Tullio Valent

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On the Notion of Potential for Mappings between Linear Spaces. A Generalized Version of the Poincaré Lemma.

TULLIO VALENT

Sunto. – *Si indica un approccio alla teoria delle forme differenziali lineari in uno spazio vettoriale X senza richiedere una struttura di spazio di Banach su X. Nelle definizioni e negli enunciati intervengono (implicitamente) solo delle topologie intrinseche (parzialmente vettoriali) di X. Successivamente si considera una funzione $F : U \subseteq X \to Y$, con X, Y spazi vettoriali reali ed U sottoinsieme radiale di X. Dopo aver mostrato un teorema di rappresentazione delle forme bilineari $\langle \cdot , \cdot \rangle$ su $X \times Y$ tali che $\langle x, y \rangle = 0 \ \forall x \in X \Rightarrow y = 0$, si osserva come l'assegnazione di una tale forma bilineare permetta di associare, in una maniera naturale, alla funzione F una forma differenziale, e ciò conduce spontaneamente alla definizione di «potenzialità» di F. Questa definizione ha interesse soprattutto quando F «descrive» un problema al contorno e, o, ai valori iniziali; nella sezione 6 si espone un esempio tratto dalla teoria dell'elasticità finita.*

Summary. – *An approach to the theory of linear differential forms in a radial subset of an (arbitrary) real linear space X without a Banach structure is proposed. Only intrinsic (partially linear) topologies on X are (implicitly) involved in the definitions and statements. Then a mapping $F : U \subseteq X \to Y$, with X, Y real linear spaces and U a radial subset of X, is considered. After showing a representation theorem of those bilinear forms $\langle \cdot , \cdot \rangle$ on $X \times Y$ for which $\langle x, y \rangle = 0 \ \forall x \in X \Rightarrow y = 0$, we observe that the assignment of such a bilinear form allows to associate (in a natural way) a linear differential form to the mapping F; this fact spontaneously leads us to a definition of potentialness for F. This definition has a special interest in the case when the mapping F describes a boundary and, or, initial value problem; a simple example, originated from finite elasticity, is explained in sect.* 6.

1. – Introduction.

In the first part of this article we deal with linear differential forms in a radial subset of an (arbitrary) real linear space X without a Banach structure . (Note that, in any linear topological space, every open set is radial). Only intrinsic (partially linear) topologies of the linear space X will be (implicitly) involved in the definitions and statements. These topologies (essentially, the radial topology of X and the n-radial topology of X) are introduced in section 2: they have an intrinsic character, in the sense that they are determined by the linear structure of X. It seemed to us that (in our algebraic approach) the right notion of locally exactness of a linear differential form $\omega : U \subseteq X \to X^*$ would be referred to the 2-radial topology of X; actually, one can prove that if ω (supposed to be weakly of class C^1 on the planes) is locally exact with respect to the 2-radial topology of X then it is closed (see Theorem 3.1). The main result in section 3 is Theorem 3.2: it is a generalized version of the Poincaré Lemma. A similar theorem was proved in Vainberg [5] (Th.5.1) within Banach spaces. (In the Vainberg's theorem U is an open subset of a Banach space, while here U is only a radial subset of the linear space X).

In the second part of the paper we first prove that for any bilinear form $(x, y) \mapsto \langle x, y \rangle$ on $X \times Y$, with X, Y real linear spaces, such that

$$\langle x, y \rangle = 0 \qquad \forall x \in X \Rightarrow y = 0 \tag{1.1}$$

there are an inner product $\cdot$ on Y and a linear map $\tau : X \to Y$ such that

$$\langle x, y \rangle = \tau(x) \cdot y \qquad \forall (x, y) \in X \times Y.$$

In other words, any bilinear form on $X \times Y$ satisfying (1.1) can be described through an inner product on Y and a linear map from X into Y. Obviously, the roles of X and Y can be exchanged.

Then, we observe that the assignment of a bilinear form $\langle \cdot, \cdot \rangle$ on $X \times Y$ satisfying (1.1) allows to associate (in a natural way) to any mapping $F : U \subseteq X \to Y$, with U a radial subset of X, a linear differential form $\widetilde{F} : U \subseteq X \to X^*$. Thus it is spontaneous to say that the mapping F admits a potential if there is a bilinear form $\langle \cdot, \cdot \rangle$ on $X \times Y$ satisfying (1.1) such that the linear differential form $\widetilde{F}$ associated with F through $\langle \cdot, \cdot \rangle$ is exact.

