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Note on the Wijsman Hyperspaces of Completely Metrizable Spaces.

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Sunto. – *Consideriamo sugli spazi $CL(X)$ dei sottoinsiemi chiusi e non vuoti di uno spazio X completamente metrizzabile la topologia di Wijsman τ_{W_d}. Se X è separabile, mostriamo che, per ogni metrica d, e su X, ogni insieme chiuso e numerabile in $(CL(X), \tau_{W_e})$ ha punti isolati in $(CL(X), \tau_{W_d})$. Se $d = e$, questo implica il teorema di Costantini sulla completezza topologica di $(CL(X), \tau_{W_d})$. Per X non-separabili, rispondiamo ad una questione sollevata da Zsilinszky, mostrando che in molti casi gli spazi $(CL(X), \tau_{W_d})$ contengono copie chiuse dei razionali.*

Summary. – *We consider the hyperspace $CL(X)$ of nonempty closed subsets of completely metrizable space X endowed with the Wijsman topologies τ_{W_d}. If X is separable and d, e are two metrics generating the topology of X, every countable set closed in $(CL(X), \tau_{W_e})$ has isolated points in $(CL(X), \tau_{W_d})$. For $d = e$, this implies a theorem of Costantini on topological completeness of $(CL(X), \tau_{W_d})$. We show that for nonseparable X the hyperspace $(CL(X), \tau_{W_d})$ may contain a closed copy of the rationals. This answers a question of Zsilinszky.*

1. – Introduction.

Our terminology and notation follows Beer [1]. Given a topological space X, we denote by $CL(X)$ the collection of nonempty closed sets in X. Let X be a metrizable space and $\mathcal{D}(X)$ be the family of metrics on X generating the topology. For each $d \in \mathcal{D}(X)$, the Wijsman topology τ_{W_d} on $CL(X)$ is the weakest topology making all functionals $A \to \mathrm{dist}_d(z, A)$ continuous, where $z \in X$ and $dist_d(z, A) = \inf\{d(z, x) : x \in A\}$.

If X is separable metrizable, so are the spaces $(CL(X), \tau_{W_d})$, cf. [1, Theorem 2.1.5]. Costantini [2] proved that for completely metrizable separable X, the hyperspaces $(CL(X), \tau_{W_d})$ are completely metrizable. Zsilinszky [7] showed that for any completely metrizable X, $(CL(X), \tau_{W_d})$ are strongly Choquet. In presence of separability, this yields the Costantini's result, and in general case one still concludes that the Wijsman hyperspaces are Baire. Zsilinszky asked if the space $(CL(X), \tau_{W_d})$ is hereditarily Baire, provided X is completely metrizable.

In this note we give two results concerning the topic. Theorem 1.1 contains a result from which the Costantini's theorem follows easily by a classical Hurewicz's theorem. Theorem 1.2 provides an answer to the question of Zsilinszky.

THEOREM 1.1. – *Let X be a separable completely metrizable space, let d, e be metrics generating the topology of X, and let τ_{W_d}, τ_{W_e} be the corresponding Wijsman topologies on the hyperspace $CL(X)$. If $\mathcal{A} \subset CL(X)$ is a countable set that has no isolated points, with respect to τ_{W_d}, then $\mathcal{A}$ is not closed with respect to τ_{W_e}.*

Letting in this theorem $d = e$, we see that for completely metrizable separable X, the space $(CL(X), \tau_{W_d})$ contains no closed copy of the rationals. Since τ_{W_d} is a subfamily of the Effros Borel structure on $CL(X)$, the Wijsman hyperspace is absolutely Borel, and in effect, by Hurewicz's theorem, completely metrizable, cf. [6], Theorem 12.6 and Corollary 21.21.

In the next result, $\mathbb{N}^{2^{\aleph_0}}$ is the Tychonoff product of $2^{\aleph_0}$ copies of natural numbers $\mathbb{N}$.

THEOREM 1.2. – *Let X be a metrizable space such that the set of points in X without any compact neighborhood has weight $2^{\aleph_0}$. Then for any metric d generating the topology of X, $\mathbb{N}^{2^{\aleph_0}}$ embeds as a closed subspace in $(CL(X), \tau_{W_d})$. In particular, the Wijsman hyperspace contains a closed copy of the rationals.*

This gives also an alternative justification of Costantini's result [3] that the Wijsman hyperspace of the complete metric space may not be Čech complete.

2. – Proof of Theorem 1.1.

Let X be a separable completely metrizable space. Let us make first a few remarks.

REMARK A. – *For any $e \in \mathcal{D}(X)$ there is a totally bounded $e^* \in \mathcal{D}(X)$ such that for any $z \in X$, $A \in CL(X)$ and $r > 0$, if $\mathrm{dist}_e(z, A) > r$, then $\inf\{e^*(x, y) : x \in A,\ e(z, y) \leq r\} > 0$.*

Indeed, let C be a countable dense set in X and let Ω be the set of all triples $\omega = (c, p, q)$ with $c \in C$, and $0 < p < q$ rational. For each triple ω, let $f_\omega : X \to [0, 1]$ be a continuous map such that $f_\omega(x) = 0$ if $e(c, x) \leq p$ and $f_\omega(x) = 1$ if $e(c, x) \geq q$. Then for any injection $\nu : \Omega \to \mathbb{N}$, the metric $e^*(x, y) = \sum_{\omega \in \Omega} 2^{-\nu(\omega)} |f_\omega(x) - f_\omega(y)|$ has the required properties.

