration space D is an open and bounded subset of R^N, $N \geq 1$. The velocity space V is an arbitrary open subset of R^N. The unbounded operator A_0 (i.e., $H = 0$) is studied in the Banach space $L_p(D \times V)$. Its domain is

$$D(A_0) = D(T_0) = \left\{ \psi \in L_p(D \times V) \text{ such that } v \frac{\partial \psi}{\partial x} \in L_p(D \times V),\ \psi_{|\Gamma_-} = 0 \right\}$$

where

$$\Gamma_- = \{(x, v) \in \partial D \times V \text{ such that } v \text{ is ingoing at } x \in \partial D\}.$$

It is well known that

$$\sigma(T_0) = \{\lambda \in \boldsymbol{C} \text{ such that } \operatorname{Re} \lambda \leq -\lambda^*\}$$

(see, for instance, [16], Corollary 12.11, p. 272). More precisely we have

$$\sigma_{\text{ess}}(T_0) = \sigma C(T_0) = \{\lambda \in \boldsymbol{C} \text{ such that } \operatorname{Re} \lambda \leq -\lambda^*\} \tag{3.15}$$

(see [19], p. 6211).

The existence of the eigenvalues of $T_0 + K$ in the half-plan $\{\lambda \in \boldsymbol{C}$ such that $\operatorname{Re} \lambda > -\lambda^*\}$ is related to the compactness of some iterate of $(\lambda - T_0)^{-1} K$ (see [16], Chap. 12). Unfortunately, this does not prevent from the appearance of holes, included in the resolvent set of A_0, in the region $\{\lambda \in \boldsymbol{C}$ such that $\operatorname{Re} \lambda \leq -\lambda^*\}$. However, if K is compact on $L_1(V)$ then $(\lambda - T_0)^{-1} K$ is compact on $L_p(D \times V)$ $(1 < p < \infty)$ and weakly compact on $L_1(D \times V)$ (see [32], Lemma 2.1) and consequently we have the following result.

THEOREM 3.3. – *Suppose that K is compact on $L_p(V, dv)$. Then*

$$\sigma_{\text{ess}}(A_0) = \{\lambda \in \boldsymbol{C} \textit{ such that } \operatorname{Re} \lambda \leq -\lambda^*\}. \quad \blacksquare$$

PROOF. – The hypothesis on K together with Lemma 3.1 implies that $K \in \mathcal{G}_{T_0}(L_p(D \times V))$. Now the result follows from Eq. (3.15) and Theorem 2.1 (i). Q.E.D.

We close this section by discussing briefly the essential spectrum of the transport operator arising in growing cell populations. It is about the following partial differential equation

$$(3.16)\qquad A_H\psi(\mu,v) = -v\frac{\partial\psi}{\partial\mu}(\mu,v) - \sigma(\mu,v)\,\psi(\mu,v) + \int_a^b r(\mu,v,v')\,\psi(\mu,v')\,dv' = S_H\psi + K\psi$$

where $\mu\in[0,1]$, v, $v'\in[a,b]$ with $0\leq a<b<\infty$ and $\sigma(\mu,v)=\int_a^b r(\mu,v,v')\,dv'$. This equation describes the number density $\psi(\mu,v)$ of cell population as a function of the degree of maturation μ, the maturation velocity v. The degree of maturation is defined so that $\mu=0$ at birth and $\mu=1$ at the death of a cell. The transition rate $r(\mu,v,v')$ specifies the transition of cells from a maturation velocity to another one while $\sigma(\mu,v)$ denotes the total transition cross section.

The boundary conditions are given by

$$(3.17)\qquad \psi_{|\Gamma_0} = H(\psi_{|\Gamma_1})$$

where $\Gamma_0=\{0\}\times[a,b]$ and $\Gamma_1=\{1\}\times[a,b]$, $\psi_{|\Gamma_0}$ (resp. $\psi_{|\Gamma_1}$) denotes the restriction of ψ to Γ_0 (resp. Γ_1) while K is a linear operator from a suitable function space on Γ_1 to a similar one on Γ_0.

