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Pointwise Decay for Solutions of the 2D Neumann Exterior Problem for the Wave Equation.

PAOLO SECCHI

Sunto. – *In questo articolo si considera il problema esterno nel piano per le equazioni delle onde con una condizione di Neumann al bordo. Lo studio riguarda il comportamento per tempi grandi della soluzione, con particolare attenzione per la dipendenza dalla norma dei dati iniziali nella stima del tasso di decadimento puntuale. Nell'articolo si prova una tale stima, mediante una combinazione della stima di decadimento dell'energia locale e stime per la soluzione in tutto il piano.*

Summary. – *We consider the exterior problem in the plane for the wave equation with a Neumann boundary condition. We are interested to the asymptotic behavior for large times for the solution, and in particular to the dependence on the norms of the initial data in the estimate for the pointwise decay rate. In the paper we prove such an estimate, by a combination of the estimate of the local energy decay and decay estimates for the free space solution.*

1. – Introduction.

Let Ω be an exterior domain in $\boldsymbol{R}^2$; the boundary $\partial\Omega$ is a smooth, convex and compact hypersurface. Given $r>0$, we denote $\Omega_r = \Omega \cap B_r$, where $B_r = \{x \in \boldsymbol{R}^2 \,|\, |x| < r\}$. Below, $r_0 > 0$ is a fixed constant such that $\Omega^c \subset B_{r_0}$ (Ω^c is the complement of Ω). We set $Q = [0,\, \infty) \times \Omega$, $\Sigma = [0,\, \infty) \times \partial\Omega$.

In this paper we study the decay property of solutions to the mixed problem for the wave equation with Neumann boundary condition

$$
(1) \qquad
\begin{aligned}
&(\partial_{tt}^2 - \Delta)\, u = 0 && \text{in } Q,\\
&\partial_\nu u = 0 && \text{on } \Sigma,\\
&u(0,\, x) = f(x), && \\
&\partial_t u(0,\, x) = g(x) && \text{in } \Omega.
\end{aligned}
$$

There are many papers dealing with the asymptotic behavior of solutions to the exterior problem for the wave equation, see [8] and the references thereinto. However we were not able to find in the literature the result we are interested in, namely the estimate of the pointwise decay of solutions in our particular case $n = 2$, under a Neumann boundary condition. Our proof is a combination by a cut-off argument of the estimate of the local energy decay following from the analysis of Kleinman and Vainberg [5], Morawetz [7], Vainberg [11] and decay estimates for the free space solution, in particular Klainerman's inequality, see [4]. In order to get a decay rate of local energy, some assumption on the shape of the obstacle should be taken, in order to exclude the existence of closed ray solutions. In fact, for the Dirichlet problem, Ralston [10] has shown that if there is a closed ray solution, there is no rate of decay. For the Dirichlet problem the obstacle should be *non-trapping*, see [8]; for the Neumann problem Morawetz [7] obtains the decay rate for convex bodies. This is the reason why in this paper we take the boundary convex.

Let us introduce some notation. For a multi-index $\alpha = (\alpha_1, \alpha_2)$ we set $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}$, $|\alpha| = \alpha_1 + \alpha_2$, where $\partial_1 = \partial/\partial x_1$, $\partial_2 = \partial/\partial x_2$. Let $W^{m,p}(\Omega)$ be the usual Sobolev space of order m, $m = 1, 2, \ldots$ and order of integrability $p \geq 1$, and let $\|\cdot\|_{W^{m,p}}$ denote its norm. If $p = 2$ we set $W^{m,p}(\Omega) = H^m(\Omega)$ with norm $\|\cdot\|_{H^m}$. The norm of $L^2(\Omega)$ is denoted by $\|\cdot\|$, the norm of $L^p(\Omega)$, $1 \leq p \leq \infty$, by $|\cdot|_p$. For simplicity we use the abbreviated notation $W^{m,p}$, H^m, L^p. We will also use the same symbol for spaces of vector valued functions. Let us define the weighted Sobolev space

$$\widehat{H}^m = \widehat{H}^m(\Omega) := \{f \in L^2 \colon \|f\|_{\widehat{H}^m} < \infty\}$$

where

$$\|f\|_{\widehat{H}^m} := \Big(\sum_{|\alpha| \leq m} \|(1 + |\cdot|)^{|\alpha|}\,\partial^\alpha f(\cdot)\|_{L^2}^2\Big)^{1/2}.$$

Similarly we introduce the spaces $W^{m,p}(\boldsymbol{R}^2)$, $H^m(\boldsymbol{R}^2)$, $L^p(\boldsymbol{R}^2)$ and $\widehat{H}^m(\boldsymbol{R}^2)$ whose norms are denoted by $\|\cdot\|_{W^{m,p}(R^2)}$, $\|\cdot\|_{H^m(R^2)}$, $\|\cdot\|_{L^p(R^2)}$ and $\|\cdot\|_{\widehat{H}^m(R^2)}$ respectively. Let us introduce the generalized derivatives

$$\begin{aligned}&\partial_t,\ \partial_1,\ \partial_2,\quad \omega = x_1\partial_2 - x_2\partial_1,\\ &L_0 = t\partial_t + x_1\partial_1 + x_2\partial_2,\quad L_i = t\partial_i + x_i\partial_t \quad \text{for } i = 1, 2,\end{aligned}$$

which we denote by $\Gamma_0, \Gamma_1, \ldots, \Gamma_6$. For a multi-index $A = (A_0, A_1, \ldots, A_6)$ with nonnegative integers A_i we define

$$|A| = A_0 + A_1 + \ldots + A_6, \qquad \Gamma^A = \Gamma_0^{A_0}\Gamma_1^{A_1}\ldots\Gamma_6^{A_6}, \qquad \Gamma^0 = 1.$$

