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On a Subset with Nilpotent Values in a Prime Ring with Derivation.

VINCENZO DE FILIPPIS

Sunto. – *Siano R un anello primo, privo di nil ideali destri, d una derivazione non nulla di R, I un ideale bilatero non nullo di R. Se, per ogni $x, y \in I$, esiste $n = n(x, y) \geq 1$ tale che $(d([x, y]) - [x, y])^n = 0$, allora R é commutativo. Come conseguenza si ottiene una estensione di tale risultato per ideali di Lie di R.*

Summary. – *Let R be a prime ring, with no non-zero nil right ideal, d a non-zero drivation of R, I a non-zero two-sided ideal of R. If, for any $x, y \in I$, there exists $n = n(x, y) \geq 1$ such that $(d([x, y]) - [x, y])^n = 0$, then R is commutative. As a consequence we extend the result to Lie ideals.*

This note continues a line of investigation in the literature concerning derivations having nilpotent values. The first such result is due to Herstein [6]. He proved that if R is a prime ring and d an inner derivation of R satisfying $d(x)^n = 0$, for all $x \in R$ and n a fixed integer, then $d = 0$. Many authors extended this result to arbitrary derivations which act either on Lie ideals or on multilinear polynomials in prime and semiprime rings. In [5] Felzenswalb and Lanski considered derivations satisfying $d(x)^{n(x)} = 0$, for all $x \in I$, an ideal of R, and proved $d(I) = 0$, when R has no nil right ideal. In [1] Carini and Giambruno studied the case when $d(u)^{n(u)} = 0$, for all $u \in L$, a Lie ideal of R and they proved that $d(L) = 0$ when R is a prime ring, $\mathrm{char}(R) \neq 2$ and R contains no nil right ideal, and then obtain the same conclusion when n is fixed and R is a 2-torsion free semiprime ring. Later in [9] Lanski obtained the same results, removing both the bound on the indices of nilpotence and the characteristic assumption on R. More recently Wong [12] proved that if $f(x_1, \ldots, x_n)$ is a multilinear polynomial on a prime ring R and $d(f(r_1, \ldots, r_n))^{n(r_1, \ldots, r_n)} = 0$, for all $r_1, \ldots, r_n \in R$ and n depending on the choice of $r_1, \ldots, r_n$, then $f(x_1, \ldots, x_n)$ is central valued on R provided R contains no non-zero nil right ideal.

Our purpose here is to obtain some information on the structure of a prime ring R, when a special subset of R has nilpotent values. More precisely, let $d \neq 0$ a derivation of R, $A = \{d([x, y]) - [x, y] : x, y \in R\}$. It is known that if R is a 2-torsion free semiprime ring and any element of A is central in R, then R is commutative (see [8]). Moreover if R is semiprime and any element of A is zero or invertible in R, then either R is a division ring or R is the ring of all 2×2 matrices over a division ring (see [3]).

Here we consider the case when R is prime and, for any $a \in A$, there exists an integer $n = n(a) \geqslant 1$ such that $a^n = 0$. We prove the following:

THEOREM 1. – *Let R be a prime ring with no non-zero nil right ideal, d a non-zero derivation of R, I a non-zero ideal of R. If for any $x, y \in I$ there exists $n = n(x, y) \geqslant 1$ such that $(d([x, y]) - [x, y])^n = 0$ then R is commutative.*

As a natural consequence, we will also obtain the following extension to Lie ideals:

THEOREM 2. – *Let R be a prime ring with no non-zero nil right ideal, d a non-zero derivation of R, L a Lie ideal of R. If, for any $u \in L$, there exists $n = n(u) \geqslant 1$ such that $(d(u) - u)^n = 0$ then L is central in R, except when L is commutative,* $\mathrm{char}(R) = 2$ *and R satisfies the standard identity $S_4(x_1, \ldots, x_4)$.*

For sake of completeness, first we state some well known results:

LEMMA 1. – *Let R be a prime ring, $d \neq 0$ a derivation of R, L a Lie ideal of R such that $d(u) - u = 0$, for all $u \in L$. Then L is central in R.*

PROOF. – Let $u \in L$, $x \in R$, then $d([u, x]) = [u, x]$. Expanding this last one, we have $[d(u), x] + [u, d(x)] = [u, x]$ i.e. $[u, x] + [u, d(x)] = [u, x]$. It follows that $[L, d(R)] = (0)$, which means that $L \subseteq C_R(d(R))$, the centralizers of $d(R)$ in R. Since it is well known that $C_R(d(R)) = Z(R)$, we are done. ■

