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On Quasihomogeneous Manifolds — via Brion-Luna-Vust Theorem.

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Sunto. – *In questo lavoro si studiano varietà proiettive liscie sulle quali agisce un gruppo algebrico semplice G con una orbita aperta. In particolare si utilizza un teorema di Brion-Luna-Vust per correlare l'azione di G su X con l'azione indotta di G sul fibrato normale di una orbita chiusa. Come applicazione si ottiene una classificazione nel caso $G = SL(n)$ e $\dim X \leq 2n-2$.*

Summary. – *We consider a smooth projective variety X on which a simple algebraic group G acts with an open orbit. We discuss a theorem of Brion-Luna-Vust in order to relate the action of G with the induced action of G on the normal bundle of a closed orbit of the action. We get effective results in case $G = SL(n)$ and $\dim X \leq 2n-2$.*

Introduction.

Let G be a simply connected simple algebraic group over complex numbers acting on a smooth projective variety X. If the action is transitive then X is a homogeneous variety of type G/P, where $P \subset G$ is a parabolic group, and the classification of such varieties is know (see [Ti] or [Bou]). If the action of G on X has an open orbit then X is called quasihomogeneous and we may think about such an X as a smooth projective equivariant compactification of G/H, where $H \subset G$ is the isotropy group of a general point of X.

The classification of quasihomogeneous varieties was considered by several authors. Akhiezer studied the situation in which complement of the open orbit is either disconnected (two ends), or contains an isolated fixed point, or is a homogeneous divisor. The latter case was also considered by Huckleberry, Oeljeklaus and Brion. A discussion of these results can be found in chapter 7 of [Ak]. Interesting and complete results have been also obtained in the case of $SL(2)$-quasihomogenous 3-folds, see [L-V], [M-U] and [Na1]. The case of $SL(3)$ quasihomogenous 4-folds was then studied in [Na2] while an attempt to study this problem in higher dimension — via Mori theory, in the same way as done by [M-U] in the 3-dimensional case — can be found in [An].

In the present paper we adopt the following approach: let $Z \subset X$ be a closed orbit of the G action, then Z is a homogeneous variety of G. Also the normal bundle of Z in X, we denote it by $N_{Z/X}$, is homogeneous with respect to the tangent action of G (for a precise definition see Section 1) and the classification of such bundles is rather well understood, being directly related to the representations of the parabolic isotropic group of a point of Z, via Borel-Weil-Bott Theory.

It is natural to believe that the action of G on $N_{Z/X}$ should somehow reflect the action of G on X. A naive expectation is that if the action of G on X has an open orbit then the induced action of G on $N_{Z/X}$ has an open orbit as well. In fact, this is the case when Z is just a fixed point and it follows from the well known Luna's theorem on étale slices ([Lu]). However it turns out that, in general, the expectation is wrong, as showed by Example (2.7).

The right method of relating these two actions is via a theorem of Brion, Luna and Vust, namely Theorem 1.4 in [B-L-V], which is a corner-stone of a beautiful theory of spherical varieties, i.e. varieties on which a Borel subgroup $B \subset G$ acts with an open orbit. As an immediate application of the Brion-Luna-Vust theorem one can relate the dimension of a general orbit of an induced action of a parabolic subgroup $\tilde{P} \subset G$, opposite to the isotropy group of a point in Z, to the dimension of the orbits of the induced action on $N_{Z/X}$.

This brings us, however, to the question of relating the action of G on X to the induced action of its parabolic subgroup P, which is a weaker version of a general problem of identifying spherical varieties among quasihomogenous ones, see e.g. [Br].

In this paper we deal with quasihomogeneous manifolds of the group $G = SL(n)$. The main result is Theorem (3.4) which is about the normal bundle of a closed orbit $\simeq \boldsymbol{P}^{n-1}$. Subsequently, we can apply results of Akhiezer and Brion to understand the structure of quasihomogenous $SL(n)$-manifolds of dimension at most $2n-2$, see Corollary (3.5). We show that $\dim X$ cannot be strictly between n and $2n-4$. The classification of quasihomogeneous manifolds of dimension at most n was done previously, so we get new results for dimension $2n-4$, $2n-3$ and $2n-2$. It turns out that, in this range – with one possible exception – all quasihomogenous varieties are spherical.

In absense of suitable references concerning pertinent basics on the group actions we discuss some of them in detail in section 1, risking that the experts in the field will get annoyed.

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Notation. For a vector bundle E over Z by $\boldsymbol{P}(E)$ we denote the projective bundle of lines in E, that is $E\backslash\{0\}/\boldsymbol{C}^*$. Classes of equivalence relations will be denoted by [ ]. $SL(n)$ is the group of $n \times n$ matrices with determinant 1; $sl(n)$ is its Lie algebra of matrices of trace 0. In general, Lie algebra of a group G will be denoted by a calligraphic letter $\mathcal{G}$. For a sheaf $\mathcal{F}$ over the projective space $\boldsymbol{P}^n$ by $\mathcal{F}(a)$ we denote its twist $\mathcal{F} \otimes \mathcal{O}(a)$ by a line bundle $\mathcal{O}(a)$.

1. – Primer on homogeneous vector bundles.

In the present section we collect material on homogeneous vector bundles which we believe to be commonly known, however we have not been able to find an adequate reference for.