For a scalar function $u = u(t, x) \colon \boldsymbol{R}^2 \to \boldsymbol{R}$ and a nonnegative integer m we introduce the norm

$$|||u(t)|||_m = \max_{|A| \leq m} \left(\int_{\mathbf{R}^2} |\Gamma^A u(t, x)|^2 \, dx \right)^{1/2}, \quad \forall t \geq 0.$$

For a function $u = u(t, x)$ defined over Ω instead of $\mathbf{R}^2$, we may define a similar norm by taking the integrals over Ω:

$$|||u(t)|||_{m, \Omega} = \max_{|A| \leq m} \left(\int_{\Omega} |\Gamma^A u(t, x)|^2 \, dx \right)^{1/2}, \quad \forall t \geq 0.$$

We introduce the time-independent version of the above vector fields:

$$\Lambda = (\Lambda_1, \dots, \Lambda_4) = (\partial_1, \partial_2, \omega, x_1 \partial_1 + x_2 \partial_2).$$

For a multi-index $B = (B_1, \dots, B_4)$ with nonnegative integers B_i we define

$$|B| = B_1 + \dots + B_4, \quad \Lambda^B = \Lambda_1^{B_1} \dots \Lambda_4^{B_4}, \quad \Lambda^0 = 1.$$

We introduce the weighted space

$$L^2_{\log} = L^2_{\log}(\Omega) := \{ u \in L^2 : (1 + |x \log |x||) \, u \in L^2 \},$$
$$\|u\|_{L^2_{\log}} = \|(1 + |x \log |x||) \, u\|_{L^2}.$$

We finally introduce the following spaces, similar to $\widehat{H}^m$, but based in $L^2_{\log}$,

$$H^m_{\log} = H^m_{\log}(\Omega) := \{ f \in L^2 : \|f\|_{H^m_{\log}} < \infty \}$$

where

$$\|f\|_{H^m_{\log}} := \left(\sum_{|\alpha| \leq m} \|(1 + |\cdot|)^{|\alpha|} \, \partial^\alpha f(\cdot)\|^2_{L^2_{\log}} \right)^{1/2}.$$

Similarly we introduce the spaces $L^2_{\log}(\mathbf{R}^2)$, $H^m_{\log}(\mathbf{R}^2)$ whose norms are denoted by $\|\cdot\|_{L^2(R^2)}$, $\|\cdot\|_{H^m_{\log}(R^2)}$, respectively.

THEOREM 1.1. – *Suppose u is a solution of the exterior problem* (1.1). *Assume the initial data satisfy $f \in \widehat{H}^5 \cap W^{5,1}$, $g \in \widehat{H}^4 \cap H^2_{\log} \cap W^{4,1}$. Then there exists a constant $C > 0$ such that*

$$(2) \qquad |\partial_t u(t, \cdot)|_\infty + |\nabla u(t, \cdot)|_\infty \leq C(1+t)^{-1/2} \log^2(e+t) \times$$
$$(\|f\|_{\widehat{H}^5} + \|f\|_{W^{5,1}} + \|g\|_{\widehat{H}^4} + \|g\|_{H^2_{\log}} + \|g\|_{W^{4,1}}) \quad \forall t \geq 0.$$

Observe that the decay rate obtained in (2) is slightly slower than the optimal rate decay $t^{-1/2}$ of the free space solution. A simplication of the function spaces from which the initial data are taken occurs by taking them of compact support.

COROLLARY 1.1. – *Assume that the initial data have compact support and satisfy $f \in H^5$, $g \in H^4$. Then there exists a constant $C > 0$ depending on the support of the data such that*

$$|\partial_t u(t,\cdot)|_\infty + |\nabla u(t,\cdot)|_\infty \leq C(1+t)^{-1/2}\log^2(e+t)(\|f\|_{H^5}+\|g\|_{H^4}) \quad \forall t \geq 0. \tag{3}$$

2. – Local pointwise decay.

Let us consider the initial boundary value problem (1) with new notation

$$\begin{aligned} &(\partial^2_{tt} - \Delta)\, w = 0 && \text{in } Q, \\ &\partial_\nu w = 0 && \text{on } \Sigma, \\ &w(0,x) = w_0(x), && \\ &\partial_t w(0,x) = w_1(x) && \text{in } \Omega. \end{aligned} \tag{4}$$

LEMMA 2.1. – *Let (w_0, w_1) have compact support and satisfy $(\nabla w_0, w_1) \in H^2$. Then the solution w of* (4) *satisfies the estimate*

$$|\partial_t w(t)|_{L^\infty(\Omega_R)} + |\nabla w(t)|_{L^\infty(\Omega_R)} \leq C_R(1+t)^{-1}(\|\nabla w_0\|_{H^2} + \|w_1\|_{H^2}) \tag{5}$$

for every $R > r_0$ and $t \geq 0$, where C_R depends on R, the support of the initial data and the geometry of $\partial\Omega$.

