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Pronormal and Subnormal Subgroups and Permutability.

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Dedicated to Professor G. Zacher on the occasion of his seventy-fifth birthday.

Sunto. – *Trattiamo gruppi finiti che soddisfano una delle condizioni seguenti:* (1) *I sottogruppi massimali permutano con i sottogruppi subnormali,* (2) *I sottogruppi massimali ed i p-sottogruppi di Sylow* ($p < 7$) *permutano con i sottogruppi subnormali.*

Summary. – *We describe the finite groups satisfying one of the following conditions: all maximal subgroups permute with all subnormal subgroups,* (2) *all maximal subgroups and all Sylow p-subgroups for* $p < 7$ *permute with all subnormal subgroups.*

1. – Introduction and statement of results.

The subnormal subgroups and the pronormal subgroups of a group are in some sense opposite families of subgroups. Only normal subgroups are both pronormal and subnormal, and both embedding properties are inherited to quotient groups and subgroups. In finite groups prominent examples of pronormal groups include Sylow p-subgroups, their normalizers, and maximal subgroups. Permutability of subnormal subgroups with certain classes of pronormal subgroups in finite groups has been the subject of many publications, we mention here only Agrawal [1] for Sylow subgroups, and the authors [5] for the Carter subgroups of a soluble group, together with permutability of subnormal subgroups with every subgroup (see Zacher [17] and Beidleman, Brewster and Robinson [4]).

As a first case of pronormal subgroups we will consider here the maximal subgroups, they are always pronormal, no restriction on the structure of the group is needed. R. Maier has considered subgroups that are permutable with all maximal subgroups in finite groups and developed a supersolubility criteri-

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on (see [12]). Our first aim will be to characterize the finite groups in which all maximal subgroups and all subnormal subgroups permute with each other. As was to be expected, this leads to a characterization of the Frattini quotient group. In fact, we obtain the following

THEOREM A. – *Let G be a finite group. Then the following statements are equivalent:*

(i) *All subnormal subgroups of G permute with all maximal subgroups of G;*

(ii) all chief factors of the Frattini quotient group $G/\Phi(G)$ are simple;

(iiia) *All soluble quotient groups of G are supersoluble*,

(iiib) *all perfect subnormal subgroups of G are normal*,

(iiic) *$G/\Phi(G)$ is an extension of a direct product of non-abelian simple groups by a supersoluble group.*

Notice that there is, for example, a non-split extension of an elementary abelian group of order 16 by A_5, it is therefore difficult to say something about the Frattini subgroup. In cases of solubility, however, we can characterize the whole group.

COROLLARY A. – *All subnormal subgroups of the polycyclic group G permute with all maximal subgroups if and only if G is supersoluble.*

In certain other cases the maximal subgroups alone do not suffice to determine the structure of G below $\Phi(G)$. We add permutability with certain Sylow subgroups.

THEOREM B. – Let G be a finite group. Then the following statements are equivalent:

(i) All subnormal subgroups of G permute with all maximal subgroups and with all Sylow p-subgroups of G, where $p < 7$;

(iia) All chief factors of G are simple groups, the product of all perfect subgroups is a central extension of a direct product of simple groups,

(iib) If N is a normal subgroup of G and F/N the Fitting subgroup of G/N, then every subgroup of F/N is permutable with every Sylow-p-subgroup of G/N, for $p < 7$.

2. – Permutability with maximal subgroups.

PROOF OF THEOREM A. – Assume first that all subnormal subgroups permute with all maximal subgroups of G and let M be a maximal subgroup of G.

There is a minimal normal subgroup T/M_G of G/M_G, where M_G is the intersection of all conjugates of M. If T/M_G is abelian, we know that $T \cap M = M_G$. Every subgroup U satisfying $M_G \subsetneq U \subseteq T$ is subnormal, for $U \neq M_G$ we have in addition $UM = MU (= G)$ and so $U = T$. This shows that T/M_G is cyclic of order some prime p and G/T, being isomorphic to some subgroup of $Aut(T/M_G)$, is cyclic. If T/M_G is nonabelian, it is a direct product of simple nonabelian groups S_i/M_G which are all isomorphic. Assume that there are at least two such direct factors, S_1/M_G and S_2/M_G. We look for a new maximal subgroup N of G in the following way. Let $V/M_G \neq 1$ be a Sylow p-subgroup of T/M_G and K be the normalizer of V. By the Frattini argument, $KT = G$, and by construction $M_G \subseteq K$. Both statements remain true if K is substituted with a maximal subgroup N containing K, in particular, $N_G = M_G$. Also $N \cap S_i \neq M_G$ for each S_i. By maximality of N we have $G = NS_1 = NS_2$ and by the modular law $T = (T \cap N)S_1 = (T \cap N)S_2$. Let a be an element of $S_1 \setminus M_G$. Then there is an element $b \in S_2$ such that $Nb^{-1} = Na$ and so $ba \in T \cap N$. Now $[ba, (N \cap S_1)]M_G = [a, (N \cap S_1)]\, M_G \subseteq S_1$, and since a is arbitrary in the nonabelian simple group S_1 and $N \cap S_1 \neq 1$, we obtain $S_1 \subseteq (N \cap S_1)$ and $T \subseteq N$, a contradiction. So T/M_G is simple as stated. So (ii) follows from (i).