LEMMA 2. – *Let R be a division ring, $d \neq 0$ a derivation of R, L a Lie ideal of R. If, for any $u \in L$, there exists $n = n(u) \geqslant 1$ such that $(d(u) - u)^n = 0$, then L is central in R.*

PROOF. – It follows directly from Lemma 1. ■

We begin the proof of the main result with the following:

LEMMA 3. – *Let R be a primitive ring, d a non-zero derivation of R, I a non-zero ideal of R. If, for any $x, y \in I$, there exists $n = n(x, y) \geqslant 1$ such that $(d([x, y]) - [x, y])^n = 0$ then R is commutative.*

PROOF. – Let V a faithful irreducible right R-module with endomorphisms ring D, a division ring. Since I is a non-zero ideal of R, then R and I are both dense subring of D-linear transformations on V. Suppose that $\dim_D V \geqslant 2$. Let $v \in V$, $r \in I$

such that $vr = 0$. Suppose $vd(r) \neq 0$. There exists $w \in V$ such that $vd(r)$ and wr are linearly independent over D. By the density of I, there exist $s_1, s_2 \in I$ such that $vd(r)\, s_1 = w$, $wrs_2 = v$ and $vd(r)\, s_2 = 0$. Therefore $v(d([rs_1, rs_2]) - [rs_1, rs_2]) = v$. Since there exists $n = n(r, s_1, s_2) \geq 1$ such that $(d([rs_1, rs_2]) - [rs_1, rs_2])^n = 0$, then we get the contradiction $0 = v(d([rs_1, rs_2]) - [rs_1, rs_2])^n = v \neq 0$. Hence, for all $v \in V$ and $r \in I$, if $vr = 0$ then $vd(r) = 0$.

Now we prove that vr and $vd(r)$ are linearly D-dependent, for all $r \in I$ and $v \in V$. In fact, if not, by the density of I, there exists $s \in I$ such that $vrs = 0$ and $vd(r)\, s \neq 0$. On the other hand, since $vrs = 0$, by the above argument, we have that $vd(rs) = 0$ and also $vrd(s) = 0$. Then $0 = vd(rs) = vd(r)\, s + vrd(s) = vd(r)\, s \neq 0$, a contradiction. This means that, for any $r \in I$ and $v \in V$, there exists $\alpha_{r,v} \in D$, depending on the choice of r and v, such that $vd(r) = \alpha_{r,v} vr$.

Let now $r_1, r_2 \in I$. Then $vd(r_1) = \alpha_1 vr_1$ and $vd(r_2) = \alpha_2 vr_2$. If vr_1 and vr_2 are independent over D, then by $vd(r_1) + vd(r_2) = vd(r_1 + r_2) = \beta v(r_1 + r_2)$, we have $\alpha_1 = \alpha_2 = \beta$. If vr_1 and vr_2 are non-zero and linearly D-dependent, consider the element vr_3, with $r_3 \in I$, such that vr_3 is independent on vr_1 and vr_2. Since $vd(r_1) = \alpha_1 vr_1$, $vd(r_2) = \alpha_2 vr_2$ and $vd(r_3) = \alpha_3 vr_3$, as above we conclude that $\alpha_1 = \alpha_2 = \alpha_3$. Therefore, fixed $v \in V$, for all $r \in I$, there exists $\alpha = \alpha_v$, depending on the choice of v, such that $vd(r) = \alpha_v vr$.

Fix now $r \in I$, with $\operatorname{rank}(r) > 1$. For all $u, v \in I$, $ud(r) = \alpha_u ur$, $vd(r) = \alpha_v vr$, for suitable α_u and α_v in D. If ur and vr are independent over D, then $\alpha_u ur + \alpha_v vr = (u + v)d(r) = \alpha_{u+v}(u + v)r$ implies $\alpha_u = \alpha_v = \alpha_{u+v}$. If ur and vr are non-zero and linearly dependent over D, consider wr, with $w \in V$, such that wr is independent on ur and vr. Thus, since $wd(r) = \alpha_w wr$, then $\alpha_u = \alpha_v = \alpha_w$. Hence we have proved that there exists $\alpha \in D$ such that $\alpha vr = vd(r)$, for all $r \in I$ and $v \in V$. Moreover $\alpha \neq 0$. In fact, if $\alpha = 0$ then $Vd(R) = (0)$, which implies the contradiction $d(R) = 0$.