PROOF. – From the result of [5], [7], [11] for the Neumann problem for convex bodies, the local energy decays according to the estimate

$$\int_{\Omega_r} (|\partial_t w(t)|^2 + |\nabla w(t)|^2)\, dx \leq C_r(1+t)^{-2}(\|w_1\|^2 + \|\nabla w_0\|^2) \tag{6}$$

for all $r > r_0$, where C_r depends on r, the support of the initial data and the geometry of $\partial\Omega$. From (4) and time differentiation, $\partial_t w$ solves

$$\begin{aligned} &(\partial^2_{tt} - \Delta)\, \partial_t w = 0 && \text{in } Q, \\ &\partial_\nu \partial_t w = 0 && \text{on } \Sigma, \\ &\partial_t w(0,x) = w_1(x), && \\ &\partial_t(\partial_t w)(0,x) = \Delta w_0(x) && \text{in } \Omega. \end{aligned} \tag{7}$$

From application of (6) to problem (7) we have

$$\int_{\Omega_r} (|\partial^2_{tt} w(t)|^2 + |\nabla \partial_t w(t)|^2)\, dx \leqslant C_r(1+t)^{-2}(\|\Delta w_0\|^2 + \|\nabla w_1\|^2),$$

which yields

$$\int_{\Omega_r} (|\Delta w(t)|^2 + |\nabla \partial_t w(t)|^2)\, dx \leqslant C_r(1+t)^{-2}(\|\Delta w_0\|^2 + \|\nabla w_1\|^2), \tag{8}$$

for every $r > r_0$. We time differentiate once more and obtain the problem

$$\begin{aligned} &(\partial^2_{tt} - \Delta)\, \partial^2_{tt} w = 0 && \text{in } Q, \\ &\partial_\nu \partial^2_{tt} w = 0 && \text{on } \Sigma, \\ &\partial^2_{tt} w(0, x) = \Delta w_0(x), && \\ &\partial_t(\partial^2_{tt} w)(0, x) = \Delta w_1(x) && \text{in } \Omega, \end{aligned}$$

whose solution obeys the estimate

$$\int_{\Omega_r} (|\partial^3_{ttt} w(t)|^2 + |\nabla \partial^2_{tt} w(t)|^2)\, dx \leqslant C_r(1+t)^{-2}(\|\Delta w_1\|^2 + \|\nabla \Delta w_0\|^2),$$

which yields

$$\int_{\Omega_r} (|\Delta \partial_t w(t)|^2 + |\Delta \nabla w(t)|^2)\, dx \leqslant C_r(1+t)^{-2}(\|\Delta w_1\|^2 + \|\nabla \Delta w_0\|^2), \tag{9}$$

for every $r > r_0$. For any fixed $t > 0$ and given $R > r_0$, we choose $\sigma(x) \in C_0^\infty(\boldsymbol{R}^2)$ such that $\sigma(x) = 1$ if $|x| \leqslant R$ and $= 0$ if $|x| \geqslant R+1$. Let us denote $\Phi = \Delta(\sigma \partial_t w)$. Then $\sigma \partial_t w$ solves the elliptic problem

$$\begin{aligned} &\Delta(\sigma \partial_t w) = \Phi && \text{in } \Omega_{R+1}, \\ &\partial_\nu(\sigma \partial_t w) = 0 && \text{on } \partial\Omega, \\ &\sigma \partial_t w = 0 && \text{on } \partial B_{R+1}. \end{aligned}$$

We then have the estimate

$$\|\sigma \partial_t w\|_{H^2(\Omega_{R+1})} \leqslant C\|\Phi\|_{L^2(\Omega_{R+1})}. \tag{10}$$

From the Sobolev imbedding $H^2(\Omega_{R+1}) \subset L^\infty(\Omega_{R+1})$ and (10) we get

$$|\partial_t w|^2_{L^\infty(\Omega_R)} \leqslant C \int_{\Omega_{R+1}} (|\partial_t w|^2 + |\nabla \partial_t w|^2 + |\Delta \partial_t w|^2)\, dx. \tag{11}$$

From (6), (8), (9) under the choice $r = R+1$, and (11) we then obtain

$$|\partial_t w(t)|_{L^\infty(\Omega_R)} \leqslant C_R(1+t)^{-1}(\|\nabla w_0\|_{H^2} + \|w_1\|_{H^2}). \tag{12}$$

In order to estimate $|\nabla w(t)|_{L^\infty(\Omega_R)}$, we try to proceed similarly. In this case we consider the elliptic system

$$\begin{aligned} &\Delta(\sigma\nabla w) = \Psi && \text{in } \Omega_{R+1},\\ &(\sigma\nabla w)\cdot\nu = 0 && \text{on } \partial\Omega,\\ &\text{rot }(\sigma\nabla w) = 0 && \text{on } \partial\Omega,\\ &\sigma\nabla w = 0 && \text{on } \partial B_{R+1}, \end{aligned}$$

where we have set $\Psi = \Delta(\sigma\nabla w)$. Thus we have

$$(13)\qquad |\nabla w|_{L^\infty(\Omega_R)} \leq C\|\sigma\nabla w\|_{H^2(\Omega_{R+1})} \leq C\|\Psi\|_{L^2(\Omega_{R+1})} \leq$$
$$C\bigg(\int_{\Omega_{R+1}} (|\nabla w|^2 + \sum_{|\alpha|=2} |\partial^\alpha w|^2 + |\Delta\nabla w|^2)\, dx\bigg)^{1/2}.$$