The fact that the linear differential form $\widetilde{F}$ associated with the mapping F through $\langle \cdot, \cdot \rangle$ is closed expresses the «symmetry» of the Gateaux derivative of F with respect to the bilinear form $\langle \cdot, \cdot \rangle$. We recall that the problem of searching for (and constructing) a nondegenerated bilinear form on $X \times Y$ with respect to which such symmetry holds has been deeply studied, in particular, by Magri [1], Tonti [3], [4], Telega [2].

The mapping F can «describe» a boundary and, or, initial value problem, as it occurs in the example explained in section 6 and originated from elasticity; in this case the linear map τ can have the meaning of a «trace operator».

2. – Some (partially linear) intrinsic topologies of the linear spaces.

In this section we emphasize some intrinsic topologies of an arbitrary real linear space X. Such topologies are only partially compatible with the algebraic structure of X. In order to introduce them we need a few definitions. Let U be a subset of X. We say that U is *radial* at x $(\in X)$ if for each $\xi \in X$ there is a number $\lambda_{x,\,\xi} > 0$ such that $x + \lambda\xi \in U$ for all real λ with $|\lambda| \leqslant \lambda_{x,\,\xi}$ (this means that U contains a line segment through x in each direction). When U is radial at every one of its points we shall say that U is *radial*. Moreover, for any integer n with $1 \leqslant n \leqslant \dim X$, we say that U is *n-radial* at x ($\in X$) if for each $(\xi_1, \ldots, \xi_n) \in X$ there is a number $\lambda_{x,\,\xi_1,\,\ldots,\,\xi_n} > 0$ such that $x + \lambda_1\xi_1 + \ldots + \lambda_n\xi_n \in U$ for all $\lambda_1 \ldots, \lambda_n \in \mathbb{R}$ with $|\lambda_1|, \ldots, |\lambda_n| \leqslant \lambda_{x,\,\xi_1,\,\ldots,\,\xi_n}$. The set U will be called *n-radial* if it is n-radial at every one of its points. It is evident that the set of all radial subsets of X is a topology on X: we shall say that it is the *radial topology* of X. More in general, for each n, with $1 \leqslant n \leqslant \dim X$, the set of all n-radial subsets of X is a topology on X: it will be called the *n-radial topology* of X. Let us denote by $\mathcal{R}(x)$ the set of those subsets of X that are radial at x; clearly, $\mathcal{R}(x)$ is a filter on X, and, for any x_0, $x \in X$,

$$\mathcal{R}(x_0 + x) = x_0 + \mathcal{R}(x).$$

It follows that for the radial topology of X, the translations $x \mapsto x_o + x$ are continuous, and hence they are omeomorphisms. It is also evident that, for the radial topology, the scalar multiplication is separately continuous (in each of two variables). Thus, we can conclude that, *for the radial topology, vector addition and scalar multiplication are separately continuous.*

We now prove that, *if $\mathcal{T}$ is a topology on X for which (for each $x_0 \in X$) the mappings $x \mapsto x_0 + x$, $\lambda \mapsto \lambda x_0$ are continuous (from X into X, and from $\mathbb{R}$ into X, respectively), then $\mathcal{T}$ is contained in the radial topology of X.* To do this let us denote by $\mathcal{F}(x)$ the filter of neighborhoods of x for the topology $\mathcal{T}$, and observe that if for $\mathcal{T}$ the mapping $x \mapsto x_0 + x$ is continuous then $\mathcal{F}(x_0) = x_0 + \mathcal{F}(0)$. Therefore, in order to prove that $\mathcal{T}$ is contained in the radial topology of X it suffices to show that each element U of $\mathcal{F}(0)$ contains a radial subset V of X with $0 \in V$. Accordingly, we fix $U \in \mathcal{F}(0)$ and consider an open element V of $\mathcal{F}0)$ contained in U. Since $V \in \mathcal{F}(x)$ $\forall x \in V$ we have $V - x \in \mathcal{F}(0)$ $\forall x \in V$. If for the topology $\mathcal{T}$ on X the mappings $\lambda \mapsto \lambda x$, $x \in X$, are continuous at 0 then each element of $\mathcal{F}(0)$ is radial at 0, and so, for any $x \in V$ the set $V - x$ is radial at 0, namely V is radial (at any $x \in V$) and $0 \in V$.