Let G be a connected algebraic group over complex numbers. In what follows we will assume that G is simply connected and simple, but this is not needed at the beginning of this section. Let $H \subset G$ be a connected closed (algebraic) subgroup such that the (right) quotient $G/H = \{gH = [g] : g \in G\}$ is an algebraic variety; in other words G/H is a quotient of G by the right-hand-side action of H defined as $H \times G \ni (h, g) \to gh^{-1} \in G$. The group G acts on G/H from the left side $G \times G/H \ni (g, [g']) \to [gg'] \in G/H$. The variety $Z := G/H$ is called a G-homogeneous variety and H is called its isotropy group; Z is projective if and only if H is parabolic.

DEFINITION (1.1). – A rank r vector bundle $\pi : E \to Z$ is called G-homogeneous or simply homogeneous if one of the two following equivalent conditions holds:

(a) There exists an action of G on E, linear on fibers of π and compatible with the action of G on Z which means that the following diagram commutes:

$$\begin{array}{ccc} G \times E & \to & E \\ \downarrow & & \downarrow \\ G \times Z & \to & Z \end{array}$$

(b) There exists a representation $\varrho : H \to GL(V)$, with $\dim V = r$, such that $E \simeq E_\varrho$, where E_ϱ is the vector bundle defined as the quotient of $G \times V$ via the equivalence relation $\sim$, where $(gh, v) \sim (g, \varrho(h)\, v)$ for any $h \in H$; in other words E_ϱ is a quotient of $G \times V$ via a (right-hand-side) action of H defined as

$$H \times (G \times V) \ni (h, (g, v)) \to (gh^{-1}, \varrho(h)\, v) \in G \times V\,.$$

The equivalence of conditions (a) and (b) is straightforward. Given a representation ϱ from (b) we defined an action of G on E_ϱ in the following way:

$$G \times E_\varrho \ni (g, [g', v]) \to [gg', v] \in E_\varrho$$

This action in the fiber over $[1_G]$ yields the representation ϱ because $(h, [1_G, v]) \to [h, v] = [1_G, \varrho(p)\, v]$.

Conversely, given an action of G on a vector bundle E over G/H, the isotropy group H acts on the fiber of E over $[1_G]$ as a representation $\varrho : H \to GL(E_{[1_G]})$. We define $\varphi : E \to E_\varrho$ by setting for $v \in \pi^{-1}([g])$, $\varphi(v) = [g, g^{-1}(v)]$; the definition of φ does not depend on the choice of g above hence it yields an isomorphism of vector bundles.

Let us note that although our definition of the G action on E_ϱ is unique, the above defined correspondence $E \leftrightarrow E_\varrho$ does not have to preserve G action. That is: the restriction of two different actions of G on E to the action of H on $E_{[1_G]}$ may define the same representation $\varrho : H \to GL(E_{[1_G]})$. For example, take $H = \{1_G\}$ and $E = G \times V$ with the action defined by any representation $G \to GL(V)$. What is more, two different representations ϱ_1 and ϱ_2 of the isotropy group H may define isomorphic bundles $E_{\varrho_1} \simeq E_{\varrho_2}$ on G/H.

EXAMPLE (1.2). – Take $G = SL(n)$ with $Z = SL(n)/P \simeq \boldsymbol{P}^{n-1} = \boldsymbol{P}(V)$, where $V = \boldsymbol{C}^n$, and the associated isotropy group $P \subset SL(n)$ consisting of matrices whose first column has zeroes outside the highest row. Then take ϱ_1 being the trivial representation on V so that $E_{\varrho_1} \simeq \boldsymbol{P}^{n-1} \times V$ is the trivial bundle. If we take ϱ_2 being the restriction to P of the standard representation of $SL(n)$ on V and ϱ_3 its dual then the bundles E_{ϱ_2} and E_{ϱ_3} are trivial as well but the associated actions of $SL(V)$ on $\boldsymbol{P}^{n-1} \times V$ in these three cases are different (except when $n \leqslant 3$). This is a special case of the following somewhat more general observation, for a proof see ([Sl], p. 25).

LEMMA (1.3). – *Let G be an algebraic group with a closed subgroup $H \subset G$. Suppose that a representation $\varrho : H \to GL(V)$ extends to $\varrho : G \to GL(V)$. Then the bundle E_ϱ over G/H is trivial and the induced action of G on the trivialization $G/H \times V \simeq E_\varrho$ is as follows*

$$G \times (G/H \times V)) \ni (g, ([g'], v)) \to ([gg'], \varrho(g)\, v) \in G/H \times V.$$

Another formulation of the preceding lemma could be as follows: *If the right-hand-side action of H on $G \times V$, which gives rise to E_ϱ, extends to the whole G then the bundle E_ϱ is trivial.* Accordingly the argument could be reduced to the following observation: the right-hand-side action of G on $G \times V$ defines a bundle E_0 over G/G and, by the functoriality of this construction, E_ϱ is a pull-back of E_0.