REMARK B. – *Let* q, e, $e^* \in \mathcal{D}(X)$, *where* q *is complete and* e^* *is the metric associated with* e *in Remark A. Let* $B_1, B_2, \ldots$ *be in* $CL(X)$ *and, for* $i \geq 1$, *let* F_i *be a finite* 2^{-i}*-net in* B_i *with respect to* e^* (*i.e.,* $F_i \subset B_i$ *and for any* $x \in B_i$ *there is* $y \in F_i$ *with* $e^*(x, y) < 2^{-i}$). *Furthermore, assume that for any* $x \in F_i$ *there is* $y \in F_{i+1}$ *with* $q(x, y) + e^*(x, y) < 2^{-i}$, *and let* $K = \bigcap_{n=1}^{\infty} (\overline{F_n \cup F_{n+1} \cup \ldots})$. *Then* $B_i \to K$ *in* $(CL(X), \tau_{W_e})$.

To see this, let us consider any open sets V, U in X with $V \cap K \neq 0$ and $\inf\{e^*(x, y): x \in K, y \notin U\} > 0$. Then all but finitely many B_i hit V and are contained in U. The connection between e and e^* described in Remark A makes it clear that $B_i \to K$ with respect to τ_{W_e}.

REMARK C. – *Let* d, $e \in \mathcal{D}(X)$, A, $B \in CL(X)$ *and* $A \neq B$. *Then there exist* $\mathcal{U} \in \tau_{W_d}$ *and* $\mathcal{V} \in \tau_{W_e}$ *such that* $A \in \mathcal{U}$, $B \in \mathcal{V}$ *and* $\mathcal{U} \cap \mathcal{V} = \emptyset$.

If there is $z \in A \backslash B$, we set $r = 1/2 \inf\{e(z, x): x \in B\}$ and we let $\mathcal{U} = \{E \in CL(X): E$ hits the ball $\{x : e(z, x) < r\}\}$, $\mathcal{V} = \{E \in CL(X): \mathrm{dist}_e(z, E) > r\}$. Notice that $\mathcal{U} \in \tau_{W_d}$ for any $d \in \mathcal{D}(X)$, and $\mathcal{V} \in \tau_{W_e}$. If there is $z \in B \backslash A$, we proceed similarly, changing the role of e and d.

Now, passing to the proof of Theorem 1.1, let d, e and $\mathcal{A}$ be as in this theorem, and let us assume that $\mathcal{A}$ has no isolated points with respect to τ_{W_d}. Let e^* be the totally bounded metric associated with e in Remark A, and let $A_1, A_2, \ldots$ be the elements of $\mathcal{A}$ listed without repetitions. We shall fix a complete metric $q \in \mathcal{D}(X)$.

Any $\mathcal{U} \in \tau_{W_d}$ hitting $\mathcal{A}$ contains infinitely many A_i. Therefore, using Remark C, one can pick inductively $\mathcal{U}_i \in \tau_{W_d}$, $\mathcal{V}_i \in \tau_{W_e}$, $n(i) > i$, and finite sets $F_i \subset A_{n(i)}$ such that

(1) $$A_{n(i)} \in \mathcal{U}_i, \qquad A_i \in \mathcal{V}_i, \qquad \mathcal{U}_i \cap \mathcal{V}_i = \emptyset\,,$$

(2) $$\mathcal{U}_1 \supset \mathcal{U}_2 \supset \ldots,$$

(3) $$F_i \text{ is a } 2^{-i}\text{-net in } A_{n(i)} \text{ with respect to } e^*,$$

(4) $$\text{for any } x \in F_i \text{ there is } y \in F_{i+1} \text{ with } q(x, y) + e^*(x, y) < 2^{-i}.$$

By Remark B, conditions (3) and (4) guarantee that $A_{n(i)} \to K$ in $(CL(X), \tau_{W_e})$. On the other hand, by (1) and (2), $K \neq A_i$ for all i. Therefore K is not in $\mathcal{A}$, but it is in the closure of $\mathcal{A}$ with respect to τ_{W_e}, which completes the proof.

3. – Proof of Theorem 1.2.

Let $d \in \mathcal{D}(X)$ and let $\operatorname{dist}(z, A)$ denote the distance function given by d. The assumptions about X yield $r > 0$, and $S \subset X$ of cardinality $2^{\aleph_0}$ such that

$$(1), \qquad d(s, t) > r \quad \text{for} \quad s, t \in S, \quad s \neq t,$$

and no $s \in S$ has a compact neighborhood.