Assume now that G satisfies (ii) and consider a soluble quotient group G/L. Then all non-Frattini chief factors of G/L are of prime order and so all maximal subgroups of G/L have index a prime. This is known to be equivalent to being supersoluble (see Huppert [10]. This shows (iiia).

If D/E is a nonabelian chief factor of G, there is a maximal subgroup M of G such that $M_G = E$ and consequently $MD = G$, and D/E must be simple. If X is a perfect subnormal subgroup of G possessing only one maximal normal subgroup, Y say, then X^G/Y^G is a nonabelian chief factor of G, and by its simplicity we have that X is a normal subgroup. This shows (iiib).

The construction of all finite simple groups has shown as well that the Schreier conjecture is true: If Z is a finite simple group, then $Aut(Z)/Inn(Z)$ is soluble (for this see Gorenstein [9], Theorem 1.46). So if D/E is a nonabelian chief factor of G, then $(G/E)/C_{G/E}(D/E)$ is soluble. Consequently, if D/E is a chief factor which is supplemented, i. e. G possesses a maximal subgroup M such that $DM =$ G and $E \subseteq M_G$, then G/DM_G is isomorphic to a subgroup of $Aut(D/E)$ and hence soluble. We obtain that all chief factors belonging to the soluble radical of the maximal perfect subgroup of G are not supplemented, and (iiic) follows.

Assume now that (iiia),(iiib), and (iiic) are satisfied for the group G. We consider any subnormal subgroup S of G and consider a maximal subgroup $M \not\supseteq S$. If the minimal normal subgroup T/M_G of G/M_G is abelian, it is the only minimal normal subgroup of this quotient group, and it is self-centralizing and of order a prime p. The subnormal subgroup SM_G/M_G possesses a p-subgroup K/M_G as minimal normal subgroup, otherwise T/M_G can not be self-centralizing. Again since T/M_G is self-centralizing, it is the maximal normal p-subgroup of G/M_G and so $T/M_G = TK/M_G$, further $T \subseteq SM_G$. Now $MS = MM_G S = MT = TM = SM_G M = SM$. This in particular clears the soluble case. If, on the other hand, T/M_G is nonabelian, then arguing as before we see that SM_G/M_G possesses a normal subgroup K/M_G isomorphic to the quotient group T/M_G. By condition (iiib), K/M_G is a normal subgroup of G/M_G. T/M_G need not be the only minimal normal subgroup of G/M_G, but there is obviously no proper normal subgroup of G/M_G which is contained in M/M_G. So $SM = SM_G M = KM = MK = MM_G S = MS$. This shows (i).

Proof of Corollary A. – Finite soluble groups H are supersoluble if and only if all subnormal subgroups permute with all maximal subgroups of H, by Theorem A. Now by a theorem of Baer (see Satz 3.2 of [3]) polycyclic groups G are supersoluble if and only if all finite quotient groups are. This proves Corollary A.

Remark 1. – We now consider three classes of groups which are very much related to Theorems A and B. A subgroup H of G is said to be *Sylow* or *S-permutable*, if $HP = PH$ for all Sylow subgroups P of G. Kegel [11] proved that S-permutable subgroups of G are subnormal. Therefore, S-permutability is transitive in G if and only if every subnormal subgroup of G is S-permutable. Groups of this type are called PST-groups (see [1, 2, 5, 16,]). Two related classes of groups are the classes of PT- and T-groups. A group G is called a PT (resp. T)-group provided that every subnormal subgroup of G is permutable (resp. normal) (see [4, 5, 7, 13, 14, 15, 16, 17]).

