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Existence of Multiple Principal Eigenvalues for some Indefinite Linear Eigenvalue Problems.

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Sunto. – *Si studia l'esistenza di autovalori principali per operatori differenziali del secondo ordine non necessariamente in forma di divergenza. Otteniamo risultati sulla molteplicità degli autovalori principali, sia nel caso variazionale che per operatori in forma generale. Si utilizza sistematicamente il teorema di Krein-Rutman e poi un argomento di punto unito per il raggio spettrale di alcuni problemi ausiliari. La caratterizzazione variazionale è usata nel caso auto-aggiunto e anche in quello generale.*

Summary. – *We study the existence of principal eigenvalues for differential operators of second order which are not necessarily in divergence form. We obtain results concerning multiplicity of principal eigenvalues in both the variational and the general case. Our approach uses systematically the Krein-Rutman theorem and fixed point arguments for the spectral radius of some associated problems. We also use a variational characterization for both the self-adjoint and the general case.*

Introduction.

The eigenvalues of linear differential operators and their properties play an important role as well in the development of the corresponding linear theory as for their applications since nonlinear operators can be locally approximated by their derivatives in some function spaces. In the case of ordinary differential equations, Sturm-Liouville theory gives a complete and satisfying description of the (countable) set of the eigenvalues and of the nodal sets of the associated eigenfunctions; it shows that each eigenvalue is simple and its dependence with respect to the coefficients of the operator and on the interval of definition.

In the case of domains of dimension greater than one, there is no completely satisfying theory on nodal lines of eigenfunctions. In fact, there are difficult open problems there. However, some properties of the previous case are still valid. In particular, if Ω is a regular bounded domain in $\mathbb{R}^N$, with boundary $\partial\Omega$, the eigenvalue problem:

$$-\Delta u = \lambda u \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega,$$

possesses an infinite sequence of positive eigenvalues:

$$0 < \lambda_1 < \lambda_2 \leqslant \dots \lambda_k \leqslant \dots; \quad \lambda_k \to \infty, \text{ as } k \to \infty$$

with finite multiplicity. Moreover λ_1 is simple and its associate eigenfunction φ_1 is positive in Ω and $\partial\varphi_1/\partial n < 0$ on the boundary. The same properties hold for the following problem:

$$-\Delta u + a_0(x)u = \lambda m(x)\, u \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

if a_0 and m are positive on $\overline{\Omega}$ and smooth enough. In the following we denote by $\lambda_1 = \lambda_1(-\Delta + a_0, m, \Omega)$ the first eigenvalue of the above problem.

The same results are valid for general second order linear operators in divergence form with regular coefficients with suitable sign. Existence of eigenvalues with positive eigenfunction is important from the viewpoint of nonlinear problems where positive solutions are interesting; this is the case for many reaction-diffusion systems in population dynamics, chemical reactions, combustion, ... (see [Sm]). These eigenvalues are called principal eigenvalues.

The classical reference for this theory is the book of Courant and Hilbert ([CH]) where the theory is developed for continuous coefficients and also valid for bounded coefficients. The variational characterization of the eigenvalues leads to the continuous and monotone dependence with respect to the coefficients, and also with respect to the domain Ω, of the spectrum. The main tool here is the abstract theory of linear compact self-adjoint operators in Hilbert spaces.

This theory can be extended to the case of unbounded coefficients and also when a_0 changes sign in Ω. If $a_0 > 0$ in Ω, (or more generally if $\lambda_1(-\Delta + a_0, m, \Omega) > 0$) with $m, a_0 \in L^r(\Omega)$, $r > N/2$ and $m > 0$ (resp. $m < 0$) on a subdomain of positive measure, then there is exactly one principal eigenvalue $\lambda_1^+ > 0$ (resp. $\lambda_1^- < 0$), with positive eigenfunction. This result is established in [MM] (see also [dF], [BL] and [W]).

This approach does not work any more for general second-order differential operators L which are not in divergence form. The celebrated Krein-Rutman Theorem [KR] becomes instrumental there in order to exhibit a principal eigenvalue, which is actually simple (see [Am] [H] [Sc] [Sm] [T1]). The preceeding result was extended in this framework by Hess and Kato [HK] for $m \in \mathcal{C}(\overline{\Omega})$. Some of these ideas can also be extended to periodic-parabolic problems, see the book by Hess [H] and the references therein.

The above results concern the case of operators $-\Delta + a_0$ (or L) having a positive principal eigenvalue. The case $\lambda_1(-\Delta + a_0, m, \Omega) \leqslant 0$ is considerably more involved, some results in this direction can be found in [Al] [AS] [FM]. It is very easy to find examples of problems having two principal eigenvalues with the same sign (see Remark 1.9). Recently some non trivial examples were given in [LG1] [LG2] for operators in general form satisfying rather strong regularity assumptions. On the other hand, it was proved in [BNV], among many other interesting results, the existence of a principal eigenvalue in the case of a general domain Ω, with no smoothness property (see also [Bi]; [Ch] [BMS] for the extension to systems), for a_0 and m positive and bounded. Moreover, a variational characterization of the principal eigenvalue is given. That there are at most two principal eigenvalues follows from the concavity of the spectral radius corresponding to the operator $-\Delta + a_0(x) - \lambda m(x)$ (see [K], [H], [LG1] and [LG2]).

We only deal here with the case of one linear equation defined on a bounded domain, but the same approach can be extended to more general situations including unbounded domains, systems (even if they are not of potential type) and also to some quasilinear equations involving the so-called p-Laplacian (see, e.g., [An]). Indeed, the main properties of the eigenvalue $r(\lambda)$ arising in our fixed point arguments follow from the corresponding variational characterization, something which is still available even for operators which are not in divergence form in the linear case (even for systems, see [BMS]), but also for the nonlinear p-Laplacian. Some of the results in this paper can be extended to this situation and an interesting nonlinear version of the Krein-Rutman theorem due to Takac [T2] can be used. This will be studied elsewhere. Another possible extension is to linear problems in general form with singular coefficients (see [HMV] and its references). Different linear boundary conditions, for example third type boundary conditions, for which a nice existence and uniqueness theory is available (as well as the maximum principle) can be treated by similar arguments.

In this paper, we extend most of the above quoted results for changing sign (indefinite) coefficients in a much more unified way. The general strategy consists in writing an equivalent form of the problem in such a way that all coefficients are now positive and use the Krein-Rutman Theorem. Here it is very convenient to replace the usual version in cones with non-empty interior by a more general one given in [DKM], Theorem 12.3 (see also [BMS], [D1], [D2], [dP], [Sc] [Z]), in terms of quasi-interior points, which allows us to work in the spaces $L^p(\Omega)$ $(1 < p < \infty)$ whose cones have empty interior. Then the proof of existence of principal eigenvalues is reduced to a fixed point problem for the associated spectral radius $r(\lambda)$, whose properties follow from the classical variational characterization of the corresponding eigenvalue in the variational case, but also from the one in [BNV] for general operators.

However, in order to exhibit the power and flexibility of the approach to deal with more general situations (elliptic systems, quasi-linear operators like the p-Laplacian, ...), we start in Section 1 with the variational case for $a_0 \equiv 0$ and m bounded and indefinite. Here we improve slightly the results in [MM] and [dF] in the sense that we show that the eigenvalues are the only ones with a positive eigenfunction, a result which is not included in [MM].

In Section 2, we allow m unbounded and $\lambda_1(-\Delta + a_0, 1, \Omega) \leqslant 0$. We thus consider three different cases. Case (I), $\lambda_1(-\Delta + a_0, 1, \Omega) > 0$, is very similar to the case $a_0 \geqslant 0$. Case (II), $\lambda_1(-\Delta + a_0, 1, \Omega) = 0$, raises some new features, but in any case we have (at most) one positive principal eigenvalue. The most intriguing situation, Case (III), corresponds to $\lambda_1(-\Delta + a_0, 1, \Omega) < 0$. Here we need a very interesting result, by Dancer ([D3]) where the smoothness of a subdomain of Ω plays an important role. Then, by introducing a parameter t in the weight m, we are able to provide a careful description of the continuous transition from two principal eigenvalues to no one by crossing a «critical» value of t with exactly one principal eigenvalue. This gives a much more general (less smoothness on both coefficients and domains) and systematic view than in [LG1], [LG2].

In Section 3, we consider finally the case of an operator in general form L and we show how to extend the results in [BNV] and [Bi] to the indefinite case. In the same vein, we obtain an extension of most of the results in [HK] under weaker assumptions (see [HKS] for some related work). However, since the result by Dancer stated in Proposition 2.3 is any more available for operators in general form and for non-smooth domains, we cannot extend Theorem 2.5 to this broader context. But all results which do not rely on it can still be proved by completely similar arguments.

The main results have been announced in [FHT].

1. – The variational case for bounded indefinite weights.

In this section we consider the linear model case:

$$-\Delta u = \lambda m(x) u, \qquad x \in \Omega; \tag{1.1}$$

$$u(x) = 0, \qquad x \in \partial\Omega, \tag{1.2}$$

where Ω is a smooth bounded domain in $\boldsymbol{R}^N$, $N \geqslant 2$ and

$$m \in L^\infty(\Omega). \tag{1.3}$$

We consider the case where the weight m changes sign, which is usually referred as an «indefinite weight». We define the following subsets:

$$\Omega^+ = \{x \in \Omega \,|\, m(x) > 0\}, \tag{1.4a}$$

$$\Omega^- = \{x \in \Omega \,|\, m(x) < 0\}, \tag{1.4b}$$

$$\Omega_0 = \{x \in \Omega \,|\, m(x) = 0\}. \tag{1.4c}$$

Without other statement, we assume that

$$|\Omega^+| > 0, \; |\Omega^-| > 0,$$

where $|A|$ is the Lebesgue measure of the set A.

REMARK 1.1. – The smoothness properties of the bounded domain Ω and its subdomains will play an important role in all which follows. We assume that all subdomains considered here are such that the comparison results used below are applicable to them and that Ω^+ and Ω^- have positive measure.