Let us recall the definition of the tangent bundle of a homogeneous variety. Let G be a connected algebraic group with the associated Lie algebra $\mathcal{G}$ of left-invariant fields on G. Thus we have a natural trivialization of the tangent bundle of G, $TG \simeq G \times \mathcal{G}$, under which the left-hand-side action of G is trivial on $\mathcal{G}$. If, however, we consider the right-hand-side action of the group (or its subgroup) then the resulting action will be described by the adjoint representation $Ad : G \to GL(\mathcal{G})$. That is, let $H \subset G$ be a closed algebraic subgroup with the associated Lie algebra $\mathcal{H} \subset \mathcal{G}$. Let us consider the right-hand-side action of H on G: $H \times G \ni (h, g) \to gh^{-1} \in G$. Then the resulting action of H on $TG = G \times \mathcal{G}$ is as follows: $(h, (g, v)) \to (gh^{-1}, Ad(h)\, v)$.

Now take $gH \subset G$; then, because of the left invariance of the trivialization of the tangent bundle, $T(gH) \simeq gH \times \mathcal{H} \subset gH \times \mathcal{G} \simeq TG_{|gH}$. Therefore, since the map $G \to G/H$ is smooth, its relative tangent bundle may be identified with the trivial G-invariant sub-bundle $G \times \mathcal{H} \subset G \times \mathcal{G}$. So the quotient $G \times (\mathcal{G}/\mathcal{H})$ is the pull-back of the tangent bundle $T(G/H)$ and therefore the latter is obtained from the former via a right-hand-side action of H which, as we explained in the previous paragraph, comes from Ad. That is, $T(G/H) = E_{\widehat{Ad_H}}$, where $\widehat{Ad_H} : H \to GL(\mathcal{G}/\mathcal{H})$ is the quotient of the restriction of the adjoint representation to the subgroup H, that is $Ad_{|H} : H \to GL(\mathcal{G})$.

On the other hand, since the left action of G on itself descends to the left action of G on G/H, the left invariant fields from $\mathcal{G}$ descend to fields over G/H. Thus the surjective morphism $G \times \mathcal{G} \to G \times (\mathcal{G}/\mathcal{H})$ descends, via the right-hand-side action of H, to a surjective morphism of homogeneous vector bundles

$$\varepsilon : E_{Ad_{|H}} \to E_{\widehat{Ad_H}} = T(G/H), \qquad \varepsilon([g, v]) = [g, [v]]$$

which is just the evaluation of these fields, as sections of $T(G/H)$. We note that the trivialization $\varphi : E_{Ad_H} \to G/H \times \mathcal{G}$, defined in the previous lemma, identifies the kernel of ε over a point $[g] \in G/H$ to $Ad(g)\, \mathcal{H} \subset \mathcal{G}$.

As the result of the discussion we observe the following

LEMMA (1.4). – *Let $H \subset G$ be as above. Let $H' \subset G$ be another connected closed algebraic subgroup of G with the associated Lie algebra $\mathcal{H}' \subset \mathcal{G}$. Then the left action of H' on G/H has an open orbit if and only if for some $g \in G$ we have $\mathcal{H}' + Ad(g)(\mathcal{H}) = \mathcal{G}$.*

PROOF. – In view of the preceding discussion the condition $\mathcal{H}' + Ad(g)(\mathcal{H}) = \mathcal{G}$ is equivalent to say that vector fields tangent to the automorphisms of G/H, arising from the action of H', span $T(G/H)$ at the point $[g]$, hence the result.

Note that the above statement is symmetric. That is, in the notation of the Lemma, the action of H' on G/H has an open orbit if and only if the action of H on G/H' has an open orbit.

In the remainder of this section we discuss $SL(n)$-homogeneous vector bundle over $\boldsymbol{P}^{n-1}$. Let $P \subset SL(n)$ be a subgroup consisting of matrices whose entries in the first column are zero, except the one on the diagonal. Hence $G/P \simeq \boldsymbol{P}^{n-1}$. Let $\chi : P \to \boldsymbol{C}^*$ be the character generating the group of characters $\mathrm{Hom}(P, \boldsymbol{C}^*)$ such that $E_\chi = \mathcal{O}(1)$. Thus, the line bundle $\mathcal{O}(k)$ is associated to the character χ^k. The proofs of the following results are standard and we omit them.

LEMMA (1.5). – *Let* $E = \bigoplus_{i=1}^{m} \mathcal{O}(a_i)^{\oplus r_i}$ *be a decomposable bundle over* $\boldsymbol{P}^{n-1}$ *with* $a_1 > a_2 > \ldots > a_m$. *If, for* $i = 1 \ldots m$, *we have* $r_i < n$ *then the* $SL(n)$ *action on* E *is uniquely defined and comes from the natural representation* $\varrho = r_1 \chi^{a_1} + \ldots + r_m \chi^{a_m}$ *of* P *on a fiber of* E.

PROPOSITION (1.6). – *Let* E *be a homogeneous bundle over* $\boldsymbol{P}^{n-1}$ *of rank* r. *If* $r < n-1$ *then* E *is decomposable into a sum of line bundles. If* $r = n-1$ *then* E *is either decomposable or twisted tangent, or twisted cotangent bundle. In all the cases the action of* $SL(n)$ *on* E *is uniquely defined.*

COROLLARY (1.7). – *Let* E *be a* $SL(n)$*-homogeneous vector bundle of rank* r *over* $\boldsymbol{P}^{n-1}$, *with* $n \geqslant 3$. *Suppose that* $r \leqslant n-1$. *Then the associated isotropy group representation* $P \to GL(r)$ *has an orbit of dimension* r *if and only if one of the following occurs*

(a) $r = 1$ *and* E *is non-trivial,*

(b) $r = n-1$ *and* $E \simeq T\boldsymbol{P}^{n-1}(a)$ *or* $E \simeq \Omega\boldsymbol{P}^{n-1}(a)$, *with* $a \in \boldsymbol{Z}$.