Next we consider certain localizations of PST, PT and T-groups. For the prime p we introduce the classes N_p, M_p and L_p as follows: N_p is the class of all groups G such that if N is a normal subgroup of G, then every subgroup of $O_p(G/N)$ permutes with all the Sylow subgroups of G/N;

M_p is the class of all groups G such that if N is a normal subgroup of G, then every subgroup of $O_p(G/N)$ is a permutable subgroup of G/N;

L_p is the class of all finite groups G such that if N is a normal subgroup of G, then every subgroup of $O_p(G/N)$ is a normal subgroup of G/N.

From Theorem A of [5] and the proofs of Theorems 3.1 and 3.2 of Robinson [16] the following two theorems can be proven.

Theorem 1. – *Let G be a finite group and let D be the limit of the derived series of G. Then*

(i) *G is a PST-group if and only if G is an N_p-group for all primes p and $D/Z(D)$ is a direct product of G-invariant simple groups;*

(ii) *G is a PT-group if and only if G is a PST-group and an M_p-group for all primes p;*

(iii) *G is a T-group if and only if G is a PST-group and an L_p-group for all primes p.*

THEOREM 2. – *Let G be a finite group. Then G is a PST-group if and only if its non-abelian chief factors are simple and it is an N_p-group for all primes p.*

As a consequence of Theorem A and part (i) of Theorem 1 we obtain

COROLLARY 1. – *Let G be a finite PST-group. Then all the subnormal subgroups of G permute with all the maximal subgroups of G.*

3. – Additional permutability: Sylow subgroups.

PROOF OF THEOREM B. – Assume that statement (i) holds for the group G. Then all chief factors of $G/\Phi(G)$ are simple groups by Theorem A. Choose some chief factor H/K of G where $K \subseteq \Phi(G)$. The subgroups S satisfying $K \subseteq S \subseteq H$ are subnormal subgroups of G. If H/K is of odd order, the permutability of all these subgroups S with all Sylow-2-subgroups of G yields that all S are normalized by all Sylow-2-subgroups of G and especially by the Sylow 2-subgroups of the maximal perfect subgroup D of G. So every element of a Sylow 2-subgroup of D, and every element of D itself, induces a power automorphism in H/K. So $H/K \subseteq Z(D/K)$ if H/K is of odd order. If, on the other hand, H/K is of order a power of 2, we have $D = D_1 D_2$ where D_1 is the maximal perfect subgroup of D that is generated by its Sylow-3-subgroups and D_2 is the maximal perfect subgroup of D that is generated by its Sylow 5-subgroups. In analogy to the argument in the beginning we find that the elements of all Sylow 3-subgroups and of all Sylow 5-subgroups of G/K and consequently the elements of all Sylow 3-subgroups of $D_1 K/K$ and of all Sylow 5-subgroups of $D_2 K/K$ induce power automorphisms in H/K. By perfectness of $D_1 K/K$ and of $D_2 K/K$ we have $(D_1 K/K)(D_2 K/K) \subseteq C(H/K)$ and $H/K \subseteq Z(D/K)$. We deduce that $D \cap \Phi(G)$ is contained in the hypercenter of D, and since D is perfect, we have that the hypercenter and the center of D coincide; so $D/Z(D)$ is a direct product of simple normal subgroups of $G/Z(D)$.

For the remaining statement we have to use the detailed information given by the Atlas [6]. Consider now a minimal perfect normal subgroup P of G. By the preceding we know that $Z(P)$ is the only maximal normal subgroup of P, in other words, $Z(P)$ is a quotient group of the Schur multiplier of $P/Z(P)$. Let H/K be a chief factor of G where $H \subseteq Z(P)$. Considering Table 1 (page viii) of the Atlas [6] for the sporadic groups and Table 5 (page xvi) for the Chevalley groups and finally Gorenstein [9; p. 302], we find that elementary abelian quotients of the Schur multiplier are cyclic or isomorphic to $C_2 \times C_2$ or $C_3 \times C_3$. In the first noncyclic case the permutability property with Sylow 3-subgroups yields that a chief factor of G of this form does not appear, likewise the permutability property with Sylow 2-subgroups leads to the fact that chief factors isomorphic to $C_3 \times C_3$ do not appear. So all G-chief factors H/K with $H \subseteq Z(P)$ are cyclic. Since the maximal perfect subgroup D of G is the product of all minimal perfect normal subgroups P, we obtain as well that all G-chief factors H/K with $H \subseteq Z(D)$ are cyclic. We have shown (iia).