We can change (1.1) in

$$-\Delta u + \lambda m^-(x)\, u = \lambda m^+(x)\, u, \qquad x \in \Omega, \tag{1.1'}$$

where $m^+ = \max(m, 0)$; $m = m^+ - m^-$. More conveniently, for given λ, we rewrite (1.1′) as an eigenvalue problem with parameter σ:

$$-\Delta u + \lambda\big(m^-(x) + \chi_{\Omega_0 \cup \Omega^-}\big)\, u = \sigma\big(m^+(x) + \chi_{\Omega_0 \cup \Omega^-}\big)\, u, \qquad x \in \Omega; \tag{1.5}$$

$$u(x) = 0, \qquad x \in \partial\Omega.$$

We notice that in (1.5) the functions $m^{\pm}(x) + \chi_{\Omega_0 \cup \Omega^-}$ are both non negative and bounded; here χ_C denotes the characteristic function of the set C. We will use the following classical result

LEMMA 1.1. – *Let $a \in L^\infty(\Omega)$, $a \geqslant 0$ in Ω; for all $h \in L^p(\Omega)$, $1 < p < \infty$, there exists a unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ to the problem:*

$$-\Delta u + a(x)\, u = h, \; x \in \Omega, \qquad u(x) = 0, \; x \in \partial\Omega.$$

Moreover, if $h \geqslant 0$ then $u \geqslant 0$; if $h \geqslant 0$, $h \not\equiv 0$ then $u > 0$. The «solution operator» S defined by $u = S(h)$ is linear and compact in L^p for all $1 < p < \infty$.

PROOF. – The first part follows immediately from the classical L^p theory (see [ADN], [GT]) and the maximum principle for weak solutions ([GT], [Tr]). The compactness of S is an immediate consequence of the L^p estimates ([ADN]) and the Sobolev imbedding theorem. ■

We define for any $\widehat{m}$ satisfying (1.3) the multiplication operator $M : L^p(\Omega) \to L^p(\Omega)$ in the usual manner: $Mu(x) = \widehat{m}(x)\, u(x)$. It is easy to see that M is continuous and bounded in the sense that $M(A)$ is bounded for bounded A. We deduce from here that $T = SM$, where $S = (-\Delta + \lambda(m^-(x) + \chi_{\Omega_0 \cup \Omega^-})\, I)^{-1}$ and $\widehat{m}(x) = m^+(x) + \chi_{\Omega_0 \cup \Omega^-}$ is the solution operator defined above, is compact in L^p for any fixed $1 < p < \infty$. It follows from Lemma 1.1 and from the remark on the positivity of $m^+(x) + \chi_{\Omega_0 \cup \Omega^-}$ that the operator T is strongly positive in the sense of quasi-interior points in $L^p(\Omega)$ ([DKM], [Z]), and not in the classical sense of interior points as in ([Am]). In particular, this implies that $(\mu I - T)^{-1}$ is strongly positive (again in the sense of ([DKM])), for $\mu > r(T)$, and this implies that T is irreducible. Here $r(T)$ denotes as usual the spectral radius of T. We can apply the Krein-Rutman theorem, more precisely Theorem 12.3 from ([DKM]) with hypothesis i), and with Banach lattice $L^p(\Omega)$ (see also [dP]). If we set $\sigma(\lambda)$ in place of $r(T)$, we deduce that $\sigma(\lambda) > 0$ is an eigenvalue of T which is algebraically simple; its associated eigenfunction φ_1 is a quasi-interior point in $L^p(\Omega)$, that is $\varphi_1 > 0$, $a.\ e.$ on Ω. Moreover $\sigma(\lambda)$ is the only eigenvalue associated with positive eigenfunction; hence $\sigma(\lambda)$ is an eigenvalue of (1.5) and we have:

$$(1.6) \qquad -\Delta\varphi_1 + \lambda(m^-(x) + \chi_{\Omega_0 \cup \Omega^-})\,\varphi_1 = \sigma(\lambda)(m^+(x) + \chi_{\Omega_0\cup\Omega^-})\,\varphi_1, \quad x \in \Omega;$$

$$\varphi_1(x) = 0, \qquad x \in \partial\Omega.$$

REMARK 1.2. – By use of regularity results in $L^p(\Omega)$ ([ADN]) and Morrey's lemma, we deduce that φ_1 is in $W^{2,p}(\Omega)$ for all $1 < p < \infty$ and then in $\mathcal{C}^{1,\alpha}(\overline{\Omega})$ for all $0 < \alpha < 1$. By the strong maximum principle for weak solutions ([Tr]), we also have $\frac{\partial\varphi_1}{\partial n} < 0$ on $\partial\Omega$.

We consider now the properties of $\sigma(\lambda)$ as a function of $\lambda \geqslant 0$ and with respect to the domain Ω. These properties follow in our case from the variational caracterization of $\sigma(\lambda)$ below but they are still valid in more general cases (operators in general form, systems,...) as we will see later. Since the coefficients $m^{\pm}(x) + \chi_{\Omega_0\cup\Omega^-}$ are both non negative and bounded, we have:

$$(1.7) \qquad \sigma(\lambda) = \inf_{\phi \in H_0^1(\Omega);\ \phi \not\equiv 0} \frac{\int_\Omega |\nabla\phi|^2 + \lambda\int_\Omega (m^-(x) + \chi_{\Omega_0\cup\Omega^-})\,\phi^2}{\int_\Omega (m^+(x) + \chi_{\Omega_0\cup\Omega^-})\,\phi^2}.$$

Indeed, it is well-known that, if we call for a moment λ_1^* the right-hand side in (1.7), $\lambda_1^* > 0$, is an eigenvalue to (1.6) having a positive eigenfunction. Since by the above argument $\sigma(\lambda)$ is the only eigenvalue with this property, and taking into account that the eigenvalues are the same in $H_0^1(\Omega)$ and in $L^p(\Omega)$ for $1 < p < \infty$, we have $\sigma(\lambda) = \lambda_1^*$ and (1.7) holds.

We know in particular that $\sigma(\lambda)$ is increasing and continuous with respect to λ; (indeed it is piecewise analytic ([K])); moreover it depends monotonically on the domain Ω.

From now on let us denote by $\lambda_1(-\Delta + q, g, D)$ the principal eigenvalue of the operator $-\Delta + q$ with weight g defined on the domain D with Dirichlet boundary conditions; here D is smooth and $q \geqslant 0$ in D and $g > 0$ a.e. in D. With this notation, $\sigma(\lambda) = \lambda_1(-\Delta + \lambda(m^- + \chi_{\Omega_0\cup\Omega^-}), m^+ + \chi_{\Omega_0\cup\Omega^-}, \Omega)$ and $\sigma(0) = \lambda_1(-\Delta, m^+ + \chi_{\Omega_0\cup\Omega^-}, \Omega)$.

REMARK 1.3. – The classical reference for these properties of the eigenvalues is ([CH]). This book deals with smooth coefficients but the properties for bounded coefficients can be derived from them. The dependence with respect to the domain for Dirichlet boundary conditions is also in ([CH]); of course this continuity with respect to the domain is not valid any more for Neumann boundary conditions. It is still possible to get a variational characterization of the eigenvalues when the weight changes sign (see e.g. [MM], [dF], [W]).

We deduce from the monotonicity with respect to the domain that

$$(1.8)\qquad \lambda_1(-\Delta+\lambda(m^-+\chi_{\Omega_0\cup\Omega^-}), m^++\chi_{\Omega_0\cup\Omega^-}, \Omega)<$$

$$\lambda_1(-\Delta+\lambda(m^-+\chi_{\Omega_0\cup\Omega^-}), m^++\chi_{\Omega_0\cup\Omega^-}, \Omega^+)=$$

$$\lambda_1(-\Delta, m^+, \Omega^+),$$

since $(m^-+\chi_{\Omega_0\cup\Omega^-})(x)=0$, $(m^++\chi_{\Omega_0\cup\Omega^-})(x)=m^+(x)$ for $x\in\Omega^+$. Notice that this estimate is uniform in λ.

It is clear that $\lambda>0$ is an eigenvalue of (1.1) if and only if $\sigma(\lambda)=\lambda$. If the function $\lambda\to\sigma(\lambda)$ is continuous, increasing and since (1.8) holds, it is possible to show that there is at least one such a λ if $\sigma(0)=\lambda_1(-\Delta, m^++\chi_{\Omega_-\cup\Omega_0}, \Omega)>0$. The uniqueness is a consequence of the strict concavity of the function σ except for a finite number of points (see [BNV], [K], [LG2]). However we will follow a much simpler way. We do not use neither the concavity of σ, the much simpler argument which follows suffices.

First, it follows from $\sigma(0)>0$, (1.8) and the continuity of σ that there is at least a $\tilde\lambda<\lambda_1(-\Delta, m^+, \Omega^+)$ such that $\sigma(\tilde\lambda)=\tilde\lambda$. Let us see that it is unique.