Therefore we see the necessity to estimate all double x-derivatives of w over Ω_{R+1}. We choose $\sigma'(x) \in C_0^\infty(\boldsymbol{R}^2)$ such that $\sigma'(x) = 1$ if $|x| \leq R+1$ and $= 0$ if $|x| \geq R+2$. Consider the elliptic system

$$\begin{aligned} &\text{div}\,(\sigma'\nabla w) = \sigma'\Delta w + \nabla\sigma'\cdot\nabla w && \text{in } \Omega_{R+2},\\ &\text{rot}\,(\sigma'\nabla w) = \nabla\sigma'\times\nabla w && \text{in } \Omega_{R+2},\\ &(\sigma'\nabla w)\cdot\nu = 0 && \text{on } \partial\Omega_{R+2}. \end{aligned}$$

It follows that

$$(14)\qquad \|\nabla w\|_{H^1(\Omega_{R+1})} \leq \|\sigma'\nabla w\|_{H^1(\Omega_{R+2})} \leq$$
$$C(\|\sigma'\Delta w + \nabla\sigma'\cdot\nabla w\| + \|\nabla\sigma'\times\nabla w\|) \leq C\bigg(\int_{\Omega_{R+2}} (|\nabla w|^2 + |\Delta w|^2)\, dx\bigg)^{1/2}.$$

From (6), (9) under the choice $r = R+1$, we obtain

$$(15)\qquad \int_{\Omega_{R+1}} (|\nabla w|^2 + |\Delta\nabla w|^2)\, dx \leq C_R(1+t)^{-2}(\|\nabla w_0\|_{H^2}^2 + \|w_1\|_{H^2}^2).$$

From (6), (8) under the choice $r = R+2$, and (14) we obtain

$$(16)\qquad \sum_{|\alpha|=2} \int_{\Omega_{R+1}} |\partial^\alpha w|^2\, dx \leq C_R(1+t)^{-2}(\|\nabla w_0\|_{H^1}^2 + \|w_1\|_{H^1}^2).$$

Finally from (13), (15) and (16) we obtain

$$|\nabla w(t)|_{L^\infty(\Omega_R)} \leq C_R(1+t)^{-1}(\|\nabla w_0\|_{H^2} + \|w_1\|_{H^2}). \qquad \blacksquare$$

Let us consider the initial boundary value problem

$$(17)\qquad \begin{aligned} &(\partial^2_{tt} - \Delta)\, w = G && \text{in } Q,\\ &\partial_\nu w = 0 && \text{on } \Sigma,\\ &w(0, x) = 0, && \\ &\partial_t w(0, x) = 0 && \text{in } \Omega. \end{aligned}$$

LEMMA 2.2. – *Let $G(t, \cdot)$ have compact support and satisfy $G(t, \cdot) \in H^2$ for each $t > 0$. Then the solution w of* (17) *satisfies the estimate*

$$(18)\qquad |\partial_t w(t)|_{L^\infty(\Omega_R)} + |\nabla w(t)|_{L^\infty(\Omega_R)} \leqslant C_R \int_0^t (1 + t - s)^{-1} \|G(s, \cdot)\|_{H^2}\, ds,$$

for every $R > r_0$ and $t \geqslant 0$, where C_R depends on R, the support of G and the geometry of $\partial\Omega$.

PROOF. – It is a simple consequence of Duhamel's principle. We write w as $w(t, x) = \int_0^t V(t - s, s, x)\, ds$, where, for each fixed $s \geqslant 0$, V solves

$$\begin{aligned} &(\partial^2_{tt} - \Delta)\, V(t, s, x) = 0 && \text{in } Q,\\ &\partial_\nu V(t, s, x) = 0 && \text{on } \Sigma,\\ &V(0, s, x) = 0, && \\ &\partial_t V(0, s, x) = G(s, x) && \text{in } \Omega. \end{aligned}$$

We have

$$|\partial_t w(t)|_{L^\infty(\Omega_R)} + |\nabla w(t)|_{L^\infty(\Omega_R)} \leqslant \int_0^t |\partial_t V(t - s, s, \cdot)|_{L^\infty(\Omega_R)} + |\nabla V(t - s, s, \cdot)|_{L^\infty(\Omega_R)}\, ds\,.$$

The thesis follows from application of (5). ■

At last we consider the nonhomogeneous initial boundary value problem

$$(19)\qquad \begin{aligned} &(\partial^2_{tt} - \Delta)\, w = G && \text{in } Q,\\ &\partial_\nu w = 0 && \text{on } \Sigma,\\ &w(0, x) = w_0, && \\ &\partial_t w(0, x) = w_1 && \text{in } \Omega. \end{aligned}$$

From Lemma 2.1 and Lemma 2.2, by linearity we have

COROLLARY 2.1. – *Let* (w_0, w_1) *have compact support and satisfy* $(\nabla w_0, w_1) \in H^2$. *Let* $G(t, \cdot)$ *have compact support and satisfy* $G(t, \cdot) \in H^2$ *for each* $t > 0$. *Then the solution* w *of* (19) *satisfies the estimate*

$$(20)\qquad |\partial_t w(t)|_{L^\infty(\Omega_R)} + |\nabla w(t)|_{L^\infty(\Omega_R)} \leq C_R (1+t)^{-1} (\|\nabla w_0\|_{H^2} + \|w_1\|_{H^2}) + C_R \int_0^t (1+t-s)^{-1} \|G(s, \cdot)\|_{H^2}\, ds,$$

for every $R > r_0$ *and* $t \geq 0$, *where* C_R *depends on* R, *the support of the data and the geometry of* $\partial\Omega$.

The rest of this section is devoted to an estimate of the pointwise decay of w.