Rotenberg studied essentially the Fokker-Plank approximation of (3.16) for which he obtained numerical solutions. Using eigenfunction expansion technique, Van der Mee and Zweifel [39] obtained analytical solutions for a variety of linear boundary conditions. Using Lebowitz and Rubinow's boundary conditions (cf. [28] or [35]), Boulanouar and Leboucher [1] proved that the associated Cauchy problem to the Rotenberg model is governed by a positive C^0-semigroup and they gave sufficient conditions guaranteeing its irreducibility. Similar results were also obtained in [8], Chap. 13. Recently, Latrach and Jeribi [25, 26] gives some existence results of the stationary problem (3.16) supplemented the boundary conditions (3.17).

Arguing as above we have:

THEOREM 3.4. – *Let $p\in[1,\infty)$ and suppose that the collision operator K is compact on $L_p([a,b];dv)$. Then*

$$\sigma_{\text{ess}}(A_H) = \sigma_{\text{ess}}(S_H)\,.$$

Further, if the boundary operator H is strictly singular then

$$\sigma_{\text{ess}}(A_H) = \sigma_{\text{ess}}(S_0) = \{\lambda\in\mathbf{C} \text{ such that } \operatorname{Re}\lambda \leq -\text{ess-inf}\,\sigma(.,.)\}\,. \qquad ■$$

REFERENCES

[1] M. Boulanouar - L. Leboucher, *Une équation de transport dans la dynamique des populations cellulaires*, C. R. Acad. Sci. Paris, Serie I, **321** (1995), 305-308.

[2] R. Dautray - J. L. Lions, *Analyse Mathématique et Calcul Numérique*, Tome 9, Masson, Paris, 1988.

[3] J. Diestel, *A survey of results related to Dunfor-Pettis property*, in «Contemporary Math. 2, Amer. Math. Soc. of Conf. on Integration, Topology and Geometry in Linear Spaces» (1980), 15-60.

[4] N. Dunford and Pettis, *Linear operations on summable functions*, Trans. Amer. Math. Soc., **47** (1940), 323-392.

[5] N. Dunford - J. T. Schwartz, *Linears operators*, Part 1, Interscience Publishers Inc., New York, 1958.

[6] I. Bohberg - A. Markus - I. A. Feldman, *Normally solvable operators and ideals associated with them*, Amer. Math. Soc. Transl. ser. 2, **61** (1967), 63-84.

[7] S. Goldbert, *Unbounded Linear Operators*, McGraw-Hill, New-York, 1966.

[8] W. Greenberg - G. Van der Mee - V. Protopopescu, *Boundary Value Problems in Abstract Kinetic Theory*, Birkhauser, Basel, 1987.

[9] G. Greiner, *Spectral properties and asymptotic behaviour of the linear transport equation*, Math. Z., **185** (1984), 167-177.

[10] A. Grothendieck, *Sur les applications linéaires faiblement compactes d'espaces du type $C(K)$*, Canad. J. Math., **5** (1953), 129-173.

[11] K. Gustafson - J. Weidmann, *On the essential spectrum*, J. Math. Anal. Appl., **25**, **6** (1969), 121-127.

[12] A. Jeribi, *Quelques remarques sur les opérateurs de Frédholm et application à l'équation de transport*, C. R. Acad. Sci. Paris, Série I, **325** (1997), 43-48.

[13] A. Jeribi, *Quelques remarques sur le spectre de Weyl et applications*, C. R. Acad. Sci. Paris, Série I, **327** (1998), 485-490.

[14] A. Jeribi - K. Latrach, *Quelques remarques sur le spectre essentiel et application à l'équation de transport*, C. R. Acad. Sci. Paris, Série I, **323** (1996), 469-474.

[15] K. Jörgen, *Linear integral operator*, Pitman. Advanced publishing program (1982).

[16] H. G. Kaper - C. G. Lekkerkerker - J. Hejtmanek, *Spetral Methods in Linear Transport Theory*, Birkhauser, Basel, 1982.

[17] T. Kato, *Perturbation theory for nullity, deficiency and other quantities of linear operators*, J. Anal. Math., **6** (1958), 261-322.

[18] K. Latrach, *Compactness properties for linear transport operator with abstract boundary conditions in slab geometry*, Trans. Theor. Stat. Phys., **22** (1993), 39-65.