Let $\lambda>\mu$ be two points on the corresponding interval. If we denote by φ_λ, $\varphi_\mu>0$ the associated (normalized) eigenfunctions then we have

$$-\Delta\varphi_\lambda+\lambda(m^-(x)+\chi_{\Omega_0\cup\Omega^-})\,\varphi_\lambda=\sigma(\lambda)(m^+(x)+\chi_{\Omega_0\cup\Omega^-})\,\varphi_\lambda,\ x\in\Omega;$$

$$-\Delta\varphi_\mu+\mu(m^-(x)+\chi_{\Omega_0\cup\Omega^-})\,\varphi_\mu=\sigma(\mu)(m^+(x)+\chi_{\Omega_0\cup\Omega^-})\,\varphi_\mu,\ x\in\Omega;$$

$$\varphi_\lambda=\varphi_\mu=0,\ x\in\partial\Omega.$$

From the variational characterization (1.7), it follows

$$\int_\Omega|\nabla\varphi_\mu|^2+\lambda\int_\Omega(m^-(x)+\chi_{\Omega_0\cup\Omega^-})\,\varphi_\mu^2>\sigma(\lambda)\int_\Omega(m^+(x)+\chi_{\Omega_0\cup\Omega^-})\,\varphi_\mu^2,$$

$$\int_\Omega|\nabla\varphi_\mu|^2+\mu\int_\Omega(m^-(x)+\chi_{\Omega_0\cup\Omega^-})\,\varphi_\mu^2=\sigma(\mu)\int_\Omega(m^+(x)+\chi_{\Omega_0\cup\Omega^-})\,\varphi_\mu^2,$$

and this yields

$$(\lambda-\mu)\int_\Omega (m^-(x)+\chi_{\Omega_0\cup\Omega^-})\,\varphi_\mu^2 > (\sigma(\lambda)-\sigma(\mu))\int_\Omega (m^+(x)+\chi_{\Omega_0\cup\Omega^-})\,\varphi_\mu^2.$$

Hence we obtain

$$\frac{\sigma(\lambda)-\sigma(\mu)}{\lambda-\mu} < \frac{\int_\Omega (m^-(x)+\chi_{\Omega_0\cup\Omega^-})\,\varphi_\mu^2}{\int_\Omega (m^+(x)+\chi_{\Omega_0\cup\Omega^-})\,\varphi_\mu^2}.$$

For $\mu=\tilde\lambda$, we get

$$\frac{\sigma(\lambda)-\tilde\lambda}{\lambda-\tilde\lambda} < \frac{\int_\Omega (m^-(x)+\chi_{\Omega_0\cup\Omega^-})\,\varphi_{\tilde\lambda}^2}{\int_\Omega (m^+(x)+\chi_{\Omega_0\cup\Omega^-})\varphi_{\tilde\lambda}^2} < 1,$$

where the last inequality is obtained by multiplying (1.6) for $\lambda=\tilde\lambda$ and $u=\varphi_{\tilde\lambda}$ by $\varphi_{\tilde\lambda}$ and integrating by parts. This shows that $\sigma(\lambda)<\lambda$ for $\lambda>\tilde\lambda$ and, in the same way, that $\sigma(\lambda)>\lambda$ for $\lambda<\tilde\lambda$, and this gives the uniqueness of $\tilde\lambda$.

Moreover $\tilde\lambda$, which is the unique principal eigenvalue of (1.1), is simple. This is shown by contradiction; if $\tilde\lambda=\sigma(\tilde\lambda)$ is not simple as an eigenvalue of (1.5), we get a contradiction. A similar argument shows that it is the only eigenvalue of (1.1) with associated positive eigenfunction.

REMARK 1.4. – An alternative simple proof of the uniqueness of the principal eigenvalue can be given if some more regularity for the eigenfunctions is available, namely if they are $\mathcal{C}^1$, something which happens if $r>N$. Assume to simplify the matter that

$$-\Delta u=\lambda u \text{ in } \Omega,\qquad u=0 \text{ on } \partial\Omega,$$

$$-\Delta v=\mu v \text{ in } \Omega,\qquad v=0 \text{ on } \partial\Omega,$$

with $0<\lambda<\mu$ and $u>0$ in Ω, $\dfrac{\partial u}{\partial n}<0$ on $\partial\Omega$, and the same for v. Then, using a device due to Brezis and Oswald and taking into account that $\dfrac{u}{v}$ is bounded, we have for any $k>0$, that

$$0\leqslant \int_\Omega\left[-\frac{\Delta v}{v}+\frac{\Delta ku}{ku}\right](v^2-k^2u^2)=(\mu-\lambda)\int_\Omega m(x)(v^2-k^2u^2)=$$

$$(\mu-\lambda)\left[\frac{1}{\mu}\int_\Omega |\nabla v|^2-\frac{k^2}{\lambda}\int_\Omega |\nabla u|^2\right]<0$$

for $k>0$ large enough, a contradiction. Then $\lambda=\mu$.

We derive immediately the results for $\lambda < 0$ by changing m^+ with m^- since $\lambda . m = (-\lambda)(-m)$. Therefore we have

THEOREM 1.1. – *Assume that m satisfies* (1.3) *and that* $|\Omega^+| > 0$, $|\Omega^-| > 0$. *Hence Problem* (1.1) *possesses a unique positive* (*resp. negative*) *eigenvalue* $\lambda_1^+(m)$ (*resp.* $\lambda_1^-(m)$); *this eigenvalue is such that*

$$0 < \lambda_1^+(m) < \lambda_1(-\Delta, m^+, \Omega^+), \quad (\text{resp. } 0 > \lambda_1^-(m) > \lambda_1(-\Delta, m^-, \Omega^-));$$

moreover λ_1^+ (*resp.* λ_1^-) *is algebraically simple and is the unique positive* (*resp. negative*) *eigenvalue associated with a positive eigenfuntion.*

REMARK 1.5. – Theorem 1.1 can be extended, with the same proof, to a second order differential operator in divergence form which is uniformly elliptic:

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)\, u, \tag{1.9}$$

with smooth and bounded coefficients a_{ij} and $a_0 \geq 0$.

REMARK 1.6. – We do not deal here with the case $m^- \equiv 0$, $|\Omega^+| > 0$ and $|\Omega_0| > 0$, Ω_0 smooth enough. The reason is that the treatment is completely analogous to the one below for Case (III), where it plays a relevant role. We do not include this case here in order to avoid repetitions.

REMARK 1.7. – The same approach can be used for different boundary conditions. In particular, we can obtain results for boundary conditions of the third type

$$\frac{\partial u}{\partial n} + a(x)\, u = 0 \text{ on } \partial\Omega$$

where $a(x) > 0$ is smooth. Moreover, by writing the eigenvalue problem as

$$\frac{\partial u}{\partial n} + a^+(x)\, u = a^-(x)\, u \text{ on } \partial\Omega,$$

we can study the problem when $a(x)$ changes sign and get similar results.

REMARK 1.8. – Theorem 1.1 shows that for the equation

$$-\Delta u + km(x)\, u = \sigma m(x)\, u, \tag{1.10}$$

the principal eigenvalues are $\sigma_1^+ = \lambda_1^+(m) + k$ (resp. $\sigma_1^- = \lambda_1^-(m) + k$) which can be of any sign. We shall meet these situations in the next sections.

2. – The variational case for unbounded indefinite coefficients.

We will now extend Theorem 1.1 in two directions: we will allow the indefinite weight m to be unbounded and we add a coefficient a_0 which is not necessarily positive. Let us consider the eigenvalue problem:

$$-\Delta u + a_0(x)\, u = \lambda m(x)\, u, \qquad x \in \Omega, \tag{2.1}$$

$$u(x) = 0, \qquad x \in \partial\Omega,$$

where Ω is as above and where

$$a_0,\, m \in L^r(\Omega), \qquad r > \frac{N}{2}. \tag{2.2}$$

As in Section 1, we can rewrite equation (2.1) as

$$-\Delta u + a_0^+(x)\, u + \lambda m^-(x)\, u = (\lambda m^+(x) + a_0^-(x))\, u, \qquad x \in \Omega,$$

and we are led to study the following eigenvalue problem

$$-\Delta u + (a_0^+(x) + 1)\, u + \lambda m^-(x)\, u = r(\lambda)\left(m^+(x) + \frac{a_0^-(x)+1}{\lambda}\right) u, \quad x \in \Omega, \tag{2.3}$$

$$u(x) = 0, \qquad x \in \partial\Omega,$$

Now we can replace Lemma 1.1 by another well-known result:

LEMMA 2.1. – *Let $a \geq 0$ satisfying (2.2). For all $h \in L^{\frac{2N}{N+2}}(\Omega)$, there exists a unique solution $u \in H_0^1(\Omega)$ to problem*

$$-\Delta u + a(x)\, u = h, \quad x \in \Omega; \qquad u(x) = 0, \quad x \in \partial\Omega.$$

Moreover $u \geq 0$ when $h \geq 0$, and $u > 0$ when $h \geq 0$; $h \not\equiv 0$. The solution operator $P : L^{\frac{2N}{N+2}}(\Omega) \to L^\theta(\Omega)$, defined by $u = P(h)$ is compact for all $1 < \theta < \frac{2N}{N-2}$.

PROOF. – Since $L^{\frac{2N}{N+2}}(\Omega)$ is continuously embedded in $H^{-1}(\Omega)$, the existence and uniqueness of the solution is a simple consequence of the variational theory and of Lax-Milgram Lemma, combined with (2.2), (see [dF], [S]). The positivity follows from the Maximum Principle by Stampacchia ([S]). To show the compactness, we write

$$-\Delta Pu + a(x)\, Pu = u, \quad x \in \Omega; \qquad Pu(x) = 0 \quad x \in \partial\Omega;$$

by multiplication by Pu and integration, we obtain

$$\int_\Omega |\nabla Pu|^2 + a(x)\,|Pu|^2 \leqslant \int_\Omega u\,.\,Pu\,.$$

By Hölder inequality applied to the right hand-side, we derive

$$\|Pu\|^2_{H^1_0(\Omega)} \leqslant \|u\|_{L^{\frac{2N}{N+2}}(\Omega)} \|Pu\|_{L^{\frac{2N}{N-2}}}(\Omega),$$

and then

$$\|Pu\|_{H^1_0(\Omega)} \leqslant C\|u\|_{L^{\frac{2N}{N+2}}(\Omega)};$$

combining this with the compactness of the embedding $H^1_0(\Omega) \hookrightarrow L^\theta(\Omega)$ we get the result. The continuity follows in the same way. ∎

The multiplication operator acts between different L^p spaces. We have:

LEMMA 2.2. – *Assume that* $s,\, t > 1$ *and that* m *satisfies* (2.2). *Then the multiplication operator* $M : L^s(\Omega) \to L^t(\Omega)$ *is well defined and continuous for* $t = \dfrac{rs}{r+s}$; *in particular, if* $s \geqslant \dfrac{2Nr}{N(r-2)+2r}$, *then* $t \geqslant \dfrac{2N}{N+2}$.