2. – Brion-Luna-Vust Theorem.

Let G be a simply connected, simple algebraic group acting on a smooth algebraic variety X. Assume that there exists a closed orbit $G \cdot z := Z$ which is projective, that is the isotropy group G_z is a parabolic subgroup of G. The group G acts algebraically by tangent maps on the bundle $TX_{|Z}$. This action preserves the sub-bundle $TZ \hookrightarrow TX_{|Z}$ and thus it descends to the normal bundle $N_{Z/X}$. More precisely: for a given $g \in G$ we have the tangent map $Tg : N_{Z/X} \to N_{Z/X}$ which dominates the automorphism of the base $g : Z \to Z$. According to the definition (1.1.a), the bundle $N_{Z/X}$ is homogeneous with respect to this action. The associated representation of the isotropy group G_z on the fiber $(N_{Z/X})_z$ (as in the definition (1.1.b)) we call a normal representation of G_z.

Let us blow X along Z to get $\widehat{X}$ with the exceptional divisor $\widehat{Z} = \boldsymbol{P}(N_{Z/X})$. The action of G lifts up to $\widehat{X}$ and its restriction to $\widehat{Z}$ comes from the above action on $N_{Z/X}$.

PROPOSITION (2.1). – *In the above situation let* $G = SL(n)$ *and* $Z \simeq \boldsymbol{P}^{n-1}$. *If* $\dim X \leqslant 2n-2$ *then the tangent action of* $SL(n)$ *on* $N_{Z/X}$ *is uniquely defined and as described in Proposition* (1.6).

EXAMPLE (2.2). – Ruled quasihomogeneous manifolds. Let $Z = G/P$ be a homogeneous G manifold with the isotropy parabolic group $G_x = P$. Let $\lambda_i\colon G_x \to \boldsymbol{C}^*$, $i = 1, 2$ be two characters and consider the representation $\varrho = \lambda_1 + \lambda_2\colon G_x \to GL(2)$. Let $E = E_\varrho = E_{\lambda_1} \oplus E_{\lambda_2}$ be the associated homogeneous bundle with the induced action of G. If $\lambda_1 \neq \lambda_2$ then the projective bundle $\boldsymbol{P}(E)$ is quasi-homogeneous under the induced G action. It has two closed orbits Z_1 and Z_2 which are sections of $\boldsymbol{P}(E) \to Z$ with normal bundle E_{λ_1/λ_2} and E_{λ_2/λ_1}, respectively.

EXAMPLE (2.3). – Let us consider the action of $G = SL(2)$ on $X = \boldsymbol{P}^3$ which comes from the third symmetric product of the standard representation. The only closed orbit of the action is a rational twisted cubic $Z \subset \boldsymbol{P}^3$. The action has moreover an open orbit and a unique invariant 2-dimensional subset $Y \supset Z$ which is swept by lines tangent to Z. If we blow up $\boldsymbol{P}^3$ along Z then the result is a ruled 3-fold $\widehat{X} = \boldsymbol{P}(\mathcal{E})$, with $\mathcal{E}$ a stable rank 2 vector bundle over $\boldsymbol{P}^2$ such that $c_1(\mathcal{E}) = 2$ and $c_2(\mathcal{E}) = 3$, see [S-W]. The induced action of $SL(2)$ on $\widehat{X}$ has two invariant divisors: the exceptional divisor of the blow-down $\widehat{X} \to \boldsymbol{P}^3$, let us call it $\widehat{Z}$, and a divisor $\widehat{Y}$, the strict transform of the divisor Y. The map $\widehat{Z} \to \boldsymbol{P}^2$ is a double covering and $\widehat{Y}$ is a pull-back, via $\boldsymbol{P}(\mathcal{E}) \to \boldsymbol{P}^2$, of its branch divisor which is a conic in $\boldsymbol{P}^2$. The action of $SL(2)$ on $\widehat{X} = \boldsymbol{P}(\mathcal{E})$ descends to the action on $\boldsymbol{P}^2$ whose only closed orbit is the conic in question. The intersection $B = \widehat{Z} \cap \widehat{Y}$ is the unique closed orbit of the action of $SL(2)$ on $\widehat{X}$.

From the above it follows that $\widehat{Z} \simeq \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and $B \subset \widehat{Z}$ is the ramification of a double cover $\boldsymbol{P}^1 \times \boldsymbol{P}^1 \to \boldsymbol{P}^2$, hence a curve of bidegree $(1, 1)$. Therefore $N_{Z/\boldsymbol{P}^3}$ is a twisted trivial bundle (computing by adjunction its degree we get $N_{Z/\boldsymbol{P}^3} \simeq \mathcal{O}(5) \oplus \mathcal{O}(5)$) but the tangent $SL(2)$ action on the normal bundle induces a (twisted) representation of P on the fiber coming from the standard representation of $SL(2)$ (which, of course, is not $\chi^5 + \chi^5$; notation as in the previous section). Thus we are in the situation of Example (1.2).