If N is some normal subgroup of G and F/N is the Fitting subgroup of G/N, then every subgroup S satisfying $N \subseteq S \subseteq F$ is subnormal in G, and the permutability with the Sylow p-subgroups, $p < 7$, of G entails the permutability of S/N with the Sylow p-subgroups of G/N. This shows (iib).

Assume now that (iia) and (iib) hold for a group G. Then all subnormal subgroups of G permute with all maximal subgroups of G by Theorem A. It remains to show that all Sylow p-subgroups of G permute with all subnormal subgroups of G, $p < 7$. Assume now that this is false and that G is a counterexample of smallest order. Consider a subnormal subgroup S, again of smallest order, which is not permutable with a given Sylow p-subgroup W of G. By minimality of G we know that S does not contain a proper normal subgroup of G, in particular, there is no subnormal perfect subgroup in S, since such a subgroup would be normal in G by (iia) and Theorem A. Therefore S is soluble and even supersoluble. Also, by the minimality of S, there is only one maximal normal subgroup of S. So S/S' is a cyclic r-group for some prime r and S' is a supersoluble group of order prime to r.

We will show first that $r \neq p$: We know that $W \cap S$ is a Sylow p-subgroup of S. If $r = p$, we obtain $S = S'(W \cap S)$ and $WS = WS'(W \cap S) = S'W(W \cap S) = S'(W \cap S)W = SW$, the desired contradiction.

We may assume now $r \neq p$. Let $T = (S')^G$ and $U = S^G$. Then T is an r'-group and U/T is an r-group, so U/T is contained in the Fitting subgroup of G/T. By (iib), the subgroup ST/T of G/T is permutable with WT/T, and since $p \neq r$ this means that W normalizes ST. But also S is characteristic in ST, it is the smallest normal subgroup of ST whose index in ST is prime to r. We deduce that W normalizes S, the final contradiction. So (i) follows from (iia), (iib); the proof is complete.

We now describe the groups of Theorem B in more detail.

PROPOSITION C. – *Let G be a finite soluble group. Then all the subnormal subgroups of G permute with all the maximal subgroups of G and all the Sylow p-subgroups for $p < 7$ if and only if the set of primes π dividing the order of G is partitioned into three (not necessarily non-empty) subsets Σ_1, Σ_2 and Σ_3 with the following properties:*

(i) *G is supersoluble,*

(ii) $\{2\} \cap \pi \subseteq \Sigma_1 \subseteq \{2, 3, 5\}$,

(iii) $\{2, 3, 5\} \cap \Sigma_3$ *is empty,*

(iv) *If* H_i *is a* Σ_i*-Hall subgroup of* G*, then*

(iva) H_2 *is a normal abelian subgroup of* G *and* H_1 *induces power automorphisms by conjugation in* H_2,

(ivb) H_1 *is nilpotent and* $[H_1, H_3] = 1$,

(ivc) $H_2 H_1$ *is a normal PST-subgroup of* G.

PROOF. – Assume that the subnormal subgroups of G permute with all the maximal subgroups of G and the Sylow p-subgroups of G, where $p < 7$. By Theorem A G is supersoluble. Let $K = V^G$ be the normal hull of some $\{2, 3, 5\}$-Hall subgroup V of the supersoluble group G. Since also K is supersoluble, it is 2-nilpotent. Denote the nilpotent residual of K by L. By definition of K we have that K/L is a $\{2, 3, 5\}$-group. Consider a prime $p > 5$ dividing the order of G and denote by S some Sylow p-subgroup of G. There is a normal subgroup T (the Hall subgroup for the primes $q > p$) of G such that $S \cap T = 1$ and ST is normal in G; all subgroup U satisfying $T \subseteq U \subseteq ST$ are subnormal in G and are normalized by V, so V induces power automorphisms in ST/T. We have two possibilities: either $[ST, V]T = T$ or $[ST, V]T = ST$, in the second case we have that $ST/T \cong S$ is abelian and $S \subseteq L$; these primes are defined to belong to Σ_2. The other primes $p > 5$ are defined to belong to Σ_3; also 3 or 5 belong to Σ_2 if and only if K/L is of order prime to 3 or 5 respectively. Now $L = H_2$, H_1 is nilpotent and $[H_1, H_3] = 1$. $L = H_2$ is nilpotent as nilpotent residual of the supersoluble group K and abelian since its Sylow subgroups are abelian. That $H_2 H_1$ is a soluble PST-group follows by Theorem 1 of [1].

Conversely, assume that G satisfies (i)-(iv). By (i) and Theorem A the subnormal subgroups of G permute with all the maximal subgroup of G.