LEMMA 2.3. – *Let* (w_0, w_1) *have compact support and satisfy* $(w_0, w_1) \in H^2 \times H^1$. *Let* $G(t, \cdot)$ *have compact support and satisfy* $G(t, \cdot) \in H^1$ *for each* $t > 0$. *Then the solution* w *of* (19) *satisfies the estimate*

$$(21)\qquad |w(t)|_{L^\infty(\Omega_R)} \leq C_R (1+t)^{-1} (\|w_0\|_{H^2} + \|w_1\|_{H^1}) + C_R \int_0^t (1+t-s)^{-1} \|G(s, \cdot)\|_{H^1}\, ds,$$

for every $R > r_0$ *and* $t \geq 0$, *where* C_R *depends on* R, *the support of the data and the geometry of* $\partial\Omega$.

PROOF. – We decompose the solution by using the linearity of (19).

(i) We start by considering the case $w_1 = 0$, $G = 0$. Let w' denote the solution in this case and let us set $v'(t, x) = \int_0^t w'(s, x)\, ds$. Then v' solves

$$\begin{aligned} &(\partial_{tt}^2 - \Delta) v' = 0 && \text{in } Q, \\ &\partial_\nu v' = 0 && \text{on } \Sigma, \\ &v'(0, x) = 0, && \\ &\partial_t v'(0, x) = w_0 && \text{in } \Omega. \end{aligned}$$

From (20) we get

$$(22)\qquad |w'(t)|_{L^\infty(\Omega_R)} = |\partial_t v'(t)|_{L^\infty(\Omega_R)} \leq C_R (1+t)^{-1} \|w_0\|_{H^2}.$$

(ii) Next we consider the case $w_0 = 0$, $G = 0$; the solution is denoted by w''. Let us set $v''(t, x) = \int_0^t w''(s, x)\, ds + \varphi(x)$, where the corrector φ will be choosen below. Then v'' solves

$$\begin{aligned} &(\partial^2_{tt} - \Delta)\, v'' = w_1 - \Delta\varphi && \text{in } Q, \\ &\partial_\nu v'' = \partial_\nu \varphi && \text{on } \Sigma, \\ &v''(0, x) = \varphi, && \\ &\partial_t v''(0, x) = 0 && \text{in } \Omega. \end{aligned}$$

Thus we choose φ as a solution of

(23) $$\Delta\varphi = w_1 \quad \text{in } \Omega, \qquad \partial_\nu \varphi = 0 \quad \text{on } \partial\Omega.$$

From (20) we get

(24) $$|w''(t)|_{L^\infty(\Omega_R)} = |\partial_t v''(t)|_{L^\infty(\Omega_R)} \leqslant C_R(1+t)^{-1}\|\nabla\varphi\|_{H^2}.$$

The conclusion of the proof in case (ii) is a consequence of the following result, whose proof is postponed to the end of this proof.

LEMMA 2.4. – *There exists a solution φ of* (2.33) *such that*

(25) $$\|\nabla\varphi\|_{H^2} \leqslant C\|w_1\|_{H^1},$$

where C depends on the support of w_1.

Admitting this estimate for the moment, we obtain from (24) and (25)

(26) $$|w''(t)|_{L^\infty(\Omega_R)} \leqslant C_R(1+t)^{-1}\|w_1\|_{H^1}.$$

(iii) The last case is when $w_0 = 0$, $w_1 = 0$; denote the solution by w'''. By the Duhamel's principle we write $w'''(t, x) = \int_0^t V(t-s, s, x)\, ds$, where V is as in the proof of Lemma 2.2. From (26) it follows that

(27) $$|w'''(t)|_{L^\infty(\Omega_R)} \leqslant \int_0^t |V(t-s, s, \cdot)|_{L^\infty(\Omega_R)}\, ds \leqslant C_R \int_0^t (1+t-s)^{-1}\|G(s, \cdot)\|_{H^1}\, ds.$$

The final thesis follows by addition of (22), (26) and (27). ■

PROOF OF LEMMA 2.4. – Here the main difficulty is to show

(28) $$\|\nabla\varphi\| \leqslant C\|w_1\|,$$

because the Poincaré inequality doesn't hold over the unbounded domain Ω. When this is done, we use potential theoretic arguments combined with the

Calderon-Zygmund theorem in order to show that $\|\partial^2 \varphi\|_{H^1} \leqslant C\|w_1\|_{H^1}$; therefore, adding the previous estimate gives (25). Fix any $R > r_0$ and let B be a constant such that $B|x| \geqslant 2$ as $x \in \Omega$. Since φ is defined up to a constant, we may assume that φ has mean value over Ω_{2R} equal to zero. Then, by the Poincaré inequality over Ω_{2R} we get

$$\int_{\Omega_R} |\varphi(x)|^2 |x \log(B|x|)|^{-2} dx \leqslant C_R \int_{\Omega_R} |\varphi(x)|^2 dx \leqslant C_R \int_{\Omega_{2R}} |\nabla\varphi(x)|^2 dx. \tag{29}$$

For $|x| \geqslant R$ we proceed as in [1], Lemma 2.1 and show that

$$\int_{|x| \geqslant R} |\varphi(x)|^2 |x \log(B|x|)|^{-2} dx \leqslant C_R \int_{|x| \geqslant R} |\nabla\varphi(x)|^2 dx. \tag{30}$$

We need the zero mean value over Ω_{2R} (instead of Ω_R) for (30), because we need to apply the Poincaré inequality over Ω_{2R}. Adding (29), (30) gives

$$\int_{\Omega} |\varphi(x)|^2 |x \log(B|x|)|^{-2} dx \leqslant C_R \int_{\Omega} |\nabla\varphi(x)|^2 dx. \tag{31}$$

Recalling that w_1 has compact support, we multiply (23) by φ and integrate over Ω, to obtain

$$\|\nabla\varphi\|^2 \leqslant \int_{\Omega} |w_1| |\varphi| dx \leqslant$$
$$\left(\int_{\Omega} |w_1(x)|^2 |x \log(B|x|)|^2 dx\right)^{1/2} \left(\int_{\Omega} |\varphi(x)|^2 |x \log(B|x|)|^{-2} dx\right)^{1/2} \leqslant$$
$$C_R \|w_1\| \|\nabla\varphi\|,$$

where the last inequality follows from (31) and C_R depends also on the support of w_1. This completes the proof of (28). ■

In conclusion we also state two results which are an easy consequence of (6).