Let now $r, s \in I$ and $v \in V$. Since $vd(rs) = \alpha vrs$ and also $vd(rs) = vd(r)s + vrd(s) = 2\alpha vrs$, then $\alpha vrs = 0$, i.e. $\alpha VR^2 = (0)$ and we get the contradiction $R = 0$.

All the previous arguments say that $dim_D V = 1$, that is R is a division ring and, by Lemma 2, we are done. ■

LEMMA 4. – *Let R be a semiprimitive ring, d a non-zero derivation of R, I a non-zero ideal of R. If, for any $x, y \in I$, there exists $n = n(x, y) \geq 1$ such that $(d([x, y]) - [x, y])^n = 0$ then R is commutative.*

PROOF. – Since R is semiprimitive, the Jacobson's radical $J(R)$ is zero. Then R is a subdirect product of primitive rings. For any P primitive ideal of R, let $\overline{R} = \frac{R}{P}$, which is primitive. Consider the following partition:

$$K_1 = \{P : d(I^2) \subseteq P\}$$

$K_2 = \{P : d(P) \subseteq P,\ d(I^2) \nsubseteq P\}$

$$K_3 = \{P : d(P) \nsubseteq P,\ d(I^2) \nsubseteq P\}.$$

In addition let $J_i = \cap P$, for $P \in K_i$, $i = 1, 2, 3$. Moreover $J_1 J_2 J_3 \subseteq J_1 \cap J_2 \cap J_3 = (0)$ and, by the primeness of R, one of the J_i must be zero.

If $J_1 = 0$ then $d(I^2) = 0$ and so $d = 0$, a contradiction.

Suppose $J_2 = 0$. Let $\overline{d}$ the derivation of $\overline{R} = \frac{R}{P}$ induced by d as follows: $\overline{d}(\overline{x}) = \overline{d(x)}$, for all $\overline{x} = x + P$, $x \in R$. Moreover $\overline{I} = I + P$ is an ideal of $\overline{R}$ and $(\overline{d}([\overline{x}, \overline{y}]) - [\overline{x}, \overline{y}]) = \overline{d([x, y]) - [x, y]}$ is nilpotent in $\overline{R}$, for all $x, y \in I$. By the primitive case, $\overline{R} = \frac{R}{P}$ is commutative, for all $P \in K_2$, that is R is commutative.

Suppose now $J_3 = 0$. For all $P \in K_3$, $\overline{d(I^2 P)}$ is an ideal of $\overline{R}$ and $\overline{d(I^2 P)} \neq 0$. For all $x, y \in I^2 P$ we have that $d([x, d(y)]) - [x, d(y)] = [d(x), d(y)] (mod\ P)$. Hence $[\overline{d(x)}, \overline{d(y)}]$ is nilpotent, for all $x, y \in I^2 P$, i.e. $[X, Y]$ is nilpotent in $\overline{d(I^2 P)}$. It is well known that in this case $\overline{d(I^2 P)}$ is commutative, that is $\overline{R} = \frac{R}{P}$ is commutative, for all $P \in K_3$, and so R is commutative. ■

LEMMA 5. – *Let R be a prime ring with no non-zero nil right ideal, $d \neq 0$ a derivation of R, I a non-zero ideal of R such that, for any $x, y \in I$, $(d([x, y]) - [x, y])^n = 0$, for a suitable $n = n(x, y) \geq 1$ depending on the choice of x, y. Let $a, b \in I$. If $ab = 0$ then $ad(b) = d(a)b = 0$.*

PROOF. – Let $x \in R$. Then there exists $n \geq 1$ such that

$$0 = (d([ba, xba]) - [ba, xba])^n = (bd(axb)a + baxbd(a) + d(b)\, axba - baxba)^n$$

and right multiplying by b, we have: $(baxbd(a))^n b = 0$, i.e. $(bd(a)\, bax)^{n+1} = 0$. By the arbitrariety of $x \in R$ and since R does not contain any non-zero nil right ideal, it follows that $bd(a)\, ba = 0$. Let now $s \in R$ such that $s^2 = 0$. If $c = xs$ and $f = sy$ for $x, y \in I$, then $cf = 0$ and, by above argument, $0 = syd(xs)\, syxs = syxd(s)\, syxs$. Since $d(s)\, s = -sd(s)$, then we have $(d(s)\, syx)^3 = 0$, and as above it follows $d(s)\, s = 0$. Moreover, since $ab = 0$, then $(bxa)^2 = 0$, for any $x \in R$. Hence $0 = bxad(bxa) = bxad(b)\, xa$, i.e. $(ad(b)x)^3 = 0$, so $ad(b) = 0$. ■

Now we are ready to prove:

THEOREM 1. – *Let R be a prime ring with no non-zero nil right ideal, d a non-zero derivation of R, I a non-zero ideal of R. If for any $x, y \in I$ there exists $n = n(x, y) \geq 1$ such that $(d([x, y]) - [x, y])^n = 0$ then R is commutative.*

PROOF. – Let $S = \{s \in R : s^2 = 0\}$, J the Jacobson's radical of R.