Thus the following characterizations of the radial topology of X have been proved.

REMARK 2.1. – *The radial topology of X is the strongest topology on X for which, for each $x_0 \in X$, the mappings $x \mapsto x_0 + x$, $\lambda \mapsto \lambda x_0$ are continuous.*

REMARK 2.2. – *The radial topology of X is the strongest topology on X for which addition and scalar multiplication are separately continuous (in each of the two variables).*

Moreover, it is easy to realize that the following further characterizations of the radial topology hold.

REMARK 2.3. – *The radial topology of X is the strongest topology on X for which all affine curves in X (i.e., affine functions from $\mathbb{R}$ into X) are continuous.*

REMARK 2.4. – *The radial topology of X is the strongest topology on X that induces the usual (Hausdorff, linear) topology over each one-dimensional linear submanifold of X.*

Using analogous arguments one can prove the following characterizations of the n-radial topology of X, $(1 \leqslant n \leqslant \dim X)$.

REMARK 2.5. – *The n-radial topology of X is the strongest topology on X for which, for each $(x_0, x_1, \ldots, x_n) \in X^{1+n}$, the mappings*

$$x \mapsto x_0 + x, \qquad (\lambda_1, \ldots, \lambda_n) \mapsto \lambda_1 x_1 + \ldots + \lambda_n x_n$$

are continuous.

REMARK 2.6. – *The n-radial topology of X is the strongest topology on X that induces the usual (Hausdorff, linear) topology over each linear submanifold of X with dimension $\leqslant n$.*

3. – Linear differential forms in linear spaces.

Let X be a real linear space, and let U be a radial subset of X. We are especially interested in the case when X is an infinite dimensional linear space; however, we suppose that the dimension of X is $\geqslant 2$. We consider a *linear differential form*

$$\omega : U \to X^*$$

defined in U. (Of course, X^* denotes the dual of the linear space X). We shall say that ω is exact if there is a mapping $f : U \to \mathbb{R}$ such that for every $(x, \xi) \in U \times X$ the limit

$$\partial_\xi f(x) := \lim_{\mathbb{R} \ni \lambda \to 0} \frac{1}{\lambda}(f(x + \lambda\xi) - f(x))$$

exists in $\mathbb{R}$ and

$$\partial_\xi f(x) = \omega(x)(\xi).$$

Moreover, ω will be said to be *locally exact* if for each $x \in U$ there is a 2-radial subset U_x of U with $x \in U_x$ and such that $\omega_{|U_x}$ is exact.

If, for $(x, \xi) \in U \times X$, the limit

$$\lim_{\mathbb{R} \ni \lambda \to 0} \frac{1}{\lambda}(\omega(x + \lambda\xi) - \omega(x))$$

exists for the weak topology on X^* (i.e., for the pointwise convergence topology), then this limit will be denoted by $\partial_\xi \omega(x)$; thus $\partial_\xi \omega(x)$ is the element of X^* defined by putting

$$\partial_\xi \omega(x)(\eta) = \lim_{\mathbb{R} \ni \lambda \to 0} \frac{1}{\lambda}(\omega(x + \lambda\eta)(\eta) - \omega(x)(\eta))$$

for all $\eta \in X$. A linear differential form $\omega : U \to X^*$ for which $\partial_\xi \omega(x)$ exists for every $(x, \xi) \in U \times X$ will be said to be *closed* if, for any $x \in U$, the bilinear form

$$X \times X \ni (\xi_1, \xi_2) \mapsto \partial_{\xi_1} \omega(x)(\xi_2)$$

is symmetric.

We shall say that a linear differential form $\omega : U \to X^*$ is *weakly of class C^1 on the planes* if the restriction of ω to the intersection of U with each two-dimensional linear submanifold of X is of class C^1 for the weak topology on X^*.