LEMMA 2.5. – *Let* (w_0, w_1) *have compact support and satisfy* $(w_0, w_1) \in L^2 \times L^2$. *Let* $G(t, \cdot)$ *have compact support and satisfy* $G(t, \cdot) \in L^2$ *for each* $t > 0$. *Then the solution* w *of* (19) *satisfies the estimate*

$$\|w(t)\|_{L^2(\Omega_R)} \leqslant C_R (1+t)^{-1} (\|w_0\|_{L^2} + \|w_1\|_{L^2}) + C_R \int_0^t (1+t-s)^{-1} \|G(s, \cdot)\|_{L^2} ds. \tag{32}$$

Moreover, if $(\nabla w_0, w_1) \in H^2$, $G(t, \cdot) \in H^2$ *for each* $t > 0$, *then*

$$\|w(t)\|_{H^3(\Omega_R)} \leq C_R (1+t)^{-1} (\|w_0\|_{H^3} + \|w_1\|_{H^2}) + C_R \int_0^t (1+t-s)^{-1} \|G(s, \cdot)\|_{H^2} ds. \tag{33}$$

(32) *and* (33) *hold for every* $R > r_0$ *and* $t \geq 0$; C_R *depends on* R, *the support of the data and the geometry of* $\partial\Omega$.

PROOF. – The proof of (32) is a consequence of (6) and the arguments employed in the proof of Lemma 2.3. (33) follows from (13), (15), (16), the Duhamel's principle and (32).

3. – Proof of Theorem 1.1.

Let us take functions $\tilde{f}, \tilde{g} : \boldsymbol{R}^2 \to \boldsymbol{R}$ such that $\tilde{f} = f$, $\tilde{g} = g$ on Ω, and such that $\tilde{f} \in \widehat{H}^5(\boldsymbol{R}^2) \cap W^{5,1}(\boldsymbol{R}^2)$, $\tilde{g} \in \widehat{H}^4(\boldsymbol{R}^2) \cap H^2_{\log}(\boldsymbol{R}^2) \cap W^{4,1}(\boldsymbol{R}^2)$,

$$\|\tilde{f}\|_{\widehat{H}^5(R^2)} + \|\tilde{g}\|_{\widehat{H}^4(R^2)} + \|\tilde{g}\|_{H^2_{\log}(R^2)} \leq C(\|f\|_{\widehat{H}^5} + \|g\|_{\widehat{H}^4} + \|g\|_{H^2_{\log}}),$$

$$\|\tilde{f}\|_{W^{5,1}(R^2)} + \|\tilde{g}\|_{W^{4,1}(R^2)} \leq C(\|f\|_{W^{5,1}} + \|g\|_{W^{4,1}}).$$

For this, observe that it's enough to take extensions over the bounded set Ω^c with the regularity $\tilde{f} \in H^5(\boldsymbol{R}^2)$, $\tilde{g} \in H^4(\boldsymbol{R}^2)$, since the required behavior at infinity is already furnished by f, g.

Let u_1 be the solution of the Cauchy problem

$$\begin{aligned} &(\partial^2_{tt} - \Delta)\, u_1 = 0 \qquad \text{in } [0, \infty) \times \boldsymbol{R}^2, \\ &u_1(0, x) = \tilde{f}(x), \\ &\partial_t u_1(0, x) = \tilde{g}(x) \quad \text{in } \boldsymbol{R}^2. \end{aligned} \tag{34}$$

From [9], Theorem 2.1, we have

$$|\partial_t u_1(t)|_{L^\infty(R^2)} + |\nabla u_1(t)|_{L^\infty(R^2)} \leq C(1+t)^{-1/2} \|(\nabla\tilde{f}, \tilde{g})\|_{W^{2,1}(R^2)} \leq C(1+t)^{-1/2} \|(\nabla f, g)\|_{W^{2,1}}. \tag{35}$$

Choosing $r > r_0$ and $\chi(x) \in C_0^\infty(\boldsymbol{R}^2)$ so that $\chi(x) = 1$ if $|x| \leq r$ and $= 0$ if $|x| \geq r+1$, we put

$$u_2 = u - (1-\chi)\, u_1, \qquad G = -u_1 \Delta\chi - 2\nabla u_1 \cdot \nabla\chi\,.$$

The function u_2 is the solution of the initial boundary value problem

$$(36)\qquad \begin{aligned} &(\partial^2_{tt} - \Delta)\, u_2 = G && \text{in } Q, \\ &\partial_\nu u_2 = 0 && \text{on } \Sigma, \\ &u_2(0, x) = \chi f(x), && \\ &\partial_t u_2(0, x) = \chi g(x) && \text{in } \Omega. \end{aligned}$$