The task now is to compare the original action of G on a smooth projective variety X with the tangent action of G on the normal bundle $N_{Z/X}$ of a closed orbit. The motivation for this is an étale linearization (corollary to Luna's slice theorem).

LOCAL LINEARIZATION THEOREM (2.4). – [Lu] *Suppose that a reductive group G acts on a smooth quasiprojective variety X with a fixed point $z \in X$. Then, up to an étale covering the action of G in an affine G-invariant neighborhood of z is the same as its tangent action on the tangent space $T_z X$. That is, there exists an open affine G-invariant neighborhood $z \in S \subset X$ and an étale G-equivariant map $S \to T_z X$.*

COROLLARY (2.5). – *In the situation of the above Theorem, the dimension* *of a general orbit of the action of G on X is equal to the dimension of a general orbit of the tangent representation* $G \to GL(T_z X)$.

EXAMPLE (2.6). – If G is not reductive then the corollary is not true. Take the unipotent subgroup of $SL(2)$ consisting of upper-triangular matrices with units on the diagonal and $t \in \boldsymbol{C}$ in the upper-right corner – this is just the additive group $\boldsymbol{C}^+$. Consider its standard action on $\boldsymbol{P}^1$ around with the only fixed point $[1, 0]$; in the affine coordinate z around $[1, 0]$ it is given by $(t, z) \mapsto z/(tz+1)$. The derivative at $z=0$ is identity, though, certainly, $\boldsymbol{P}^1 \setminus [1, 0]$ is an open orbit.

It would be very nice to have an instant extension of the above corollary to the case of an arbitrary closed orbit. A naive expectation is as follows: *Let a reductive (or even simple,* 1*-connected) group G act on a projective manifold X with a closed orbit Z. If the action of G on X has an open orbit then its tangent action* $G \times N_{Z/X} \to N_{Z/X}$ *has an open orbit as well.* This is however wrong:

EXAMPLE (2.7). – Let us extend Example (2.3). That is: $\widehat{X}$ is the blow-up of $\boldsymbol{P}^3$ along a twisted cubic, with the induced $SL(2)$ action and two invariant divisors $\widehat{Z}$ and $\widehat{Y}$ whose intersection B is the unique closed orbit of the action. Both divisors are smooth but their intersection along B is non-transversal, as we have noticed that $\widehat{Y}$ is the pull-back of the branching conic related to the double cover $\widehat{Z} \to \boldsymbol{P}^2$. Now let us blow up $\widehat{X}$ along B, call the result $\widetilde{X}$ with the exceptional divisor $\widetilde{B}$ and $\widetilde{Z}$, $\widetilde{Y}$ denoting strict transforms of the respective divisors. We lift up the action of $SL(2)$ to $\widetilde{X}$. The above three named divisors are clearly $SL(2)$ invariant and they intersect transversally along a 1-dimensional orbit C of the action – transversality of the intersection can be verified along a 2-dimensional slice of $\widetilde{X}$ over a curve in $\boldsymbol{P}^2$. In fact, let us note that $\widetilde{B}$ and $\widetilde{Y}$ play a symmetric role in this construction: if we contract $\widetilde{Y}$ then we get a projective bundle over $\boldsymbol{P}^2$. Thus the induced action of $SL(2)$ on $\boldsymbol{P}(N_{C/\widetilde{X}})$ has three non-meeting orbits at least, which is possible only if $\boldsymbol{P}(N_{C/\widetilde{X}}) = \boldsymbol{P}^1 \times \boldsymbol{P}^1$ and the action is nontrivial along one coordinate only.

The right way to generalize the corollary goes through the following theorem, due to Brion, Luna and Vust:

BLV THEOREM (2.8). – [Brion-Luna-Vust, Thm 1.4] *Assume that G is an connected algebraic reductive group acting on a smooth quasiprojective variety X with a closed orbit* $Z = G \cdot z$, *whose isotropy group* $P \subset G$ *is parabolic. Let* $\widetilde{P} \subset G$ *be a subgroup of G opposite to P with* $\widetilde{P}^u$ *denoting its radical unipotent part and* $L = P \cap \widetilde{P}$ *its Levi subgroup, so that* $\widetilde{P} = \widetilde{P}^u \cdot L$. *Then there exists a locally closed affine subvariety* $W \subset X$ *such that*

(1) $z \in W$ *and W in L-invariant,*

(2) $\widetilde{P}^u \cdot W$ *is an open subset of X,*

(3) *the action of* $\tilde{P}^u$ *on* X *induces an isomorphism of algebraic varieties* $\tilde{P}^u \times W \simeq \tilde{P}^u \cdot W$.

Before deriving an application of the theorem let us make some comments on its contents. For simplicity, let $X_0 = \tilde{P}^u \cdot W = \tilde{P} \cdot W \subset X$ and $Z_0 = Z \cap X_0$. Then both X_0 and Z_0 are invariant with respect to the $\tilde{P}$ action. Next, $\dim \tilde{P}^u = \dim P^u = \dim Z$ (see Proposition 14.21 and its proof in [Bor]) hence, because of property (3) above, $Z_0 = \tilde{P}^u \cdot z$ and W and Z_0 intersect transversally at z. The point z is fixed by the action of L and, because both W and Z_0 are L invariant, we have the L equivariant splitting $T_z X = T_z Z \oplus T_z W$ where $T_z W$ can be naturally identified with the normal space $(N_{Z/X})_z$ together with the induced normal representation (restricted from the isotropy group P to L), that is $L \to GL((N_{Z/X})_z)$.