PROOF. – The first part is an immediate consequence of the value of t: $\dfrac{1}{r} + \dfrac{1}{s} = \dfrac{1}{t}$ and of Hölder inequality. To prove the second part, we write:

$$\frac{1}{r} + \frac{1}{s} = \frac{1}{t} \leqslant \frac{N(r-2)+2r}{2Nr} + \frac{1}{r} = \frac{N+2}{2N}\,.$$

We choose s satisfying

$$\frac{2Nr}{N(r-2)+2r} \leqslant s < \frac{2N}{N-2}\,, \tag{2.4}$$

which is precisely equivalent to (2.2). ∎

From now on we fix s satisfying this condition (2.4). From above we know that the operator $T = SM$, where S is the solution operator defined by $S = (-\Delta + a_0^+ + 1 + \lambda m^-)^{-1}$ and where M is the operator of multiplication by the function $M(x) = m^+(x) + \dfrac{1}{\lambda}(1 + a_0^-(x))$ is well defined; moreover it is compact from $L^s(\Omega)$ into $L^s(\Omega)$. By Lemmas 2.1 and 2.2, and since $M(x) > 0$ in Ω, S is strongly positive (always in the sense of quasi-interior points of $L^s(\Omega)$), and we can apply the same version of Krein-Rutman Theorem in the Banach lattice $L^s(\Omega)$. By using the same notation, we can show the existence of a positive eigenvalue $r(\lambda) > 0$ to problem (2.3); this eigenvalue is algebraically simple and associated with a positive eigenfunction. Taking into account that the imbedding $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$ is compact, that the coefficients in (2. 1) are positive, reasoning as above (in (1.7)), we obtain the variational characterization:

$$(2.5) \qquad r(\lambda) = \inf_{\phi \in H_0^1(\Omega);\ \phi \not\equiv 0} \frac{\int_\Omega |\nabla\phi|^2 + \int_\Omega (a_0^+(x)+1)\,\phi^2 + \lambda \int_\Omega m^-(x)\,\phi^2}{\int_\Omega m^+(x)\,\phi^2 + \frac{1}{\lambda}\int_\Omega (a_0^-(x)+1)\,\phi^2} =$$

$$\lambda \inf_{\phi \in H_0^1(\Omega);\ \phi \not\equiv 0} \frac{\int_\Omega |\nabla\phi|^2 + \int_\Omega (a_0^+(x)+1)\,\phi^2 + \lambda \int_\Omega m^-(x)\,\phi^2}{\lambda\int_\Omega m^+(x)\,\phi^2 + \int_\Omega (a_0^-(x)+1)\,\phi^2}.$$

It can be seen easily that $r(\lambda)$ is increasing in λ, depends monotonically on the domain and is continuous ([MM], [dF]). Thus we can define:

$$r(0) = \lim_{\lambda\to 0} r(\lambda) = 0.$$

We study here the case where a_0^- is not necessarily $\equiv 0$. In this situation the presence of an additional term depending on λ in (2.5) makes non obvious (in fact it can be false) that $r(\lambda)$ has the same properties than when $a_0^- \equiv 0$. In any case, it follows from (2.5) that $r(\lambda)$ depends continuously on λ for $\lambda > 0$, is increasing with respect to λ and depends monotonically with respect to the domain. But it is not clear if $r(\lambda)$ is concave or satisfies the weaker property used before.

In any case, (2.5) provides several useful **estimates** for $r(\lambda)$. The **first** one is obtained by using the monotone dependence with respect to both the coefficients and the domain

$$r(\lambda) = \lambda_1\left(-\Delta + a_0^+ + 1 + \lambda m^-,\ m^+ + \frac{1}{\lambda}(a_0^- + 1),\ \Omega\right) <$$

$$\lambda_1\left(-\Delta + a_0^+ + 1 + \lambda m^-,\ m^+ + \frac{1}{\lambda}(a_0^- + 1),\ \Omega^+\right) =$$

$$\lambda_1\left(-\Delta + a_0^+ + 1,\ m^+ + \frac{1}{\lambda}(a_0^- + 1),\ \Omega^+\right) < \lambda_1(-\Delta + a_0^+ + 1,\ m^+,\ \Omega^+),$$

and this gives the uniform estimate:

$$(2.6) \qquad r(\lambda) < \lambda_1(-\Delta + a_0^+ + 1,\ m^+,\ \Omega^+).$$

REMARK 2.1. – In fact estimate (2.6) could be replaced by

$$r(\lambda) \leqslant \lambda_1(-\Delta + a_0^+, m^+, \Omega^+)$$

according to the observation in Remark 2.2 below.

Now we need the following auxiliary result:

LEMMA 2.3. – *With the notations above we have:*

$$\lambda_1(-\Delta + a_0^+, a_0^-, \Omega) > 1 \iff \lambda_1(-\Delta + a_0, 1, \Omega) > 0, \tag{2.7}$$

$$\lambda_1(-\Delta + a_0^+, a_0^-, \Omega) = 1 \iff \lambda_1(-\Delta + a_0, 1, \Omega) = 0, \tag{2.8}$$

$$\lambda_1(-\Delta + a_0^+, a_0^-, \Omega) < 1 \iff \lambda_1(-\Delta + a_0, 1, \Omega) < 0. \tag{2.9}$$

PROOF. – If $-\Delta\varphi + a_0^+ \varphi = r a_0^- \varphi$ with $r > 1$ and $\varphi > 0$ then $-\Delta\varphi + a_0 \varphi = (r-1) a_0^- \varphi$ and $r - 1 = \lambda_1(-\Delta + a_0, a_0^-, \Omega)$ which implies $\lambda_1(-\Delta + a_0, 1, \Omega) > 0$ by the variational caracterization (2.5). Conversely if $-\Delta\psi + a_0 \psi = r\psi$ with $r > 0$ and $\psi > 0$, we have $-\Delta\psi + a_0^+ \psi = a_0^- \psi + r\psi$ and a well-known result gives us $1 < \lambda_1(-\Delta + a_0^+, a_0^-, \Omega)$, which proves (2.7). The proof of (2.8) and (2.9) is analogous. ■

REMARK 2.2. – The preceding Lemma is still valid if we replace $a_0^\pm$ by $a_0^\pm + 1$, or more generally by $a_0^\pm + \alpha$ for any $\alpha > 0$. If $\alpha \to 0$, we obtain the result in Remark 2.1. We will use this often in the following.

Since we have

$$r'(0) = \lim_{\lambda \to 0^+} \frac{r(\lambda)}{\lambda} = \lambda_1(-\Delta + a_0^+ + 1, a_0^- + 1, \Omega),$$

we derive from Lemma 2.3 and Remark 2.2 that we have to consider separately the three following cases:

$$\lambda_1(-\Delta + a_0, 1, \Omega) > 0, \tag{I}$$

$$\lambda_1(-\Delta + a_0, 1, \Omega) = 0, \tag{II}$$

$$\lambda_1(-\Delta + a_0, 1, \Omega) < 0. \tag{III}$$

2.1. CASE I. $\lambda_1(-\Delta + a_0, 1, \Omega) > 0$.

Obviously if $a_0 \geqslant 0$, we are in this case. From (2.7) and Remark 2.2, we deduce $r'(0) > 1$. From the continuity of $r(\lambda)$, (2.6) and (2.7), $r(\lambda)$ intersects at least once the diagonal; we will see the uniqueness of this point.

Assume that $r(\lambda) = \lambda > 0$. Then:

$$-\Delta\varphi_\lambda + a_0(x)\,\varphi_\lambda = \lambda m(x)\,\varphi_\lambda, \quad \varphi_\lambda(x) > 0, \quad x \in \Omega, \tag{2.10}$$
$$\varphi_\lambda(x) = 0, \quad x \in \partial\Omega.$$

Multiplying (2.10) by φ_λ and integrating over Ω, we get by (2.7) and the variational characterization of the eigenvalue:

$$\lambda_1(-\Delta + a_0, 1, \Omega)\int_\Omega \varphi_\lambda^2 \leq \int_\Omega |\nabla\varphi_\lambda|^2 + \int_\Omega a_0\varphi_\lambda^2 = \lambda\int_\Omega m\varphi_\lambda^2.$$

We deduce from the above equation

$$\int_\Omega m\varphi_\lambda^2 > 0. \tag{2.11}$$

Assume moreover that there exists μ such that $r(\mu) = \mu$, and that $0 < \mu < \lambda$, then

$$-\Delta\varphi_\mu + a_0(x)\,\varphi_\mu = \mu m(x)\,\varphi_\mu, \quad \varphi_\mu(x) > 0, \quad x \in \Omega,$$
$$\varphi_\mu(x) = 0, \quad x \in \partial\Omega.$$

From (2.10), $0 = \lambda_1(-\Delta + a_0 - \lambda m, 1, \Omega)$ and by the variational characterization of the eigenvalue we get

$$\int_\Omega |\nabla v|^2 + \int_\Omega a_0 v^2 - \lambda\int_\Omega mv^2 \geq 0, \quad \forall v \in H_0^1(\Omega);$$

and the equality occurs if and only if $v = c\varphi_\lambda$, where c is a constant. We have thus

$$\int_\Omega |\nabla\varphi_\mu|^2 + \int_\Omega a_0\varphi_\mu^2 - \lambda\int_\Omega m\varphi_\mu^2 + (\lambda - \mu)\int_\Omega m\varphi_\mu^2 = 0;$$

and it follows from above that the sum of the three first terms is positive, so $\int_\Omega m\varphi_\mu^2 < 0$ which contradicts (2.11) and proves uniqueness. We have established the following result

THEOREM 2.1. – *Assume that* a_0 *and* m *satisfy* (2.2), *that* $|\Omega^+| > 0$, $|\Omega^-| > 0$ *and that* (I) *is satisfied. Then the conclusion of Theorem* 1.1 *is still valid with* $-\Delta + a_0(x)$ *instead of* $-\Delta$.

REMARK 2.3. – The same observation as in Remark 1.4 is also pertinent here. Concerning the regularity of eigenfunctions, we have that $\varphi_1 \in H_0^1(\Omega)$ is a weak solution of $\Delta\varphi_1 + (a_0(x) - \lambda_1 m(x))\,\varphi_1 = 0$, where $a_0 - \lambda_1 m \in L^r(\Omega)$, with $r > \frac{N}{2}$. By Theorem 2.3 in [BK], $\varphi_1 \in L^s(\Omega)$ for any $s \in (1, \infty)$, and then it is easy to show that φ_1 is continuous. The same argument gives $u \in \mathcal{C}^1(\overline{\Omega})$ if $r > N$.