Assume that G is a finite soluble group of minimal order satisfying conditions (i)-(iv) but G has a subnormal subgroup T which does not permute with some Sylow p-subgroup, where $p < 7$.

Let P be a Sylow p-subgroup of G where $p \in \{2, 3, 5\}$. By (ivc) P is a subgroup of $H_2 H_1$. Assume that T is a subgroup of H_3. By (iva) and (ivb) H_3 normalizes each Sylow subgroup of $H_2 H_1$ and hence $PT = TP$. Thus we can assume that T is not a subgroup of H_3. Since T is a subnormal subgroup of G, it follows that $T = (T \cap H_2 H_1)(T \cap H_3)$ and $L = H_2 H_1 (T \cap H_3)$ is a proper subgroup of G which contains T. Notice that L satisfies the conditions (i)-(iv). By the minimal choice of G, T permutes with P. This contradiction completes the proof.

As a consequence of Proposition C we obtain

COROLLARY C. – *Let* H *be a subgroup of the finite soluble group* G. *If all the subnormal subgroups of* G *permute with the maximal subgroups* *of G and the Sylow p-subgroups of G, where* $p < 7$, *then the same is true for H.*

Let G be a finite group and let p be a prime. G is said to satisfy condition K_p if N is a normal subgroup of G, then every subgroup of $O_p(G/N)$ permutes with all the Sylow q-subgroups of G/N, where $q < 7$. By Theorem A and the proof of Proposition C we obtain

PROPOSITION D. – *Let G be a finite soluble group. Then every subnormal subgroup of G permutes with all the maximal subgroups of G and all the Sylow q-subgroups, where* $q < 7$, *if and only if*

(i) *G is supersoluble,*

(ii) *G satisfies condition* K_p *for all primes p.*

For finite not necessarily soluble groups the situation is more complicated. We can say the following.

PROPOSITION E. – *Assume that all subnormal subgroups of the finite group G are permutable with all maximal subgroups and with all Sylow p-subgroups for primes* $p < 7$. *Denote the maximal perfect subgroup of G by D. Let* Σ_1 *and* Σ_3 *be the sets of primes defined in Proposition C, here used for G/D.In addition, denote by J the normal subgroup of G which is minimal with respect to having index prime to* 30.

Then the following is true:

(i) *If r is a prime dividing* $|G'/D|$, *r-chief factors of G are isomorphic as J-modules.*

(ii) *The Sylow* 2*-subgroup of* $Z(D)$ *belongs to* $Z_\infty(G)$, *the hypercenter of G.*

(iii) *If* $p \in \Sigma_3$ *and p divides* $|G'/D|$, *then the Sylow p-subgroup of* $Z(D)$ *is centralized by J.*

PROOF. – We obtain at once that two r-chief factors H_1/K_1 and H_2/K_2 are isomorphic as J-modules if either $H_1, H_2 \subseteq Z(D)$ or $D \subseteq K_1, K_2$ since they are isomorphic as S-modules where S is any Sylow p-subgroup ($p < 7$). Assume now that there is an r-chief factor H_1/K_1 with $H_1 \subseteq Z(D)$ and H_2/K_2 with $D \subseteq K_2 \subseteq H_2 \subseteq G'$. Among the normal subgroups N of G satisfying the relation $H_1 \cap N = K_1$ we choose a maximal one and call it M. We distinguish two cases: $|MD/D|$ is divisible by r or prime to r.

If $|MD/D|$ is divisible by r, there is an r-chief factor H_2/K_2 such that $M \cap D \subseteq K_2 \subseteq H_2 \subseteq M$. Now $H_1 H_2/K_1 K_2 \cong (H_1/K_1) \times (H_2/K_2)$, and every element of a Sylow p-subgroup ($p < 7$) induces a power automorphism in this quotient. So also H_1/K_1 and H_2/K_2 are isomorphic as S-modules and as J-modules.