We use BLV Theorem to prove the following proposition; this application is very similar to the proof of corollary 1.5 in [B-L-V] (see also the proof of the theorem 1.1 in [Br]).

PROPOSITION (2.9). – *In the notation of the above BLV Theorem, suppose that a general orbit of the action of* $\tilde{P}$ *on* X *is of dimension* k. *Then a general orbit of the normal representation* $L \to GL((N_{Z/X})_z)$ *is of dimension* $\geq k - \dim Z$.

PROOF. – By assumption a general orbit of the action of $\tilde{P}$ on the open subset $X_0 = \tilde{P}^u \cdot W$ has dimension k; let O be such an orbit.

We note that, because $\tilde{P} = \tilde{P}^u \cdot L$ and W is L invariant, we have $\tilde{P} \cdot (O \cap W) = \tilde{P}^u \cdot (O \cap W) \simeq \tilde{P}^u \times (O \cap W)$ where the latter isomorphism follows by property (3) of BLV Theorem. Thus $\dim (O \cap W) = \dim O - \dim \tilde{P}^u = \dim O - \dim Z$. On the other hand (again by (3) and $\tilde{P} = \tilde{P}^u \cdot L$), the stabilizer subgroup of the set W in $\tilde{P}$ is L, hence $O \cap W$ is an orbit for the action of L on W.

Since L is reductive and W is a smooth affine L-variety with a fixed point z, we can apply Local Linearization Theorem in order to compare the action of L around $z \in W$ with its tangent action $L \to GL(T_z W)$. In particular this last action has an orbit of dimension $k - \dim Z$. But we have already noticed that the representation $L \to GL(T_z W)$ is just the restriction to L of the normal representation $P \to GL((N_{Z/X})_z)$. This concludes the proof of the proposition.

3. – $SL(n)$-quasihomogeneous manifolds.

Let G be a simply-connected simple algebraic group over $\boldsymbol{C}$. Suppose that G acts on X so that X is a G-quasihomogeneous manifold. Let $Z \subset X$ be a closed orbit of the action, hence a homogeneous variety of G; such varieties are classified by subsets of the Dynkin diagram associated to G. The normal bundle

$N_{Z/X}$ is homogeneous with respect to the action of G and a classification of such G-bundles should be pretty natural; in particular we gave some results in the case $G = SL(n)$ in Section 1. Thus one is tempted to relate the classification of G-quasihomogeneous manifolds with that of G-homogeneous bundles over homogeneous spaces.

In view of BLV Theorem an *ideal scheme of the argument* would be as follows:

(0) choose the data as in BLV theorem: $z \in Z$ with isotropy parabolic group $P \subset G$ and its opposite parabolic $\tilde{P}$, and Levi subgroup group $L = P \cap \tilde{P}$;

(1) find out the dimension of a general orbit of the action of $\tilde{P}$ on X: one may actually expect that, if the codimension of Z in X is not too large, then $\tilde{P}$ acts with an open orbit on X;

(2) using BLV Theorem and its application (2.9), relate the dimension of a general orbit of $\tilde{P}$ to the dimension of the tangent action of L on the fiber of $N_{Z/X}$ over z;

(3) confront the result with a possible action of L on a fiber of a homogeneous vector bundle over Z (see for instance (1.6) or (1.7)).

It seems that the step (1) of our ideal argument should be the hardest one; in what follows we deal with it in the case $G = SL(n)$, $Z = \boldsymbol{P}^{n-1}$ and $\dim X \leqslant 2n-2$. For this we begin with a technical result.

LEMMA (3.1). – *Let $\mathcal{F}$ be a coherent $\mathcal{O}$-subsheaf of $T\boldsymbol{P}^{n-1}$ of rank $< n-1$. Then* $\dim H^0(\mathcal{F}) \leqslant n^2 - 2n$.

PROOF. – We may assume that $\mathcal{F}$ is a saturated subsheaf of $T\boldsymbol{P}^{n-1}$ of corank 1, because otherwise we can replace $\mathcal{F}$ by a larger subsheaf of $T\boldsymbol{P}^{n-1}$ which still satisfies assumptions above. Thus $\mathcal{F}$ is reflexive and we have an exact sequence of sheaves of $\mathcal{O}$-modules:

$$0 \to \mathcal{F} \to T\boldsymbol{P}^{n-1} \to \mathcal{O}(k) \to \mathcal{R} \to 0$$

where $\mathcal{O}(k)$ is a line bundle and $\mathcal{R}$ is a torsion sheaf whose support is of codimension $\geqslant 2$ in $\boldsymbol{P}^{n-1}$. Moreover $k \geqslant 2$, since $\mathrm{Hom}\,(T\boldsymbol{P}^{n-1}, \mathcal{O}(k)) = 0$ for $k \leqslant 1$. On the other hand we have the Euler sequence on $\boldsymbol{P}^{n-1}$