THEOREM 3.1. – *Let U be a radial subset of a real linear space X, and let $\omega : U \to X^*$ be weakly of class C^1 on the planes. If ω is locally exact, then ω is closed.*

PROOF. – Assume that ω is locally exact. Then, for any fixed $x \in U$ there are a 2-radial subset U_x of U with $x \in U_x$, and a mapping $f : U_x \to \mathbb{R}$ such that

$$\partial_\xi f(y) = \omega(y)(\xi) \qquad \forall (y, \xi) \in U_x \times X .$$

In order to prove that

$$\partial_{\xi_1} \omega(x)(\xi_2) = \partial_{\xi_2} \omega(x)(\xi_1) \qquad \forall (\xi_1, \xi_2) \in X \times X \tag{3.1}$$

we essentially follow the proof of the first part of Theorem 5.1 in Vainberg [5]. Let, for any fixed $(\xi_1, \xi_2) \in X \times X$, c be a number > 0 such that $x + \lambda_1 \xi_1$, $x + \lambda_2 \xi_2 \in U$ if $|\lambda_1| \leq c$ and $|\lambda_2| \leq c$. For $|\lambda_1| \leq c$ and $|\lambda_2| \leq c$ we set

$$\varphi_{\lambda_1}(x) = f(x + \lambda_1 \xi_1) - f(x), \qquad \varphi_{\lambda_2}(x) = f(x + \lambda_2 \xi_2) - f(x),$$

and observe that

$$\varphi_{\lambda_1}(x+\lambda_2\xi_2)-\varphi_{\lambda_1}(x)=\varphi_{\lambda_2}(x+\lambda_1\xi_1)-\varphi_{\lambda_2}(x)= \\ f(x+\lambda_1\xi_1+\lambda_2\xi_2)-f(x+\lambda_1\xi_1)-f(x+\lambda_2\xi_2)-f(x)\,. \tag{3.2}$$

In view of the Lagrange formula there are numbers $\tau_1, \tau_2 \in]0, 1[$ such that

$$\begin{aligned}\varphi_{\lambda_1}(x+\lambda_2\xi_2)-\varphi_{\lambda_1}(x) &= \partial_{\xi_2}\varphi_{\lambda_1}(x+\tau_2\lambda_2\xi_2)\\ &= \partial_{\xi_2}f(x+\lambda_1\xi_1+\tau_2\lambda_2\xi_2)-\partial_{\xi_2}f(x+\tau_2\lambda_2\xi_2)=\\ &= \omega(x+\lambda_1\xi_1+\tau_2\lambda_2\xi_2)(\xi_2)-\omega(x+\tau_2\lambda_2\xi_2)(\xi_2)=\\ &= \partial_{\xi_1}\omega(x+\tau_1\lambda_1\xi_1+\tau_2\lambda_2\xi_2)(\xi_2)\,.\end{aligned}$$

Analogously we see that there are numbers $\theta_1, \theta_2 \in]0, 1[$ such that

$$\varphi_{\lambda_2}(x+\lambda_1\xi_1)-\varphi_{\lambda_2}(x)=\partial_{\xi_2}\omega(x+\theta_1\lambda_1\xi_1+\theta_2\lambda_2\xi_2)(\xi_1)\,.$$

Then from (3.2) it follows that

$$\partial_{\xi_1}\omega(x+\tau_1\lambda_1\xi_1+\tau_2\lambda_2\xi_2)(\xi_2)=\partial_{\xi_2}\omega(x+\theta_1\lambda_1\xi_1+\theta_2\lambda_2\xi_2)(\xi_1)\,,$$

and so (3.1) holds because ω has been supposed to be weakly of class C^1 on the planes. ∎

A subset V of X is said to be *star-shaped* at $x(\in X)$ if $\lambda\xi+(1-\lambda)\,x\in V$ for all $\xi\in V$ and $\lambda\in]0, 1[$. The next theorem is the main result of this section.

THEOREM 3.2. – *Let U be a radial subset of a real linear space X, and let $\omega : U \to X^*$ be weakly of class C^1 on the planes. If ω is closed, then for each $x_0\in U$ and any radial subset V of X contained in U and star-shaped at x_0 the linear differential form $\omega_{|V}$ is exact (and so ω is locally exact).*

PROOF. – Suppose that ω is closed, namely that, for each $x\in U$, (3.1) holds. We shall prove that, if $x_0\in U$ and V is a radial subset of X contained in U and star-shaped at x_0, then putting for each $x\in V$

$$f(x)=\int_0^1\omega(x_0+t(x-x_0))(x-x_0)\,dt\,,$$

we have

$$\partial_\xi f(x)=\omega(x)(\xi)\qquad \forall (x,\xi)\in V\times X\,. \tag{3.3}$$

We begin by observing that, for any $(x, \xi)\in U\times X$ such that $x_0+t(x-x_0+$