Observe that $\operatorname{supp} G(t, \cdot) \subseteq \{x \mid r \leqslant |x| \leqslant r+1\}$ for all $t \geqslant 0$, and $\operatorname{supp} \chi f \subseteq \Omega_{r+1}$, $\operatorname{supp} \chi g \subseteq \Omega_{r+1}$. From (20) with $w = u_2$, $w_0 = \chi f$, $w_1 = \chi g$, we obtain

$$(37)\qquad |\partial_t u_2(t)|_{L^\infty(\Omega_{r+2})} + |\nabla u_2(t)|_{L^\infty(\Omega_{r+2})} \leqslant C_r (1+t)^{-1} (\|\nabla(\chi f)\|_{H^2} + \|\chi g\|_{H^2}) + C_r \int_0^t (1+t-s)^{-1} \|G(s)\|_{H^2}\, ds.$$

We estimate $\|G(s)\|_{H^2}$. First of all we observe that

$$\|G(s)\|_{H^2} \leqslant C_r \sum_{|\alpha| \leqslant 3} |\partial^\alpha u_1(s)|_{L^\infty(\Omega_{r+1})}.$$

u_1 is estimated by the $L^1 - L^\infty$ decay estimate (see Klainerman [3])

$$(38)\qquad |u_1(s, \cdot)|_{L^\infty} \leqslant C(1+s)^{-1/2} (\|\tilde f\|_{W^{2,1}} + \|\tilde g\|_{W^{1,1}}).$$

To complete the estimate of $\|G(s)\|_{H^2}$, we apply (35) to $\partial^\alpha u_1(s)$, $|\alpha| \leqslant 2$, in order to obtain

$$(39)\qquad \sum_{1 \leqslant |\alpha| \leqslant 3} |\partial^\alpha u_1(s)|_{L^\infty(\Omega_{r+1})} \leqslant C(1+s)^{-1/2} \|(\nabla f, g)\|_{W^{4,1}}.$$

Thus, from (38) and (39) we get

$$(40)\qquad \|G(s)\|_{H^2} \leqslant C_r (1+s)^{-1/2} (\|f\|_{W^{5,1}} + \|g\|_{W^{4,1}}).$$

We obtain from (37), (40) and (60) in the Appendix that

$$(41)\qquad \begin{aligned} |\partial_t u_2(t)|_{L^\infty(\Omega_{r+2})} + |\nabla u_2(t)|_{L^\infty(\Omega_{r+2})} &\leqslant C_r (1+t)^{-1} (\|\nabla(\chi f)\|_{H^2} + \|\chi g\|_{H^2}) + \\ &\quad C_r \int_0^t (1+t-s)^{-1} (1+s)^{-1/2} (\|f\|_{W^{5,1}} + \|g\|_{W^{4,1}})\, ds \leqslant \\ &\quad C_r M_1 (1+t)^{-1/2} \log(e+t) \qquad \forall t \geqslant 0, \end{aligned}$$

where $M_1 = \|f\|_{W^{5,1}} + \|g\|_{W^{4,1}}$. Observe that by a Sobolev imbedding $\|f\|_{H^3} \leqslant C\|f\|_{W^{4,1}}$, $\|g\|_{H^2} \leqslant C\|g\|_{W^{3,1}}$.

Choosing $\psi(x) \in C_0^\infty(\boldsymbol{R}^2)$ so that $\psi(x) = 1$ if $|x| \geqslant r+2$ and $= 0$ if $|x| \leqslant r + 1$, we observe that

$$\psi\chi f = 0, \qquad \psi\chi g = 0, \qquad \psi G = 0.$$

Let us define

$$H = -u_2 \Delta\psi - 2\nabla u_2 \cdot \nabla\psi.$$

The function ψu_2 solves the Cauchy problem

$$(42) \qquad \begin{aligned} &(\partial_{tt}^2 - \Delta)(\psi u_2) = H \quad \text{in } [0, \infty) \times \boldsymbol{R}^2, \\ &\psi u_2(0, x) = 0, \\ &\partial_t(\psi u_2)(0, x) = 0 \qquad \text{in } \boldsymbol{R}^2. \end{aligned}$$

From (35) and the Duhamel's principle we get

$$(43) \qquad |\partial_t(\psi u_2)(t)|_{L^\infty(R^2)} + |\nabla(\psi u_2)(t)|_{L^\infty(R^2)} \leqslant$$

$$C\int_0^t (1+t-s)^{-1/2} \|H(s, \cdot)\|_{W^{2,1}(R^2)} ds \leqslant C_r \int_0^t (1+t-s)^{-1/2} \|u_2(s)\|_{H^3(\Omega_{r+2})} ds.$$

On the other hand, applying (33) to the solution u_2 of (36) yields

$$(44) \qquad \|u_2(t)\|_{H^3(\Omega_{r+2})} \leqslant C_r(1+t)^{-1}(\|\chi f\|_{H^3} + \|\chi g\|_{H^2}) +$$

$$C_r \int_0^t (1+t-s)^{-1} \|G(s)\|_{H^2} ds.$$

We recall Klainerman's inequality [4] in the plane

$$(45) \quad |u(t, x)| \leqslant C(1+t+|x|)^{-1/2}(1+|t-|x||)^{-1/2} |||u(t)|||_2 \qquad \forall t \geqslant 0$$

which holds for all smooth functions vanishing sufficiently rapidly as $|x| \to \infty$, so that the norm in the right-side is finite for each fixed $t \geqslant 0$. Applying (45) to u_1 with the restriction $x \in \Omega_{r+1}$ gives

$$(46) \qquad \|G(t)\|_{H^2} \leqslant C_r \sum_{|\alpha| \leqslant 3} |\partial^\alpha u_1(t)|_{L^\infty(\Omega_{r+1})} \leqslant C_r(1+t)^{-1} \sum_{|\alpha| \leqslant 3} |||\partial^\alpha u_1(t)|||_2.$$

To estimate the last term we use the following result, whose proof is postponed to the end of this section.