[19] K. Latrach, *Some remarks on the essential spectrum of transport operators with abstract boundary conditions*, J. Math. Phys., **35** (1994), 6199-6212.

[20] K. Latrach, *Time asymptotic behaviour for linear transport equation with abstract boundary conditions in slab geometry*, Trans. Theor. Stat. Phys., **23** (1994), 633-670.

[21] K. Latrach, *Description of the real point spectrum for a class of neutron transport operators*, Trans. Theor. Stat. Phys., **23** (1993), 595-630.

[22] K. Latrach, *Quelques propriétés spectrales d'opérateurs de transport avec des conditions aux limites abstraites*, C. R. Acad. Sci. Paris, Série I, **320** (1995), 809-814.

[23] K. Latrach, *Essential spectra on spaces with the Dunfor-Pettis property*, J. Math. Anal. Appl., **233** (1999), 607-722.

[24] K. LATRACH - A. JERIBI, *On the essential spectrum of transport operators on L_1-spaces*, J. Math. Phys., **37** (1996), 6486-6494.
[25] K. LATRACH - A. JERIBI, *Sur une équation de transport intervenanten dynamique des populations*, C. R. Acad. Sci. Paris, Série I, **325** (1997), 1087-1109.
[26] K. LATRACH A. JERIBI, *A nonlinear boundary value problem arising in growing cell populations*, Nonli. Anal., Ser. A, Theory methods, **36** (7) (1999), 843-862.
[27] K. LATRACH - A. JERIBI, *Some results on Fredholm operators, essential spectra, and application*, J. Math. Anal., **225** (1998), 461-485.
[28] J. L. LEBOWITZ - S. I. RUBINOW, *A theory for the age and generation time distribution of a microbial population*, J. Math. Biol., **1** (1974), 17-36.
[29] J. LINDENSTRAUSS - L. TZAFRIRI, *Classical Banach Spaces I*, Springer-Verlag, Berlin-Heidelberg-New York, 1977.
[30] V. D. MILMAN, *Some properties of strictly singular operators*, Functional Ana. Appl., **3** (1969), 77-78.
[31] M. MOKHTAR-KHARROUBI, *Spectral theory of the neutron transport operator in bounded geometries*, Trans. Theor. Stat. Phys., **16** (1987), 467-502.
[32] M. MOKHATAR-KHARROUBI, *Time asymptotic behaviour and compactness in neutron transport theory*, Europ. J. of Mech., B Fluid, **11** (1992), 39-68.
[33] A. PALCZEWSKI, *Spectral properties of the space nonhomogeneous linearized Boltzmann operator*, Transp. Theor. Stat. Phys., **13** (1984), 409-430.
[34] A. PELCZYNSKI, *Strictly singular and cosingular operators*, Bull. Acad. Polon. Sci., **13** (1965), 31-41.
[35] M. ROTENBERG, *Transport theory for growing cell populations*, J. Theor. Biol., **103** (1983), 181-199.
[36] M. SCHECHTER, *Principles of functional analysis*, Academic Press, 1971.
[37] M. SCHECHTER, *Spectra of Partial Differential Operators*, Horth-Holland, Amsterdam, 1971.
[38] M. SCHECHTER, *On the essential spectrum of an arbitrary operator*, J. Math. Anal. Appl., **13** (1965), 205-215.
[39] C. VAN DER MEE - P. ZWEIFEL, *A. Fokker-Plank equation for growing cell populations*, J. Math. Biol., **25** (1987), 61-72.
[40] I. VIDAV, *Existence and uniqueness of nonnegative eigenfunction of the Boltzmann operator*, J. Math. Anal. Appl., **22** (1968), 144-155.
[41] J. VOIGT, *Positivity in time dependent transport theory*, Acta Aplicandae Mathematicae, **2** (1984), 311-331.
[42] L. WEIS, *On perturbation of Fredholm operators in L_p-spaces*, Proc. Amer. Math. Soc., **67** (1977), 87-92.
[43] F. WOLF, *On the essential spectrum of partial differential boundary problems*, Comm. Pure Appl. Math., **12** (1959), 211-228.
[44] F. WOLF, *On the invariance of the essential spectrum under a change of the boundary conditions of partial differential operators*, Indag. Math., **21** (1959), 142-147.

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