$\lambda\xi) \in U \ \forall (t, \lambda) \in]0, 1[\times]0, 1[$, we have

$$f(x+\xi)-f(x)=\int_0^1 [\omega(x_0+t(x-x_0+\xi))(x-x_0+\xi)-\omega(x_0+t(x-x_0))(x-x_0)]\,dt=$$

$$\int_0^1 \left(\int_0^1 \frac{d}{d\lambda}[\omega(x_0+t(x-x_0+\lambda\xi))(x-x_0+\lambda\xi)]\,d\lambda \right) dt\,.$$

Then, since

$$\frac{d}{d\lambda}[\omega(x+t(x-x_0+\lambda\xi))(x-x_0+\lambda\xi)] =$$

$$= \partial_{t\xi}\,\omega(x_0+t(x-x_0+\lambda\xi))(x-x_0+\lambda\xi)+\omega(x_0+t(x-x_0+\lambda\xi))(\xi)\,,$$

we obtain

$$f(x+\xi)-f(x)=\int_0^1 \left(\int_0^1 \partial_{t\xi}\,\omega(x_0+t(x-x_0+\lambda\xi))(x-x_0+\lambda\xi)\,d\lambda \right) dt\,+$$

$$\int_0^1 \left(\int_0^1 \omega(x_0+t(x-x_0+\lambda\xi))(\xi)\,d\lambda \right) dt\,.$$

As ω is closed we have

$$\partial_{t\xi}\,\omega(x_0+t(x-x_0+\lambda\xi))(x-x_0+\lambda\xi)=\partial_{x-x_0+\lambda\xi}\,\omega(x_0+t(x-x_0+\lambda\xi))(t\xi)\,,$$

and hence

$$f(x+\xi)-f(x)=$$

$$\int_0^1 \left(\int_0^1 [t\partial_{x-x_0+\lambda\xi}\,\omega(x_0+t(x-x_0+\lambda\xi))(\xi)+\omega(x_0+t(x-x_0+\lambda\xi))(\xi)]\,d\lambda \right) dt\,.$$

On the other hand observe that

$$t\partial_{x-x_0+\lambda\xi}\,\omega(x_0+t(x-x_0+\lambda\xi))(\xi)+\omega(x_0+t(x-x_0+\lambda\xi))(\xi)=$$

$$\frac{d}{dt}[t\omega(x_0+t(x-x_0+\lambda\xi))(\xi)]\,,$$

and recall that ω has been supposed to be weakly of class C^1 on the planes and so, for each $(x, \xi) \in U \times X$ the function

$$(\lambda, t) \mapsto t\partial_{x-x_0+\lambda\xi}\,\omega(x+t(x-x_0+\lambda\xi))(\xi)+\omega(x_0+t(x-x_0+\lambda\xi))(\xi)$$

is continuous in $]0, 1[\times]0, 1[$. Therefore

$$f(x+\xi)-f(x)=\int_0^1\left(\int_0^1 \frac{d}{dt}[t\omega(x_0+t(x-x_0+\lambda\xi))(\xi)]\,dt\right)d\lambda=\int_0^1 \omega(x+\lambda\xi)(\xi)\,d\lambda\,.$$

Now, we arbitrarily fix $x \in V$ and $\xi \in X$, and remark that (as V is radial and star-shaped at x_0)

$$x_0+t(x-x_0+\lambda(s\xi))\in V \qquad \forall (t,\lambda)\in]0,1[\times]0,1[$$

if the absolute value of the real number s is small enough. Then, from the previous argument it follows that

$$\lim_{\mathbb{R}\ni s\to 0}\frac{f(x+s\xi)-f(x)}{s}=\lim_{\mathbb{R}\ni s\to 0}\int_0^1 \omega(x+s\lambda\xi)(\xi)\,d\lambda=\omega(x)(\xi)\,.$$

Thus (3.3) is satisfied, and the proof is concluded. ■

4. – A reprentation theorem for partially nondegenerate bilinear forms.

Let X, Y be real linear spaces and let $(x, y)\mapsto\langle x, y\rangle$, (briefly $\langle\cdot,\cdot\rangle$), be a bilinear form on $X\times Y$. We shall say that $\langle\cdot,\cdot\rangle$ is *partially nondegenerate* if one of the following conditions (i) and (ii) is satisfied:

(i) $\langle x, y\rangle = 0\ \forall x\in X \Rightarrow y=0$,

(ii) $\langle x, y\rangle = 0\ \forall y\in Y \Rightarrow x=0$.