If $|MD/D|$ is prime to r, then $|G/MD|$ is divisible by r. We have the r-chief factor $H_1/K_1 \cong H_1 M/K_1 M$, and we may consider the situation in G/M. Here $H_1 M/M$ is a minimal normal subgroup; if the minimal perfect normal subgroup E_i of G is not contained in M, then $E_i M/M$ is a minimal perfect normal subgroup of G/M. We consider all minimal perfect normal subgroups E_i of G that are not included in M. Then DM/M is the product of their images $E_i M/M$, also $C_{G/M}(DM/M) = Z(D)\, M/M = \cap C_{G/M}(E_i M/M)$. So there is at least one $E_j M/M$ such that $|(G'M/M):(E_j M/M) C_{G'M/M}(E_j M/M)|$ is divisible by r. We deduce that $|Z(E_j)|$ and $|G'/E_j C_{G'}(E_j)|$ are divisible by r, so r is a divisor of the orders of the Schur multiplier of $E_j/Z(E_j)$ and of its outer automorphism group. We consult the list of finite simple groups to find that $E_j/Z(E_j)$ can not be a sporadic group neither can it be an alternating group (for 2-chief factors nothing is to be shown.) We are left with the Chevalley groups and the twisted Chevalley groups, Consulting Table 2 and Table 5 of [6] we see that we have to consider only

$$A_n(q) = L_{n+1}(q) \text{ if } r \text{ divides } \gcd\,(n+1,\, q-1),$$

$$^2A_n(q) = U_{n+1}(q) \text{ if } r \text{ divides } \gcd\,(n+1,\, q+1),$$

$$E_6(q) \text{ if } r = 3 \text{ and } 3 \text{ divides } q-1,$$

$$^2E_6(q) \text{ if } r = 3 \text{ and } 3 \text{ divides } q+1.$$

In all of these cases we obtain the Sylow r-subgroup of $Z(E_j)$ as a subgroup of the multiplicative group of some field and the r-subgroup of $(Out(E_j/Z(E_j)))'$ as a subgroup of a normal subgroup, which can be considered isomorphic to a subgroup of the multiplicative group of the same field. On both r-groups will operate the same field automorphisms, so in fact the r-chief factors belonging to $G'C(E_j)/C(E_j)E_j$ and to $Z(E_j)$ are operator isomorphic, and they are isomorphic as J-modules. This proves (i).

Since all 2-chief factors of G are central, statement (ii) is obvious.

If $p \in \Sigma_3$, then all p-chief factors of G'/D and of G/D are trivial J-modules, and the same applies for the p-chief factors of G contained in $Z(D)$ by (i). Since J is generated by elements of order dividing 30 and since p does not divide 30, we find that the Sylow p-subgroup W of $Z(D)$ is not only in the hypercenter but already in the center of $JZ(D)$. This shows (iii) and completes the proof of Proposition E.

REMARK 2. – There is no general bound for the central height of the elements mentioned in Proposition E: If r is a prime such that the prime $p \in \{2, 3, 5\}$ divides $r-1$ and m is some power of p, then $L_m(r)$ has a Schur multiplier of order divisible by $p^k \geqslant mp$ and $Out(L_m(r))$ possesses a subgroup which is an extension of a cyclic group of order p^k by a cyclic group of order m.

This leads to a group H with maximal perfect subgroup K such that the Sylow p-subgroup of Z(K) is contained in $Z_s(H)$, but not in $Z_{s-1}(H)$, where $m = p^s$.

REMARK 3. – Notice that the Sylow theorems are no longer true in polycyclic groups; for instance, the maximal p-subgroups of $C\mathrm{wr}C_p$ are no longer pronormal, where C is an infinite cyclic group. A generalization of the results of this section to certain classes of infinite groups seems therefore to be difficult.

REMARK 4. – Corollary C only holds for soluble groups. For let $G = A_{12}$. Then $A_6\mathrm{wr}C_2$ is a subgroup of G and fails to satisfy (iia) of Theorem B.

4. – Examples.

EXAMPLE 1. – There are finite soluble groups which satisfy the condition of Proposition C but are not PST-groups. Let $M = \langle a, b \mid a^{29} = b^{29} = 1$ and $[a, b] = 1\rangle$, and $N = \langle c, d \mid c^2 = d^7 = 1$ and $[c, d] = 1\rangle$. Let N act on M as follows. $a^c = a^{-1}$, $b^c = b^{-1}$, $a^d = a^{-4}$ and $b^d = b^{25}$. Let G be the semidirect product $M \rtimes N$ with respect to this action. Let $\Sigma_1 = \{2\}$, $\Sigma_2 = \{29\}$ and $\Sigma_3 = \{7\}$. In Proposition C it follows that $H_2 = M$, $H_1 = \langle c\rangle$ and $H_3 = \langle d\rangle$. Note that G is not a PST-group since the elements of $\langle d\rangle$ do not act as power automorphisms on M (see Theorem 1 of [1]).

EXAMPLE 2. – Let $H = G \times A_5$ where G is the group of Example 1. Then H satisfies the conditions of Theorem B but is not a PST-group since G is not a PST-group.

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