2.2. CASE II. $\lambda_1(-\Delta + a_0, 1, \Omega) = 0$.

In this case r has exactly slope 1 at origin, which allows all possibilities.

By (2.8), we have:

$$-\Delta\varphi_0 + a_0(x)\,\varphi_0 = 0, \quad \varphi_0(x) > 0, \quad x \in \Omega, \qquad (2.12)$$
$$\varphi_0(x) = 0, \quad x \in \partial\Omega.$$

Assume that $r(\nu) = \nu > 0$, $\psi > 0$ is an eigenpair of (2.10). We derive $0 = \lambda_1(-\Delta + a_0 - \nu m, 1, \Omega)$ and using again the variational characterization, we get:

$$\int_\Omega |\nabla v|^2 + \int_\Omega a_0 v^2 - \nu \int_\Omega m v^2 > 0, \quad \forall v \in H_0^1(\Omega), \quad v \neq c\psi,$$

and hence

$$\int_\Omega |\nabla \varphi_0|^2 + \int_\Omega a_0 \varphi_0^2 - \nu \int_\Omega m \varphi_0^2 > 0;$$

by (2.12), the sum of the two first terms is 0 and hence

$$\int_\Omega m\varphi_0^2 < 0. \qquad (2.13)$$

which provides a necessary condition for the existence of a positive principal eigenvalue.

Hence we have to consider the two following subcases:

i) If $\int_\Omega m\varphi_0^2 \geqslant 0$, we have proved above that $\nu > 0$ cannot be an eigenvalue of (2.10). Either 0 is the unique eigenvalue or there is a negative eigenvalue, which is unique by the concavity of the spectral radius ([K]).

ii) If (2.13) holds, (2.10) can have one positive eigenvalue which will be unique. The proof of uniqueness is analogous to that in Case (I) or as in *i*).

We shall prove now that there exists effectively one positive eigenvalue; to show this, we rewrite our eigenvalue problem in a different way as:

$$-\Delta u + a_0(x)\,u + \lambda(m^-(x) + 1)\,u = \varrho(m^+(x) + 1)\,u, \quad x \in \Omega,$$
$$u(x) = 0, \quad x \in \partial\Omega.$$

Since $\lambda_1(-\Delta + a_0 + \lambda(m^- + 1), 1, \Omega) > 0$, the Maximum principle holds and the Krein-Rutman theorem implies the existence of an eigenvalue that we denote by $\varrho(\lambda)$ (since it is not the same that $r(\lambda)$); $\varrho(\lambda) > 0$ is associated to a positive eigenfunction φ_λ. It is clear that $\varrho(0)=0$. We also have

$$-\Delta\varphi_\lambda + a_0(x)\,\varphi_\lambda + \lambda(m^-(x)+1)\,\varphi_\lambda = \varrho(\lambda)(m^+(x)+1)\,\varphi_\lambda, \quad x\in\Omega, \tag{2.14}$$

$$\varphi_\lambda(x)=0, \quad x\in\partial\Omega.$$

By multiplying (2.14) by $\varphi_0>0$ and by Green formula, we get

$$\int_\Omega(-\Delta\varphi_0 + a_0(x)\,\varphi_0)\,\varphi_\lambda + \lambda\int_\Omega(m^-(x)+1)\,\varphi_\lambda\varphi_0 = \varrho(\lambda)\int_\Omega(m^+(x)+1)\,\varphi_\lambda\varphi_0.$$

Taking in account the fact that the first integral is 0, we have:

$$\frac{\varrho(\lambda)}{\lambda} = \frac{\int_\Omega(m^-+1)\,\varphi_\lambda\varphi_0}{\int_\Omega(m^++1)\,\varphi_\lambda\varphi_0},$$

and hence

$$\lim_{\lambda\to 0}\frac{\varrho(\lambda)}{\lambda} = \frac{\int_\Omega(m^-+1)\,\varphi_0^2}{\int_\Omega(m^++1)\,\varphi_0^2} > 1;$$

this last result follows immediately from (2.13) (by continuity of the eigenfunction φ_λ with respect to λ which we prove next). As in Case (I), we deduce that there exists some $\lambda>0$ such that $\varrho(\lambda)=\lambda$.

LEMMA 2.4. – *Let φ_λ and φ_0 be the eigenfunctions normalized by conditions*

$$\|\varphi_\lambda\|_{L^{2r'}(\Omega)} = \|\varphi_0\|_{L^{2r'}(\Omega)} = 1, \quad \text{where } \frac{1}{r}+\frac{1}{r'}=1.$$

Hence φ_λ converges to φ_0 when $\lambda\to 0$ in $L^{2r'}(\Omega)$ (and in $H_0^1(\Omega)$ weakly).

PROOF. – We have

$$-\Delta\varphi_\lambda + a_0(x)\,\varphi_\lambda = \beta(\lambda)\,\varphi_\lambda, \quad x\in\Omega; \quad \varphi_\lambda(x)=0, \quad x\in\partial\Omega,$$

with $\beta(\lambda)=\varrho(\lambda)(m^++1)-\lambda(m^-+1)$. Multiplying the equation by φ_λ and by Green formula, we have

$$\int_\Omega|\nabla\varphi_\lambda|^2 = \int_\Omega\beta(\lambda)\,\varphi_\lambda^2 - \int_\Omega a_0(x)\,\varphi_\lambda^2 \leqslant$$

$$\|\beta(\lambda)\|_{L^r(\Omega)}\|\varphi_\lambda\|^2_{L^{2r'}(\Omega)} + \|a_0\|_{L^r(\Omega)}\|\varphi_\lambda\|^2_{L^{2r'}(\Omega)} \leqslant$$

$$\|\beta(\lambda)\|_{L^r(\Omega)} + \|a_0\|_{L^r(\Omega)} \leqslant C.$$

Indeed, since $\lim_{\lambda \to 0} \varrho(\lambda) = 0$, we have $\lim_{\lambda \to 0} \|\beta(\lambda)\|_{L^r(\Omega)} = 0$. It follows that $\|\varphi_\lambda\|_{H_0^1(\Omega)} \leq C$, and then it follows from Sobolev imbedding theorem that there exists a subsequence φ_{λ_k} converging to $\widehat{\varphi}$ weakly in $H_0^1(\Omega)$ and strongly in $L^{2r'}(\Omega)$ as λ_k converges to 0, since $2r' < \frac{2N}{N-2}$. It follows immediately that $\varphi^2_{\lambda_k}$ converges to $\widehat{\varphi}^2$ strongly in $L^{r'}(\Omega)$ and then

$$\int_\Omega a_0(x)\, \varphi^2_{\lambda_k} \to \int_\Omega a_0(x)\, \widehat{\varphi}^2$$

as λ_k converges to 0.

Moreover, by the weak-lower-semicontinuity of the norm,

$$\liminf_{\lambda_k \to 0} \int_\Omega |\nabla \varphi_\lambda|^2 \geq \int_\Omega |\nabla \widehat{\varphi}|^2 .$$

By passing to the limit in the equation, we get

$$\int_\Omega |\nabla \widehat{\varphi}|^2 + \int_\Omega a_0(x)\, \widehat{\varphi}^2 \leq 0 .$$

But it follows from condition (2.8) that

$$\int_\Omega |\nabla \widehat{\varphi}|^2 + \int_\Omega a_0(x)\, \widehat{\varphi}^2 = 0 ,$$

by the variational characterization which also yields $\widehat{\varphi} = c\varphi_0$, and replacing in the equation, we obtain $c = 1$. ■

We have thus proved the following results

THEOREM 2.2. – *Assume that* a_0 *and* m *satisfy* (2.2), $|\Omega^+| > 0$, $|\Omega^-| > 0$ *and* (II) *is satisfied. Then problem* (2.1) *has a positive* (resp. *negative*) *principal eigenvalue if and only if*

$$\int_\Omega m\varphi_0^2 < 0 \quad (\text{resp.} > 0).$$

In this case it is unique.

THEOREM 2.3. – *Assume that* a_0 *and* m *satisfy* (2.2), $|\Omega^+| > 0$, $|\Omega^-| > 0$, (II) *is satisfied and*

$$\int_\Omega m\varphi_0^2 = 0 .$$

Then 0 *is the only possible principal eigenvalue to Problem* (2.1).

REMARK 2.4. – Similar results have been proved in [HMV] for operators in general form with singular (near $\partial\Omega$) coefficients. Several results in [LG2] are also generalized in this article.

2.3. CASE III. $\lambda_1(-\Delta + a_0, 1, \Omega) < 0$.

In this case, we have $\lim_{\lambda \to 0} \frac{r(\lambda)}{\lambda} < 1$ and we need further estimates. The *second* estimate is derived from (2.5):

$$0 < r(\lambda) < r_2(\lambda), \tag{2.15}$$

where

$$r_2(\lambda) = \lambda \inf_{\phi \in H_0^1(\Omega);\ \phi \not\equiv 0} \frac{\int_\Omega |\nabla \phi|^2 + \int_\Omega (a_0^+(x) + 1)\, \phi^2 + \lambda \int_\Omega m^-(x)\, \phi^2}{\int_\Omega (a_0^-(x) + 1)\, \phi^2}. \tag{2.16}$$

A *third* estimate still follows from (2.5). Indeed we have for $\lambda > 0$

$$r(\lambda) < \lambda_1\left(-\Delta + a_0^+ + 1, \frac{1}{\lambda}(a_0^- + 1), \Omega^+\right) < r_3(\lambda) := \lambda . \lambda_1(-\Delta + a_0^+ + 1, a_0^- + 1, \Omega^+). \tag{2.17}$$

By continuity, we define

$$r(0) = \lim_{\lambda \to 0} r(\lambda) = \lim_{\lambda \to 0} r_2(\lambda) = r_2(0) = 0 = r_3(0). \tag{2.18}$$

The derivatives of these functions at the origin will play an important role in the following. From (2.5), (2.16) and (2.18) and from the continuity of the eigenvalues we deduce

$$r'(0) = r_2'(0) = \lim_{\lambda \to 0^+} \frac{r_2(\lambda)}{\lambda} = \lambda_1(-\Delta + a_0^+ + 1, a_0^- + 1, \Omega),$$

which means that the two curves have the same tangent at the origin. We also have that $r_2(\lambda)$ is convex, since $\lambda \to \frac{r_2(\lambda)}{\lambda}$ is strictly increasing. The comparison with $r_3(\lambda)$, gives a first non existence result.