We observe that the most spontaneous way to construct a bilinear form on $X\times Y$ that satisfies (i) is the following: one fixes an inner product $\cdot$ on Y and a linear map $\tau: X\mapsto Y$ such that $\tau(X)^\circ=\{0\}$, and set

$$\langle x, y\rangle = y\cdot\tau(x) \tag{4.1}$$

for all $(x, y)\in X\times Y$.

Of course, the roles of X and Y can be exchanged. Thus, in order to obtain a bilinear form $\langle\cdot,\cdot\rangle$ on $X\times Y$ satisfying (ii) one can fix an inner product $\cdot$ on X and a linear map $\sigma: Y\to X$ such that $\sigma(Y)^\circ=\{0\}$, and set

$$\langle x, y\rangle = x\cdot\sigma(y)\,. \tag{4.2}$$

Here, of course, $\tau(X)^\circ$ denotes the orthogonal of $\tau(X)$ in the pre-Hilbert space $(Y,\cdot)$, while $\sigma(Y)^\circ$ denotes the orthogonal of $\sigma(Y)$ in the pre-Hilbert space $(X,\cdot)$. Note that the equality $\tau(X^\circ)(X)=\{0\}$ [resp. $\sigma(Y)^\circ=\{0\}$] holds if $\tau(X)$ [resp. $\sigma(Y)$] is dense in the pre-Hilbert space $(Y,\cdot)$ [resp. $(X,\cdot)$].

The following theorem shows that every bilinear form on $X\times Y$ satisfying (i) is of the type (4.1); then, symmetrically, every bilinear form on $X\times Y$ satisfying (ii) is of the type (4.2).

THEOREM 4.1. – *For any bilinear form* $\langle \cdot, \cdot \rangle$ *on* $X \times Y$ *there are a nonnegative, symmetric, bilinear form* $(y_1, y_2) \mapsto y_1 \cdot y_2$ *on* $Y \times Y$ *and a linear map* $\tau : X \to Y$ *such that* (4.1) *holds. Furthermore, if the bilinear form* $\langle \cdot, \cdot \rangle$ *satisfies* (i) *then there are an inner product* $\cdot$ *on* Y *and a linear map* $\tau : X \to Y$ *such that* (4.1) *holds,* (*and hence* $\tau(X)^\circ = \{0\}$).

PROOF. – Let $(a_i)_{i \in I}$ and $(b_j)_{j \in J}$ be Hamel bases of X and Y. For any $\xi = \sum_i \xi_i a_i \in X$ we define the element $\tau(x)$ of Y by putting

$$\tau(x) = \sum_{i,j} \langle a_i, b_j \rangle \, \xi_i b_j .$$

Then, for any $x = \sum_k x_k a_k \in X$ we have

$$\langle x, \tau(\xi) \rangle = \sum_{i,j,k} \xi_i x_k \langle a_i, b_j \rangle \langle a_k, b_j \rangle ,$$

and hence

$$(x, \xi) \mapsto \langle x, \tau(\xi) \rangle$$

is a nonnegative, symmetric, bilinear form on $X \times X$. We observe that the symmetry of this form implies

$$\operatorname{Ker} \tau \subseteq \{ \xi \in X : \langle \xi, \tau(x) \rangle = 0 \quad \forall x \in X \} ,$$

and so, for all $x \in X$ the number $\langle \xi, \tau(x) \rangle$ depends only on $\tau(\xi)$, namely

$$\tau(\xi_1) = \tau(\xi_2) \Rightarrow \langle \xi_1, \tau(x) \rangle = \langle \xi_2, \tau(x) \rangle \qquad \forall x \in X .$$

Therefore, we can define the nonnegative, symmetric, bilinear mapping $\cdot$ from $\tau(X) \times \tau(X)$ into $\mathbb{R}$ by setting

$$\tau(\xi) \cdot \tau(x) = \langle \xi, \tau(x) \rangle \qquad \forall x, \xi \in X .$$

Clearly, there is a nonnegative, symmetric, bilinear form on $Y \times Y$ which extends $\cdot$; we still denote it by $\cdot$, thus (4.1) holds. To conclude the proof it suffices to note that, if the bilinear form $\langle \cdot, \cdot \rangle$ satisfies (i) then from $\tau(x) \cdot y = 0 \; \forall \xi \in X$ it follows $y = 0$, namely the nonnegative, symmetric, bilinear form $\cdot$ on $Y \times Y$ for which (4.1) holds is nondegenerate; hence (by the Cauchy-Schwarz inequality) it is positive definite, and so it is an inner product on Y. ■

5. – Linear differential forms associated with mappings. The notion of potential for mappings.