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus n} \to T\boldsymbol{P}^{n-1} \to 0\,.$$

Combining these two sequences and twisting by $\mathcal{O}(-1)$ we get

$$\mathcal{F}(-1) = \ker\,(\mathcal{O}^{\oplus n} \to \mathcal{O}(k-1))/\mathrm{im}\,(\mathcal{O}(-1) \to \mathcal{O}^{\oplus n}).$$

Therefore $\mathcal{F}$ is completely defined by the morphism $\mathcal{O}^{\oplus n} \to \mathcal{O}(k-1)$ which is a choice of n sections of $\mathcal{O}(k-1)$. Moreover, since $\mathcal{R}$ has support in codimension

≥ 2 it follows that these n sections span $\mathcal{O}(k-1)$ in codimension 2, thus we can choose among these sections a linear pencil without base component. Therefore we have two section of $\mathcal{O}(k-1)$, (homogeneous polynomials of degree $k-1$) f_1, f_2 which have no common divisor and a commutative diagram

$$\begin{array}{ccc} & H^0(\mathcal{O}(1)^{\oplus n}) & \supset & H^0(\mathcal{O}(1)) \oplus H^0(\mathcal{O}(1)) \\ & \downarrow & & \downarrow \\ H^0(\mathcal{F}) \to & H^0(T\boldsymbol{P}^{n-1}) & \to & H^0(\mathcal{O}(k)) \end{array}$$

where the right-hand-side vertical arrow is the evaluation of the pencil twisted by $\mathcal{O}(1)$, so to a pair of linear forms (s_1, s_2) it associates $s_1 f_1 + s_2 f_2$. Now to conclude the proof we have to show that the rank of the right-hand-side horizontal map is $2n-1$ at least. Thus it is enough to show that the kernel of the right-hand-side vertical arrow is one dimensional at most. If however $(s_1, s_2) \mapsto s_1 f_1 + s_2 f_2 = 0$ then $f_1/f_2 = -s_2/s_1$ which, since we assumed that f_1 and f_2 have no common factor, is possible only if they are linear and equal (up to an invertible constant) to the respective linear forms on the right-hand-side. Thus the map in question is injective for $k \geq 3$, while for $k=2$ it has 1-dimensional kernel.

REMARK (3.1.1). – The bound $\dim H^0(\mathcal{F}) \leq n^2 - 2n$ of the lemma is optimal as one can see taking for instance $n=3$, a point $x \in \boldsymbol{P}^2$ and the exact sequence

$$0 \to \mathcal{O}(1) \to T\boldsymbol{P}^2 \to \mathcal{J}_x(2) \to 0 .$$

Combining the above lemma with the observation we have made in Section 1 we get:

LEMMA (3.2). – Let $W \subset sl(n)$ *be a linear subspace of codimension* $\leq 2n-2$ *and* $\mathcal{P} \subset sl(n)$ *be a maximal parabolic subalgebra of codimension* $n-1$. *Then for a general* $g \in SL(n)$ *we have* $Ad(g)(\mathcal{P}) + W = sl(n)$.

PROOF. – We can identify $sl(n)$ with the space of sections of $T(G/P) \simeq T\boldsymbol{P}^{n-1}$. Let $\varepsilon : \boldsymbol{P}^{n-1} \times sl(n) \to T\boldsymbol{P}^{n-1}$ be the evaluation map. The restriction $\varepsilon_W : \boldsymbol{P}^{n-1} \times W \to T\boldsymbol{P}^{n-1}$ has to be generically surjective, since otherwise we would set $\mathcal{F} := im(\varepsilon_W)$ and we contradict the previous lemma. Thus, because of the discussion preceding Lemma (1.4), $Ad(g)(\mathcal{P}) + W = sl(n)$.

PROPOSITION (3.3). – *Let* X *be a* $SL(n)$ *quasihomogeneous manifold which contains a closed orbit* $Z \simeq \boldsymbol{P}^{n-1}$. *Let* $P = G_z \subset SL(n)$ *be the isotropy group of a point* $z \in Z$ *and let* $\tilde{P} \subset SL(n)$ *be a parabolic subgroup opposite to* P. *If* $\dim X \leq 2n-2$ *then* $\tilde{P}$ *acts on* X *with an open orbit.*

PROOF. – Let $x \in X$ be a point in the general orbit with the isotropy group $H = G_x \subset SL(n)$ and the associated Lie algebra $\mathcal{H} \subset sl(n)$. Let $\widetilde{\mathcal{P}}$ be the Lie algebra of $\widetilde{P}$. Then, in view of the proceeding lemma, for a general $g \in SL(n)$ we have $\widetilde{\mathcal{P}} + Ad(g)(\mathcal{H}) = sl(n)$ and, by Lemma (1.4), $\widetilde{P}$ acts on G/H with an open orbit.

Now we can proceed with the argument explained at the beginning of this section (applying in the consecutive order (3.3), (2.9) and (1.7)) and get the following.