PROPOSITION 2.1. – *If* $\lambda_1(-\Delta + a_0, 1, \Omega^+) \leqslant 0$, *then there is no positive principal eigenvalue to* (2.1).

PROOF. – By Lemma 2.3, we have equivalently $\lambda_1(-\Delta + a_0^+ + 1, a_0^- + 1, \Omega^+) \leqslant 1$ and the result follows from (2.17) combined with (2.18). ∎

It remains to consider the case

$$\lambda_1(-\Delta + a_0^+ + 1, a_0^- + 1, \Omega^+) > 1. \tag{2.19}$$

or equivalently

$$\lambda_1(-\Delta + a_0, 1, \Omega^+) > 0, \tag{2.20}$$

which will be a necessary condition for the existence of a positive principal eigenvalue.

A better necessary condition can be obtained from estimate (2.15) given by $r_2(\lambda)$. It follows from the convexity of $r_2(\lambda)$ that it will intersect the straight line $r = \lambda_1(-\Delta + a_0^+, m^+, \Omega^+)$ in a unique point which will be denoted by λ^*.

PROPOSITION 2.2. – *If we have*

$$\lambda_1(-\Delta + a_0^+, m^+, \Omega^+) \leqslant \lambda^*,$$

then there is no positive principal eigenvalue to (2.1).

PROOF. – It follows easily from the above arguments, (2.15) and Remark 2.1.

Hence we have obtained the necessary condition

$$\lambda_1(-\Delta + a_0^+, m^+, \Omega^+) > \lambda^*, \tag{2.21}$$

for the existence of a positive eigenvalue. ■

In principle, there may be now two, one or no solutions. We are unable to give a precise explicit answer for each given m, except in the special situations $m^+ \equiv 0$ and $m^- \equiv 0$ which are considered later, but it is possible to offer a generic description of the situation.

In fact, in this case, we need also more precise estimates concerning the slope at infinity. We use the following result

PROPOSITION 2.3. – *Let D be a regular bounded domain in $\mathbb{R}^N$. Let b, q, g in $L^r(D)$, $r > N/2$, be such that $b \geqslant 0$, $q \geqslant 0$, $g > 0$ on Ω. We define $D_0 := \{x \in D / q(x) = 0\}$. Assume that*

$$D_0 = \overline{int(D_0)}, \quad |int(D_0)| > 0 \quad \textit{and} \tag{H}$$

$int(D_0)$ satisfies the cone property except may be for a set of capacity zero.

Then we have

$$\lim_{\alpha \to +\infty} \lambda_1(-\Delta + b + \alpha q, g, D) = \lambda_1(-\Delta + b, g, D_0).$$

SKETCH OF PROOF. – As in [D3] we can prove

$$\lim_{\alpha \to +\infty} \lambda_1(-\Delta + b + \alpha q, g, D) \leqslant \lambda_1(-\Delta + b, g, D_0).$$

On the other hand, if there is some $\delta > 0$ such that, for all α

$$\lambda_1(-\Delta + b + \alpha q, g, D) \leqslant K := \lambda_1(-\Delta + b, g, D_0) - \delta,$$

we choose a sequence $(w_j) \subset H_0^1(D)$ of eigenfunctions associated to $\lambda_1(-\Delta + b + jq, g, D)$ with $\int_D g w_j^2 = 1$. We can show that w_j converges weakly in $H_0^1(D)$, strongly in $L_g^2(D)$ to some $w \in H_0^1(D)$ which satisfies $\int_D q w^2 = 0$; it follows from a theorem of Hedberg that $w \in H_0^1(D_0)$ and therefore

$$\lambda_1(-\Delta + b, g, D_0) \leqslant \int_{D_0} |\nabla w|^2 + b w^2 \leqslant$$

$$\liminf_{j \to +\infty} \left(\int_D |\nabla w_j|^2 + b w_j^2 \right) \leqslant \lim_{j \to +\infty} \lambda_1(-\Delta + b + jq, g, D) \int_D g |w_j|^2 \leqslant$$

$$\lim_{j \to +\infty} \lambda_1(-\Delta + b + jq, g, D) \int_D g |w|^2 \leqslant K \int_D g |w|^2 = K,$$

a contradiction. Therefore

$$\lambda_1(-\Delta + b, g, D_0) = \lim_{\alpha \to +\infty} \lambda_1(-\Delta + b + \alpha q, g, D). \quad \blacksquare$$

REMARK 2.5. – Proposition 2.3 is a particular case of a result in ([GS]) and was applied in the study of some nonlinear problems in ([AT1], [AT2]). For the readers's convenience, we state here the particular case of the results of ([D3]) that we use; see Proposition 1 and the remarks preceding the Proposition in page 444 of ([D3]). Necessary and sufficient conditions for its validity, which are related with fine properties of Sobolev spaces were given by Dancer in the interesting paper ([D3]). For the case of operators in non-divergence form, see ([LG2]), where the smoothness assumptions on both the domain and the coefficients are stronger than in ([D3]). See also the observations made in ([D3]).

REMARK 2.6. – From Proposition 2.3 above with $b = a_0^+ + 1$, $q = m^-$, $g = a_0^- + 1$, we derive

$$r_2'(\infty) := \lim_{\lambda \to +\infty} \frac{r_2(\lambda)}{\lambda} = \lambda_1(-\Delta + a_0^+ + 1, a_0^- + 1, \Omega^+ \cup \Omega_0). \tag{2.22}$$

Hence if the convex function $r_2(\lambda)$ has a «slope at infinity», $\lambda_1(-\Delta + a_0^+ + 1, a_0^- + 1, \Omega^+ \cup \Omega_0)$ which is $\leqslant 1$, (2.21) cannot be satisfied.

We study first the particular case $m^+ \equiv 0$ and consider the existence (or not) of negative eigenvalues. It turns out that $\lambda < 0$ cannot be a principal eigenvalue to (2.1). Indeed, if $-\Delta\psi + a_0\psi = -\lambda m^-(x)\psi$ in Ω, $\psi = 0$ on the boundary and $\psi > 0$ in Ω, then, by a comparison argument $0 = \lambda_1(-\Delta + a_0 + \lambda m^-, 1, \Omega) < \lambda_1(-\Delta + a_0, 1, \Omega)$, a contradiction. Hence all possible eigenvalues are positive. Now the problem can be rewritten as

$$(2.23)\quad -\Delta u + (a_0^+(x)+1)\,u + \lambda m^-(x)\,u = \varrho\left(\frac{a_0^-(x)+1}{\lambda}\right)u, \qquad x \in \Omega,$$

$$u(x) = 0, \qquad x \in \partial\Omega.$$

The spectral radius is given by the expression

$$(2.24)\qquad \overline{r}(\lambda) = \lambda \inf_{\phi \in H_0^1(\Omega);\ \phi \not\equiv 0} \frac{\int_\Omega |\nabla\phi|^2 + + \int_\Omega (a_0^+(x)+1)\,\phi^2 + \lambda\int_\Omega m^-(x)\,\phi^2}{\int_\Omega (a_0^-(x)+1)\,\phi^2}$$

which is actually the $r_2(\lambda)$ in (2.16). Thus $\overline{r}(\lambda)$ is convex and $\overline{r}(\lambda)(0)=0$. If (H) is satisfied, it follows from Proposition 2.3, that its «slope at infinity» is given by

$$\lim_{\lambda\to+\infty} \frac{\overline{r}(\lambda)}{\lambda} = \lambda_1(-\Delta + a_0^+ + 1,\ a_0^- + 1,\ \Omega_0).$$

It turns out that, if $\lambda_1(-\Delta + a_0^+ + 1,\ a_0^- + 1,\ \Omega_0) \leq 1$, or what is equivalent by Lemma 2.3,

$$(2.25)\qquad \lambda_1(-\Delta + a_0,\ 1,\ \Omega_0) \leq 0,$$

$\overline{r}(\lambda)$ will never intersect the diagonal and the problem has no solution. On the opposite side, if $\lambda_1(-\Delta + a_0^+ + 1,\ a_0^- + 1,\ \Omega_0) > 1$, or equivalently

$$(2.26)\qquad \lambda_1(-\Delta + a_0,\ 1,\ \Omega_0) > 0,$$

then, since $\overline{r}(\lambda)$ is strictly convex, it will intersect exactly once the diagonal. We have thus proved the following result:

THEOREM 2.4. – *Assume that a_0 and m satisfy* (2.2), $m^+ \equiv 0$, $|\Omega_0| > 0$, *and* (H) *and* (III) *are satisfied. Then there is a unique positive principal eigenvalue to* (2.1) *if and only if* (2.26) *holds.*

REMARK 2.7. – *The case $m^- \equiv 0$* can be treated exactly in the same way. Under less stringent conditions on the coefficients, we obtain then an alternative proof of results stated in [LG1], [LG2]. On the other hand, the above arguments formulated as they are, are only valid for differential operators in divergence form. See however Section 3.