Let X, Y be real linear spaces, and let U be a radial subset of X. In order to make a sense to the assertion: «a mapping $F : U \subseteq X \to Y$ admits a potential» we must somehow regard F as a linear differential form defined in U, i.e., as a mapping from U into X^*. The most general way to do this is to consider a bilinear form $(x, y) \mapsto \langle x, y \rangle$ on $X \times Y$ such that

$$\langle x, y \rangle = 0 \qquad \forall x \in X \Rightarrow y = 0 , \tag{5.1}$$

and so identify the linear space Y with the linear subspace $\widetilde{Y}$ of X^* defined by putting

$$\widetilde{Y} = \{\tilde{y} \in X^* : y \in Y\}, \qquad \tilde{y}(x) = \langle x, y \rangle \qquad \forall x \in X .$$

Then we can associate to F the linear differential form $\widetilde{F}$ defined in U by writing $\widetilde{F}(x) = \widetilde{F(x)}$ $\forall x \in U$, i.e.,

$$\widetilde{F}(x)(\xi) = \langle \xi, F(x) \rangle \qquad \forall (x, \xi) \in U \times X .$$

We shall say that $\widetilde{F}$ is the *linear differential form associated with the mapping F through the bilinear form $\langle \cdot, \cdot \rangle$*. At this point it is spontaneous to say that *a mapping $F : U \subseteq X \to Y$ admits a potential with respect to a bilinear form $\langle \cdot, \cdot \rangle$ on $X \times Y$ satisfying (4.1) if the linear differential form $\widetilde{F}$ associated with F through $\langle \cdot, \cdot \rangle$ is exact.*

We remark that the fact that the linear differential form $\widetilde{F}$ is closed, namely that

$$\partial_{\xi_1} \widetilde{F}(x)(\xi_2) = \partial_{\xi_2} \widehat{F}(x)(\xi_1) \qquad \forall \xi_1, \xi_2 \in X \text{ and } \forall x \in U ,$$

means that for every $x \in U$ the following symmetry holds:

$$\langle \xi_1, \partial_{\xi_2} F(x) \rangle = \langle \xi_2, \partial_{\xi_1} F(x) \rangle \qquad \forall \xi_1, \xi_2 \in X . \tag{5.2}$$

We also observe that if F is linear then (5.2) becomes

$$\langle \xi_1, F(\xi_2) = \langle \xi_2, F(\xi_1) \rangle \qquad \forall \xi_1, \xi_2 \in X . \tag{5.3}$$

Moreover, if $\cdot$ is an inner product on Y and $\tau : X \to Y$ is a linear map such that $\langle x, y \rangle = y \cdot \tau(x)$ $\forall (x, y) \in X \times Y$ (see Theorem 4.1), then symmetry (5.3) has the form

$$F(\xi_2) \cdot \tau(\xi_1) = F(\xi_1) \cdot \tau(\xi_2) \qquad \forall \xi_1, \xi_2 \in X .$$

6. – An example from elasticity.

In this section we deal with the potentialness of a mapping which «describes» a boundary problem. For any integer $n \geqslant 1$, let us denote by $\mathbb{M}_n$ the set of real $n \times n$ matrices, and by $\mathbb{M}_n^+$ the set of $Z \in \mathbb{M}_n$ such that $\det Z > 0$. Moreover, let Ω be a bounded, smooth, open subset of $\mathbb{R}^n$, let ν be the outward, unit normal to the boundary $\partial\Omega$ of Ω, and let $s : \overline{\Omega} \times \mathbb{M}_n^+ \to \mathbb{M}_n$ be a given mapping.