THEOREM (3.4). – *Let X be a $SL(n)$ quasihomogeneous manifold which contains a closed orbit $Z \simeq \boldsymbol{P}^{n-1}$ and assume that* $\dim X \leq 2n-2$. *Then one of the following occurs:*

(a) $\dim X = n$ and $N_{Z/X} = \mathcal{O}(a)$ with $a \neq 0$, or

(b) $\dim X = 2n-2$ and either $N_{Z/X} \simeq T\boldsymbol{P}^{n-1}(a)$ or $N_{Z/X} \simeq \Omega\boldsymbol{P}^{n-1}(a)$ with $a \in \boldsymbol{Z}$.

This theorem can be nicely combined with the results of Akhiezer and others (see [Ak] or [Br]) to obtain a classification of $SL(n)$-quasihomogeneous manifolds X of dimension $\leq 2n-2$, where $n \geq 3$. For this we first notice that, since the classification of projective manifolds of dimension $\leq n$ with a nontrivial $SL(n)$-action was done in [Ma] (and more recently reproved considering also the case of other simple groups with the use of Mori theory in [An]) we can assume that $n+1 \leq \dim X \leq 2n-2$.

COROLLARY (3.5). – *Let X be a $SL(n)$ quasihomogeneous manifold and assume $n+1 \leq \dim X \leq 2n-2$. Then* $\dim X \geq 2n-4$ *and one of the following occurs.*

(a) $\dim X = 2n-4$ *and X is homogeneous,* $X \simeq Gr(2, n)$;

(b) $\dim X = 2n-3$ *and either*

  (b0) *X is homogeneous (then X is a flag manifold $F(1, 2, n)$ or $X = Gr(3, 6)$ for $n = 6$), or*

  (b1) *X contains a codimension 1 closed orbit isomorphic to $Gr(2, n)$;*

(c) $\dim X = 2n-2$ *and either*

  (c0) *X is homogeneous (then $n = 4$ and $X \simeq F(1, 2, 3, 4)$, or $n = 7$ and $X = Gr(3, 7)$),*
  or X contains a proper closed orbit Z which satisifies one of the following:

  (c1) *Z is a divisor in X (then either $n = 6$ and $Z = Gr((3, 6)$ or Z is a flag manifold $F(1, 2, n)$),* (c2) *Z is the projective space* $\boldsymbol{P}^{n-1}$ *and the blow-up of X along Z is a quasihomogeneous variety with the exceptional divisor being homogeneous;*

(c3) *Z is the Grassmann manifold* $Gr(2, n)$ *of codimension* 2.

PROOF. – We note that if a $SL(n)$ quasihomogeneous manifold of dimension $\leqslant 2n-1$ has a fixed point then it is $\boldsymbol{P}^n$, with the action extending the standard one (or its dual) via the natural inclusion $\boldsymbol{C}^n \subset \boldsymbol{P}^n$. Indeed, if $n < \dim X < 2n$ and z is a fixed point of the action of $SL(n)$ then the tangent representation $SL(n) \to GL(T_z X)$ is a sum of at most one copy of n-dimensional representation and a trivial representation of complementary dimension — therefore it has no orbit of dimension bigger than n — so X can not be quasihomogeneous by Local Linearization Theorem (Corollary 2.5).

Thus we may assume that there exists a closed orbit of positive dimension in X. But on the other hand, the only homogeneous $SL(n)$-manifolds of dimension $\leqslant 2n-2$ are the following: (1) a point, (2) $\boldsymbol{P}^{n-1}$, (3) Grassmann manifold $Gr(2, n)$, (or $Gr(n-2, n)$) of dimension $2n-4$, (4) flags $F(1, 2, n)$ (or $F(n-2, n-1, n)$ of dimension $2n-3$, (5) flags $F(1, 2, 3, 4)$ of dimension 6 if $n=4$ or $X = Gr(3, 6)$ if $n=6$, or $X = Gr(3, 7)$ if $n=7$.

These give rise to all the cases of the corollary; what is left to prove is the statement in (c2) which follows in fact from the Theorem (3.4).

REMARK (3.5.1). – The above corollary shows that, with the exception of case (c3), the classification of $SL(n)$ quasihomogeneous manifolds of dimension $\leqslant 2n-2$ can be derived from the works of Akhiezer. Indeed, if X is not homogeneous then, after possibly blowing up an orbit isomorphic to $\boldsymbol{P}^{n-1}$ (case (c2)), X has an orbit which is an homogeneous divisor. Thus either the complement of the open orbit in X is disconnected, and X is classified by Theorem 7.6 in [Ak], or the complement of the open orbit is the homogeneous divisor, and X is classified by Theorem 7.8 in [Ak].

In case (c3) the classification is not complete and extra work is needed: for instance a Lemma similar to (3.1) for $Gr(2, n)$ would be useful. In fact, it is a very interesting question to find a suitable version of Lemma (3.1) for any G homogeneous variety, where G is a simply connected, simple algebraic group over $\boldsymbol{C}$. If $n=3$, however, the cases (c2) and (c3) in the above theorem coincide so we get the following result known already by [Na2].

COROLLARY (3.6). – *If X is a* $SL(3)$ *quasihomogeneous 4-fold then, after possible blowing up 2-dimensional closed orbits of the action, we get a* $SL(3)$ *quasihomogenous manifold with all closed orbits of codimension* 1, *hence classified as an equivariant completion of rank one spherical variety, see* [Ak] *or* [Br].

To conclude, note that Corollary (3.5) proves also a conjecture stated in [An].

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