Finally, we will consider the problem for a weight function which changes sign on Ω. According to Remark 2.7, we assume that

$$(2.27)\qquad r_2'(\infty) = \lambda_1(-\Delta + a_0^+ + 1,\ a_0^- + 1,\ \Omega^+ \cup \Omega_0) > 1,$$

and we study the problem, starting from $m^+ \equiv 0$, with $\lambda > 0$. More precisely, we consider a family of eigenvalue problems

$$(2.28)\qquad \begin{aligned} -\Delta u + a_0(x)\,u &= \lambda(tm^+(x) - m^-(x))\,u, \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega, \end{aligned}$$

where $t \geqslant 0$ plays the role of a parameter and we rewrite the equation as

$$(2.29)\qquad -\Delta u + (a_0^+(x)+1)\,u + \lambda m^-(x)\,u = \varrho\left(tm^+(x) + \frac{a_0^-(x)+1}{\lambda}\right)u, \quad x \in \Omega.$$

These problems are well-defined and the corresponding eigenvalue is given by

$$(2.30)\qquad r(t,\lambda) = \lambda \inf_{\phi \in H_0^1(\Omega);\ \phi \not\equiv 0} \frac{\int_\Omega |\nabla\phi|^2 + \int_\Omega (a_0^+(x)+1)\,\phi^2 + \lambda\int_\Omega (m^-(x)\,\phi^2}{\lambda t\int_\Omega (m^+(x))\,\phi^2 + \int_\Omega (a_0^-(x)+1)\,\phi^2}.$$

By (2.27), there exists a unique point $\widehat{\lambda}_2$ where $r(0,\lambda) = r_2(\lambda)$ intersects the diagonal. For $t = 0$, we are in the case $m^+ \equiv 0$ and $\dfrac{r(0,\lambda)}{\lambda} > 1$ for any $\lambda > \widehat{\lambda}_2$. By continuity, for any $\varepsilon_0 > 0$, $\dfrac{r(t, \widehat{\lambda}_2 + \varepsilon_0)}{\widehat{\lambda}_2 + \varepsilon_0} > 1$ for $t > 0$ small. Indeed, if this is not true for any small $t_0 > 0$, there exists a sequence $0 < t_k < t_0$ such that

$$1 \geqslant \lim_{t_k \to 0} \frac{r(t_k, \widehat{\lambda}_2 + \varepsilon_0)}{\widehat{\lambda}_2 + \varepsilon_0} = \frac{r(0, \widehat{\lambda}_2 + \varepsilon_0)}{\widehat{\lambda}_2 + \varepsilon_0} = > 1,$$

contradicting the assumption.

Then, for any $\varepsilon_0 > 0$, and for $0 < t < t_0$, *t fixed* and small enough, the curve $r(t,\lambda)$ is above the diagonal for some $\lambda = \widehat{\lambda}_2 + \varepsilon_0$. Since $r(t,\lambda) \leqslant \lambda_1(-\Delta + a_0^+ + 1, tm^+, \Omega^+)$ then it intersects (at least once) the diagonal. But by the concavity of the eigenvalue of the original problem, this can only happen exactly once again. Then there will be exactly two positive principal eigenvalues for any $0 < t < t_0$, t_0 small enough. (It turns out that $t_0 > 1$ implies the existence of two positive principle eigenvalues to the initial problem (2.1)).

The curves $r(t,\lambda)$ form a family which is pointwise decreasing in t for λ fixed. Moreover

$$\lim_{t \to +\infty} \frac{r(t,\lambda)}{\lambda} = 0,$$

where the convergence is uniform on the interval $\lambda \geqslant \delta > 0$, for any $\delta > 0$. (We pick $\delta = \widehat{\lambda}_2$). Indeed if $\varphi \in H_0^1(\Omega)$, $\varphi \not\equiv 0$, with $\operatorname{supp}(\varphi) \subset \Omega^+$, we have as $t \to +\infty$

$$\frac{r(t,\lambda)}{\lambda} \leqslant \frac{\int_\Omega |\nabla\varphi|^2 + \int_\Omega (a_0^+(x)+1)\varphi^2}{\lambda t\int_\Omega (m^+(x))\,\varphi^2 + \int_\Omega (a_0^-(x)+1)\,\varphi^2} \to 0.$$

All this shows that there are two principal eigenvalues for t small and none for t large.

Hence we have shown that, for $t > 0$ small, there are two principal eigenvalues which we denote by $\lambda_1(t) < \lambda_2(t)$. Moreover, it follows easily from (2.30) and the above reasoning that $\lambda_1(t)$ (resp. $\lambda_2(t)$) is an increasing (resp. decreasing) function of t and that both are continuous. We define A as the set of positive t such that there exists two principal eigenvalues of (2.29). It is clear that A is a non empty, open (by continuity) and bounded set; let $\bar{t} := \sup A$. Then if $t_k \nearrow \bar{t}$ as $k \to +\infty$, $\lambda_1(t_k) \nearrow \lambda_1(\bar{t})$ and $\lambda_2(t_k) \searrow \lambda_2(\bar{t})$ and it follows from the monotonicity that $\lambda_1(\bar{t}) \leqslant \lambda_2(\bar{t})$.

We claim that $\lambda_1(\bar{t}) = \lambda_2(\bar{t})$. Indeed, if $\lambda_1(\bar{t}) < \lambda_2(\bar{t})$, since $r(t, \nu) > \nu$ for any $\nu \in (\lambda_1(t); \lambda_2(t))$, we have $r(\bar{t}, \nu) \geqslant \nu$ for any $\nu \in I := (\lambda_1(\bar{t}); \lambda_2(\bar{t}))$. But then either $r(\bar{t}, \bar{\nu}) = \bar{\nu}$ for some $\bar{\nu} \in I$, which is impossible by the concavity of the spectral radius of the original problem, or $r(\bar{t}, \nu) > \nu$ for any $\nu \in I$ and using once again the continuity there is a $t^* > \bar{t}$ having two principal eigenvalues, a contradiction.

Now, if we put $\bar{\lambda} := \lambda_1(\bar{t}) = \lambda_2(\bar{t})$, it follows by the continuity of $r(t, \lambda)$ (see above) that $r(\bar{t}, \bar{\lambda}) = \bar{\lambda}$. Then, by passing to the limit in

$$-\Delta\varphi_k + (a_0^+(x) + 1)\,\varphi_k + \lambda_k m^-(x)\,\varphi_k = r(t_k, \lambda_1(t_k))\left(t_k m^+(x) + \frac{a_0^-(x) + 1}{\lambda_k}\right)\varphi_k, \quad x \in \Omega,$$

$$\varphi_k(x) = 0, \quad x \in \partial\Omega,$$

we obtain

$$-\Delta\varphi + (a_0^+(x) + 1)\,\varphi + \bar{\lambda} m^-(x)\,\varphi = \bar{\lambda}\left(\bar{t} m^+(x) + \frac{a_0^-(x) + 1}{\bar{\lambda}}\right)\varphi, \quad x \in \Omega,$$

$$\varphi(x) = 0, \quad x \in \partial\Omega.$$

Since $\varphi \not\equiv 0$, and $\varphi > 0$ on Ω, we have proved that $\bar{\lambda}$ is an eigenvalue for $t = \bar{t}$ with positive eigenfunction φ. This implies that $\bar{\lambda}$ is a principal eigenvalue for $t = \bar{t}$ and it is simple. That it is actually the only one follows from the definition of $\bar{t}$ and the same continuity argument. We have thus proved the following result.

THEOREM 2.5. – *Suppose that Ω is a regular bounded domain in $\mathbb{R}^N$ such that $|\Omega^+| > 0$, $|\Omega^-| > 0$ and moreover $\Omega^+ \cup \Omega_0$ satisfies Condition (H) in Proposition* 2.3. *Assume also that conditions* (2.2), (2.21), $\lambda_1(-\Delta + a_0, 1, \Omega) < 0$, $\lambda_1(-\Delta + a_0, 1, \Omega^+ \cup \Omega_0) > 0$ *are satisfied. Then there exists a $\bar{t} > 0$ such that the eigenvalue problem* (2.28) *has two positive principal eigenvalues for any $t \in (0, \bar{t})$, exactly one for $t = \bar{t}$, and none if $t > \bar{t}$.*

REMARK 2.8. – The main theorems we have obtained in Case (III) rely heavily on the important result by Dancer [D3] stated in Proposition 2.3, which holds for $-\Delta$ or, more generally, for operators in divergence form. A result of the same type was proved in [LG2] for operators in general form with smooth coefficients. Hence our proofs do not extend automatically to the non selfadjoint case and the reader may see the remarks in [D3, p. 439] for this situation.

If $\bar{t} \geq 1$ this means that the original problem has a positive principal eigenvalue. It is an important question to compare $\bar{t}$ with 1. A partial answer is given by the following proposition.

PROPOSITION 2.4. – *Assume that the hypotheses of Theorem* 2.5 *are satisfied and let* $\varphi_0 > 0$ *be the eigenfunction associated to* $\lambda_1(-\Delta + a_0, 1, \Omega) < 0$. *Then, for any* $t > 0$ *such that* (2.28) *has a positive principal eigenvalue, we have*

$$t < \frac{\int_\Omega m^- \varphi_0^2}{\int_\Omega m^+ \varphi_0^2}.$$

PROOF. – Let $\lambda > 0$ be a principal eigenvalue of (2.28) for some $t > 0$. Since φ_0 is not an eigenfunction of (2.28), and since $\lambda_1(-\Delta + a_0 - \lambda t m^+ + \lambda m^-, 1, \Omega) = 0$, we have

$$\int_\Omega |\nabla \varphi_0|^2 + a_0 \varphi_0^2 - \lambda \int_\Omega (tm^+ - m^-)\, \varphi_0^2 > 0.$$

Since $\lambda_1(-\Delta + a_0, 1, \Omega) < 0$, we derive $\lambda \int_\Omega (tm^+ - m^-)\, \varphi_0^2 < 0$ and since $m^+ \not\equiv 0$, the result follows. ■

COROLLARY 2.1. – *If* $\int_\Omega m\varphi_0^2 \geq 0$, *then there is no positive principal eigenvalue to Problem* 2.1.

It is clear that similar results can be proved under analogous conditions and we leave this task to the interested reader.

3. – The case of a general operator.

3.1. *Smooth domains.*

We consider here the case of a differential operator in general form on a bounded domain Ω in $\mathbb{R}^N$. The corresponding eigenvalue problem can be written as

$$\begin{aligned} &Lu = \lambda m(x)\, u, \quad x \in \Omega, \\ &u(x) = 0, \quad x \in \partial\Omega, \end{aligned} \tag{3.1}$$

where

$$Lu = -\sum_{i,j} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b_i(x) \frac{\partial u}{\partial x_i} + a_0(x)\, u \tag{3.2}$$

is a second order uniformly elliptic differential operator. We assume that both Ω and the coefficients in L are such that the L^p-regularity theory in [ADN] (see also ([GT], [Tr])) applies; in particular we have

$$a_0 \in L^\infty(\Omega). \tag{3.3}$$

Moreover we assume that

$$m^- \in L^\infty(\Omega), \tag{3.4}$$

$$m^+ \in L^r(\Omega), \quad r > \frac{N}{2}, \tag{3.5}$$

are satisfied.