For $s \in S$ we set

$$(2) \qquad B_s = \{x : d(s, x) < r/4\}, \qquad E_s = \{x : d(s, x) \leqslant r/5\}.$$

The neighborhood E_s of s is not compact, hence it contains a countable closed discrete set M_s. We write

$$(3) \qquad M_s = \{a_{s, n} : n \in \mathbb{N}\} \subset E_s, \qquad M = \bigcup_{s \in S} M_s,$$

where $a_{s, n} \neq a_{s, m}$ for $n \neq m$.

Let $\mathbb{N}^S$ be the space of functions $u : S \to \mathbb{N}$ with the pointwise topology, i.e., topologically – the Tychonoff product $\mathbb{N}^{2^{\aleph_0}}$. We shall define $F : \mathbb{N}^S \to CL(X)$ by the formula, cf. (2), (3),

$$(4) \qquad F(u) = \{a_{s, u(s)} : s \in S\} \cup \Big(X \setminus \bigcup_{s \in S} B_s\Big).$$

We claim that

$$(5) \qquad \mathcal{F} = F(\mathbb{N}^S) \text{ is closed in } (CL(X), \tau_{W_d})$$

and

$$(6) \qquad F : \mathbb{N}^S \to \mathcal{F} \text{ is a homeomorphism},$$

where $\mathcal{F}$ is considered with the relative Wijsman topology.

To check (5), let us notice that, cf. (2), (3),

$$\mathcal{U} = \Big\{A \in CL(X) : \Big(X \setminus \bigcup_{s \in S} B_s\Big) \setminus A \neq \emptyset\Big\} \cup$$

$$\bigcup_{s \in S} \{A \in CL(X) : \operatorname{dist}(s, A) > r/5\} \cup$$

$$\bigcup_{s \in S} \{A \in CL(X) : |A \cap B_s| > 2\} \cup$$

$$\bigcup_{s \in S} \{A \in CL(X) : A \cap (B_s \setminus M_s) \neq 0\}$$

belongs to τ_{W_d}. The complement of $\mathcal{U}$ consists of the sets $A \in CL(X)$ that contain $X \setminus \bigcup_{s \in S} B_s$ and hit each B_s in exactly one point, and the point is in M_s. This means that $\mathcal{F} = CL(X) \setminus \mathcal{U}$, confirming (5).

To verify (6), let us fix $u \in \mathbb{N}^S$. Basic neighborhoods of $F(u)$ in $(CL(X), \tau_{W_d})$ are of the form $\mathcal{W} = \{A \in CL(X) : A \cap W_i \neq \emptyset, i = 1, \ldots, k$ and $\operatorname{dist}(z_j, A) > r_j$, $j = 1, \ldots, l\}$, where W_i are open and $r_j > 0$. Since $F(u) \in \mathcal{W}$ and we concentrate on $\mathcal{W} \cap \mathcal{F}$, one can demand in addition that $W_i \cap F(u) = \{a_{s(i), u(s(i))}\} = W_i \cap M$, $i = 1, \ldots, k$, cf. (3), (4). Moreover, by (4), $F(u) \in \mathcal{W}$ implies that $z_j \in B_{t(j)}$, and increasing k, we can assume that $\{t(1), \ldots, t(l)\} \subset \{s(1), \ldots, s(k)\}$. From (1), (2) and (3), $d(z_j, a_{t(j), u(t(j))}) < \operatorname{dist}(z_j, \bigcup_{s \neq t(j)} M_s)$. In effect, we see that for the restricted basic neighborhood $\mathcal{W}$ of $F(u)$, $F(v) \in \mathcal{W}$ if, and only if, $v(s(i)) = u(s(i))$, $i = 1, \ldots, k$. Therefore, F takes the neighborhoods of u in $\mathbb{N}^S$ onto neighborhoods of $F(u)$ in $\mathcal{F}$ equipped with the relative Wijsman topology, which proves (6).

Finally, the space of rationals $\mathbb{Q}$ embeds onto a closed subspace of $\mathbb{N}^S$, cf. [5] (for the smallest cardinality κ such that $\mathbb{Q}$ embeds as a closed subset of $\mathbb{N}^\kappa$ see [4], Theorems 8.15 and 8.16). Therefore, the hyperspace $(CL(X), \tau_{W_d})$ contains a closed copy of $\mathbb{Q}$.

REMARK 3.1. – Costantini [3] demonstrated that the Borel structure of the Wijsman hyperspaces on the non-separable completely metrizable space X depends on the choice of a metric on X. The reasoning in the proof of Theorem 1.2 can also be used to that effect. To that end, let us consider the discrete space X of cardinality $\aleph_1$. Let e be the discrete metric on X, and let d be a metric on X generating the discrete topology such that one can find in the metric space (X, d) sets with the properties (1), (2) and (3). Then $\mathbb{N}^{\aleph_1}$ embeds onto a closed subset H of $(CL(X), \tau_{W_d})$. On the other hand, $(CL(X), \tau_{W_e})$ can be identified with the Cantor Cube $\{0, 1\}^{\aleph_1}$ without the point having all coordinates zero. Since H is a Baire space without any dense Čech complete subspace, H can not be embedded as a Borel set into any compact space. In effect, H is closed with respect to τ_{W_d} but not Borel with respect to τ_{W_e}.

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