Suppose $S \neq 0$, and so $J \neq 0$. Let $T = \{t \in R : atb = 0$ if $ab = 0,\ a, b \in R\}$ and $W = S \cap T$.

If $W = 0$ then $d(J) = 0$ (see [5, theorem 5]), a contradiction.

Hence consider $W \neq 0$ and $V = \{v \in W : vRv \subseteq T\}$. By the proof of theorem 5 in [5] we get that $V \neq 0$ and $vhv = 0$, for all $v \in V$ and for all h nilpotent element of R. Thus, in particular, for all $v \in V$ and $x, y \in I$, $v(d([x, y]) - [x, y])\, v = 0$. Since R and I satisfy the same differential identities (see [11]), then $v(d([x, y]) - [x, y])\, v = 0$, for all $x, y \in R$. Let $u \in [R, R]$, $r \in R$, then $v(d([u, vr]) - [u, vr])\, v = 0$. By calculation we have $v[R, R](d(vr))\, v = 0$. In particular $v[x, vy](d(vr))\, v = 0$, for any $x, y \in R$, that is $vxvy(d(vr))\, v = 0$. Since R is prime and $v \neq 0$, it follows $d(vr)\, v = 0$, which means that $d(x)\, x = 0$ for any element x of the right ideal $\varrho = vR$ of R. By Lemma in [1] d is an inner derivation induced by $q \in Q$, the Martindale quotients ring of R, that is $d(x) = [q, x]$, for all $x \in R$, moreover $qv = 0$. Therefore, for all $r_1, r_2 \in R$, we have

$$0 = v(q[r_1, r_2] - [r_1, r_2]\, q - [r_1, r_2])\, v = vq[r_1, r_2]\, v - v[r_1, r_2]\, v = (vq - v)[r_1, r_2]\, v$$

and so $(vq - v)[R, R]\, v = 0$. Since $v \neq 0$, then $vq = v$.

By our assumption, for any $x \in R$ there exists $m \geqslant 1$ such that

$$0 = ([q, [v, x]] - [v, x])^m = ([q, vx - xv] - vx + xv)^m =$$
$$(-qxv - vxq + xvq - vx + xv)^m = (-qxv - vxq - vx + 2xv)^m.$$

Right multiplying by v we obtain that $(-vx)^m v = 0$, so $(-vx)^{m+1} = 0$, which means that vR is a nil right ideal of R. Since R has no non-zero nil right ideal, then $v = 0$.

The previous contradiction says that $S = 0$, that is R is a domain and so $d([x, y]) - [x, y] = 0$, for all $x, y \in I$. Thus we conclude, by Lemma 1, that R is commutative. ■

We conclude this paper with an extension of previuos theorem to Lie ideals. First we premit the following:

LEMMA 6. – *Let R be a prime ring and L a non-central Lie ideal of R. Then either there exists a non-zero ideal I of R such that $0 \neq [I, R] \subseteq L$ or L is commutative,* $\mathrm{char}(R) = 2$ *and R satisfies the standard identity $S_4(x_1, \ldots, x_4)$.*

PROOF. – See [7, pp. 4-5], [4, lemma 2, proposition 1], [10, theorem 4]. ■

Finally we have:

THEOREM 2. – *Let R be a prime ring with no non-zero nil right ideal, d a non-zero derivation of R, L a Lie ideal of R. If, for any $u \in L$, there exists $n =$*

$n(u) \geqslant 1$ such that $(d(u) - u)^n = 0$ then L is central in R, except when L is commutative, $\mathrm{char}(R) = 2$ *and R satisfies the standard identity $S_4(x_1, \ldots, x_4)$.*

PROOF. – Suppose L is not central. In this case R cannot be commutative. By Lemma 6 either $[I, I] \subseteq L$, for some ideal I of R, or $\mathrm{char}(R) = 2$, L is commutative and R satisfies the standard identity $S_4(x_1, \ldots, x_4)$. Since in the first case, by Theorem 1, we have the contradiction that R is commutative, we are done. ∎

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