As above (3.1) can be written equivalently as

$$\begin{gathered} Lu + \lambda(m^-(x) + \chi_{\Omega^- \cup \Omega_0})\, u = \lambda(m^+(x) + \chi_{\Omega^- \cup \Omega_0})\, u, \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \end{gathered} \tag{3.6}$$

and the associated eigenvalue problem is now

$$\begin{gathered} Lu + \lambda(m^-(x) + \chi_{\Omega^- \cup \Omega_0})\, u = \varrho(m^+(x) + \chi_{\Omega^- \cup \Omega_0})\, u, \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega. \end{gathered} \tag{3.7}$$

The multiplication operator M defined by $Mu(x) = (m^+(x) + \chi_{\Omega^- \cup \Omega_0}(x))u(x)$ is well defined as a linear operator from $\mathcal{C}(\overline{\Omega})$ into $L^r(\Omega)$ and it is continuous and bounded. If S is the solution operator corresponding to the linear problem:

$$\begin{gathered} Lu + \lambda(m^-(x) + \chi_{\Omega^- \cup \Omega_0})\, u = h, \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega. \end{gathered}$$

Then for any $h \in L^r(\Omega)$, there exists a unique solution $u \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ and $u \in \mathcal{C}(\overline{\Omega})$ by (3.5). Since $S : L^r(\Omega) \to W^{2,r}(\Omega)$ is continuous and since the embedding J of $W^{2,r}(\Omega)$ into $\mathcal{C}(\overline{\Omega})$ is compact, $T = JSM$: $\mathcal{C}(\overline{\Omega}) \to \mathcal{C}(\overline{\Omega})$ is also compact. By the Strong Maximum Principle for weak solutions ([GT], [Tr], [PW]), T is strongly positive in the sense of quasi-interior points of [DKM]. We have proved the

LEMMA 3.1. – *Suppose that the above assumptions, as well as* (3.3), (3.4) *and* (3.5) *are satisfied. Then, for any* $\lambda > 0$ *there exists a unique eigenvalue* $r(\lambda) > 0$ *to* (3.7) *which is algebraically simple and is the only one associated to a positive eigenfunction.*

If we use the same notation, $\lambda_1(L+q, g, \Omega)$, (now with operator L instead of $-\Delta$), the monotone dependence with respect to the domain yields as before:

$$r(0) = \lambda_1(L, m^+ + \chi_{\Omega^- \cup \Omega_0}, \Omega) > 0,$$

$$r(\lambda) < \lambda_1(L + \lambda(m^- + \chi_{\Omega^- \cup \Omega_0}), m^+ + \chi_{\Omega^- \cup \Omega_0}, \Omega^+) = \lambda_1(L, m^+, \Omega^+).$$

That $r(\lambda)$ is continuous and increasing as a function of λ is proved by using the variational characterization in [BNV] or as in [HK].

The uniqueness of the fixed point of r follows from the concavity of $r(\lambda)$, see ([BNV], [LG2]). This allows us to prove the

THEOREM 3.1. – *Suppose that the above assumptions, as well as* (3.3) *to* (3.5) *are satisfied. Then there exists a unique positive* (*resp. negative*) *eigenvalue* $\lambda_1^+(L, m, \Omega)$ (*resp.* $\lambda_1^-(L, m, \Omega)$) *to* (3.1); *moreover*

$$0 < \lambda_1^+(L, m, \Omega) < \lambda_1(L, m^+, \Omega^+)$$

$$(resp.\ -\lambda_1(L, m^-, \Omega^-) < \lambda_1^-(L, m, \Omega)) < 0).$$

Moreover, $\lambda_1^+(L, m, \Omega)$ (*resp.* $\lambda_1^-(L, m, \Omega)$) *is algebraically simple and it is the only positive* (*resp. negative*) *eigenvalue having a positive eigenfunction.*

REMARK 3.1. – Theorem 3.1 extends most of the results in [HK] in the sense that it allows less regular coefficients (not in $\mathcal{C}(\overline{\Omega})$) and the weight m satisfies (3.4), (3.5) instead of $m \in \mathcal{C}(\overline{\Omega})$.

An alternative proof can be given if (3.5) is replaced by

(3.8) $$m^+ \in L^r(\Omega), \quad r > N.$$

We sketch it in order to show the flexibility of the method. We work now in the function space $\mathcal{C}_0^1(\overline{\Omega})$ and the positive cone has the nonempty interior:

$$K = \left\{ u \in \mathcal{C}_0^1(\overline{\Omega}) \,|\, u > 0 \text{ in } \Omega, \ \frac{\partial u}{\partial n} < 0 \text{ on } \partial\Omega \right\}.$$

Reasoning as above, it is easy to show that $T = JSM$, where $M : \mathcal{C}_0^1(\overline{\Omega}) \to L^r(\Omega)$, $S : L^r(\Omega) \to W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$, $J : W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega) \to \mathcal{C}_0^1(\overline{\Omega})$ is a compact linear operator in $\mathcal{C}_0^1(\overline{\Omega})$. Moreover, $Tu \in K$ if $u \geq 0$ by the Strong Maximum principle for weak solutions. We use the classical version of Krein-Rutman theorem for cones with nonempty interior and, then, a similar fixed point argument.

3.2. *Nonsmooth domains*

Here Ω is a bounded domain with no smoothness property, as in ([BNV]) or ([Bi]). The differential operator L is defined by (3.2) with $a_{ij} \in \mathcal{C}(\overline{\Omega})$, b_i, $a_0 \in$

$L^\infty(\Omega)$, and $a_0 \geqslant 0$. Moreover we assume:

$$m \in L^\infty(\Omega). \tag{3.9}$$

The corresponding eigenvalue problem can be written as

$$\begin{aligned} Lu &= \lambda m(x) u, \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{3.10}$$

The boundary condition (1.2) should be understood, all along this subsection, in the sense of $u \overset{u_0}{=} 0$ in [BNV] where u_0 is the solution of the problem $Lu_0 = 1$ in Ω, $u_0 = 0$ on $\partial\Omega$, again in the sense of the paper [BNV].

As above, (3.10) can be written equivalently as

$$Lu + \lambda(m^-(x) + \chi_{\Omega^- \cup \Omega_0})\, u = \lambda(m^+(x) + \chi_{\Omega^- \cup \Omega_0})\, u, \quad x \in \Omega, \tag{3.11}$$

and the associated eigenvalue problem is now

$$Lu + \lambda(m^-(x) + \chi_{\Omega^- \cup \Omega_0})\, u = \varrho(m^+(x) + \chi_{\Omega^- \cup \Omega_0})\, u, \quad x \in \Omega. \tag{3.12}$$

The role of Lemmas 1.1 and 2.2 is now played by

LEMMA 3.2. – *For any $h \in L^t(\Omega)$, with $t \geqslant N$, there exists a unique solution $u \in L^\infty(\Omega)$ of*

$$Lu = h \text{ in } (\Omega); \quad u = 0 \text{ on } \partial\Omega .$$

Moreover, if $h \geqslant 0$, then $u \geqslant 0$. The solution operator S defined by $u = S(h)$ is a linear compact operator in $L^t(\Omega)$ for any $t \geqslant N$.

PROOF. – Existence and uniqueness were proved in ([BNV]) and positivity follows from the Refined Maximum Principle in ([BNV]). The compactness of S is proved in the same way as in ([Bi]) for $t = N$. ■

The multiplication operator M is defined as usual but now it is convenient to pick $p \geqslant N$.

The linear operator $T = SM$ where $S = (L + \lambda(m^-(x) + \chi_{\Omega^- \cup \Omega_0})\, I)^{-1}$ is compact in $L^p(\Omega)$ for any $p \geqslant N$ fixed and sends the positive cone K in $L^p(\Omega)$ into itself. Moreover by the Krylov-Safonov-Harnack inequality, (see Corollary 9.25 in [GT]), T is still strongly positive in the sense of quasi-interior points in the Banach lattice $L^N(\Omega)$ and the generalized version of Krein-Rutman theorem in [DKM] is applicable. As before we define

$$r(\lambda) := \lambda_1(L + \lambda m^- + \lambda\chi_{\Omega^- \cup \Omega_0},\, m^+ + \chi_{\Omega^- \cup \Omega_0},\, \Omega).$$

Here also r is monotone and continuous, $r(0) > 0$, and r satisfies

$$r(\lambda) < \lambda_1(L, m^+, \Omega^+).$$

Hence r has a fixed point and we have thus proved the following result

THEOREM 3.2. – *Under the above assumptions, the eigenvalue problem* (3.10) *has a positive eigenvalue* $\lambda_1 > 0$ *which is the only eigenvalue having a positive eigenfunction.*

REMARK 3.2. – An alternative proof of this result can be given by reasoning as in [Bi], where another variant of the Krein-Rutman theorem is used.

In order to extend Theorem 3.2 to the case of indefinite weights m and a zero order coefficient a_0 which changes sign, it is instrumental to have the same properties of $r(\lambda)$ concerning its dependence as a function of λ and the domain Ω. Since the principal eigenvalue obtained here should coincide with the one in [BNV] and by taking into account (3.9) and the variational characterization (1.14) in [BNV], we see that $r(\lambda)$ is continuous and monotone as a function of λ and depends monotonically on Ω. The continuous dependence on the domain is proved in [LG2] when Ω and the coefficients are very smooth; some results in [BNV], namely Theorem 2.4 and Proposition 9.3 go in this direction.

Added in proof: After the acceptation of this paper for publication, we have received a preprint by H. Amann, «*Maximum Principles and Principal Eigenvalues*», where most of the general results mentioned in the introduction have been extended.

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