For any *deformation* x of $\overline{\Omega}$ (i.e., orientation preserving diffeomorphism of

$\overline{\Omega}$ onto a subset of $\mathbb{R}^n$) consider the mapping $S(x) : \overline{\Omega} \to \mathbb{M}_n$ defined by putting

$$S(x)(t) = s(t,\, \partial x(t)) \qquad \forall t \in \overline{\Omega}\,,$$

where $\partial x(t)$ denotes the gradient at t of the mapping $x : \overline{\Omega} \to \mathbb{R}^n$. Let then, for any deformation x of $\overline{\Omega}$,

$$F(x) = (F_1(x),\, F_2(x))\,,$$

where $F_1(x) \colon \Omega \to \mathbb{R}^n$ and $F_2(x) \colon \partial\Omega \to \mathbb{R}^n$ are the functions defined by

$$\begin{cases} F_1(x)(t) = -\displaystyle\sum_{j=1}^{n} \frac{\partial}{\partial t_j} S_{ij}(x)(t)\,, & t \in \Omega\,, \\ F_2(x)(t) = \displaystyle\sum_{j=1}^{n} S_{ij}(x)(t)\, \nu_j(t)\,, & t \in \partial\Omega\,. \end{cases}$$

In the physical context ($n = 3$), Ω represent a reference configuration of a body, the function s defines an elastic response of the body in the sense that $s(t,\, \partial x(t))$ is the first Piola-Kirchhoff stress at the point t relative to the deformation x, and $x \mapsto F(x)$ is the *finite elastostatics operator.*

If the function s is smooth we can take

$$X = C^\infty(\overline{\Omega},\, \mathbb{R}^n), \qquad Y = C^\infty(\overline{\Omega},\, \mathbb{R}^n) \times C^\infty(\partial\overline{\Omega},\, \mathbb{R}^n)\,,$$

and take as U the (radial) subset of X whose elements are the orientation-preserving C^∞ diffeomorphisms of $\overline{\Omega}$ onto a subset $\mathbb{R}^n$. However, other choices of the linear spaces X, Y are possible; for example (see Valent [6], Ch. 3)

$$X = W^{m+2,p}(\Omega, \mathbb{R}^n), \quad Y = W^{m,p}(\Omega, \mathbb{R}^n) \times W^{m+1-\frac{1}{p},p}(\partial\Omega, \mathbb{R}^n) \quad \text{with } p(m+1) > n\,.$$

Consider the (nondegenerated) bilinear form $(x,\, y) \mapsto \langle x,\, y\rangle$ on $X \times Y$ defined by setting

$$\langle x,\, y\rangle = \int_{\Omega} x(t)\cdot y_1(t)\, dt + \int_{\partial\Omega} x(t)\cdot y_2(t)\, d\sigma(t) \tag{6.1}$$

for all $x \in X$ and $y = (y_1,\, y_2) \in Y$, where $\cdot$ denotes the usual inner product on $\mathbb{R}^n$. The linear differential form $\widetilde{F}$ associated with the mapping F through such bilinear form on $X \times Y$ (see section 5) is defined by

$$\widetilde{F}(x)(\xi) = -\sum_{i,j=1}^{n} \int_{\Omega} \xi_i \frac{\partial}{\partial t_j} S_{ij}(x)(t)\, dt + \sum_{i,j=1}^{n} \int_{\partial\Omega} \xi_i S_{ij}(x)(t)\, \nu_j(t)\, d\sigma(t), \qquad (x,\, \xi) \in U \times X\,.$$

Then, in view of the divergence theorem, we have

$$\tilde{F}(x)(\xi) = \sum_{i,j=1}^{n} \int_{\Omega} S_{ij}(x)(t) \frac{\partial}{\partial t_j} \xi_i(t)\, dt .$$

It follows that $\tilde{F}$ is closed if and only if

$$\partial_{Z_{hk}} s_{ij}(t, Z) = \partial_{Z_{ij}} s_{hk}(t, Z), \qquad i, j, h, k = 1, \ldots, n,$$

for all $(t, Z) \in \Omega \times \mathbb{M}_n^+$. This symmetry is satisfied provided there is a (*stored-energy*) function $w : \Omega \times \mathbb{M}_n^+ \to \mathbb{R}$ such that

$$s(t, Z) = \partial_Z w(t, Z) \qquad \forall (t, Z) \in \Omega \times \mathbb{M}_n^+ .$$

In this case, a potential of $\tilde{F}$, (i.e., a potential of F with respect to the bilinear form $\langle \cdot, \cdot \rangle$ defined by (6.1)), is the function $f : U \to \mathbb{R}$ defined by

$$f(x) = \int_{\Omega} w(t, \partial x(t))\, dt .$$

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Dipartimento di Matematica Pura ed Applicata, Università di Padova
Via Belzoni 7 - 35131 Padova

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