LEMMA 3.1. – *There exists a constant $M_2 > 0$ such that*

$$\sum_{|\alpha| \leqslant 3} |||\partial^\alpha u_1(t)|||_2 \leqslant M_2 \qquad \forall t \geqslant 0.$$

Thus, from (44), (46) and (59) in the Appendix we obtain

$$(47) \qquad \|u_2(t)\|_{H^3(\Omega_{r+2})} \leqslant C_r M_1 (1+t)^{-1} + C_r M_2 \int_0^t (1+t-s)^{-1}(1+s)^{-1} ds \leqslant$$
$$C_r M_1 (1+t)^{-1} + C_r M_2 (1+t)^{-1} \log(1+t) \leqslant$$
$$C_r (M_1 + M_2)(1+t)^{-1} \log(e+t).$$

Then from (43), (47) and (61) in the Appendix one has

$$(48) \qquad |\partial_t(\psi u_2)(t)|_{L^\infty(R^2)} + |\nabla(\psi u_2)(t)|_{L^\infty(R^2)} \leqslant$$
$$C_r (M_1 + M_2) \int_0^t (1+t-s)^{-1/2}(1+s)^{-1} \log(e+s)\, ds \leqslant$$
$$C_r (M_1 + M_2)(1+t)^{-1/2} \log^2(1+t).$$

Moreover, from (21), (40) and (59) in the Appendix we have

$$(49) \qquad |u_2(t)|_{L^\infty(\Omega_{r+2})} \leqslant C_r (1+t)^{-1} (\|\chi f\|_{H^2} + \|\chi g\|_{H^1}) +$$
$$C_r \int_0^t (1+t-s)^{-1} \|G(s)\|_{H^1} ds \leqslant$$
$$C_r M_1 (1+t)^{-1} + C_r M_1 \int_0^t (1+t-s)^{-1}(1+s)^{-1/2} ds \leqslant$$
$$C_r M_1 (1+t)^{-1/2} \log(e+t);$$

using the strongest estimate (47) and a Sobolev imbedding doesn't improve the final result. Since $u = (1-\chi)\, u_1 + u_2$, we have

$$|\partial_t u(t)|_\infty + |\nabla u(t)|_\infty \leqslant$$
$$|(1-\chi)\, \partial_t u_1(t)|_\infty + |\nabla((1-\chi)\, u_1(t))|_\infty + |\partial_t u_2(t)|_\infty + |\nabla u_2(t)|_\infty \leqslant$$
$$|\partial_t u_1(t)|_{L^\infty(R^2)} + |\nabla u_1(t)|_{L^\infty(R^2)} + C|u_1(t)|_{L^\infty(R^2)} +$$
$$|\partial_t(\psi u_2(t))|_{L^\infty(R^2)} + |\nabla(\psi u_2(t))|_{L^\infty(R^2)} +$$
$$|\partial_t u_2(t)|_{L^\infty(\Omega_{r+2})} + |\nabla u_2(t)|_{L^\infty(\Omega_{r+2})} + C|u_2(t)|_{L^\infty(\Omega_{r+2})}.$$

From (35), (38), (41), (48), (49) we finally obtain

$$(50)\qquad |\partial_t u(t)|_\infty + |\nabla u(t)|_\infty \leqslant CM_1(1+t)^{-1/2} + C_r M_1 (1+t)^{-1/2}\log(e+t) + C_r(M_1+M_2)(1+t)^{-1/2}\log^2(1+t) \leqslant C_r(M_1+M_2)(1+t)^{-1/2}\log^2(e+t) \qquad \forall t \geqslant 0.$$

This estimate gives the required decay rate. The final dependence on the norms of the data is given after the following proof.

PROOF OF LEMMA 3.1. – For the sake of brevity here we write u, f, g instead of $u_1, \tilde{f}, \tilde{g}$ and $\|\cdot\|$ instead of $\|\cdot\|_{L^2(R^2)}$. We also set $\partial_0 = \partial_t$, $D = (\partial_0, \partial_1, \partial_2)$. Let us recall the commutation relations [2]

$$(\partial^2_{tt} - \Delta)\Gamma_i - \Gamma_i(\partial^2_{tt} - \Delta) = 2\delta_{0i}(\partial^2_{tt} - \Delta) \quad \text{for } i = 0, \dots, 6$$

$$\Gamma_i \Gamma_j - \Gamma_j \Gamma_i = \sum_{k=0}^{6} c_{ijk}\Gamma_k \quad \text{for } i, j = 0, \dots, 6$$

$$\Gamma_i \partial_j - \partial_j \Gamma_i = \sum_{k=0}^{2} c^*_{ijk}\partial_k \quad \text{for } i = 0, \dots, 6;\ j = 0, 1, 2$$

with certain numerical coefficients c_{ijk}, c^*_{ijk}. Because of the noncommutativity of the Γ_i one has product rules of the type

$$\Gamma^A \Gamma^B = \Gamma^{A+B} + \sum_C \gamma_{ABC}\Gamma^C \quad \text{with } |C| < |A| + |B|,$$

$$[D, \Gamma^A] = \sum_{|B| \leqslant |A|-1} \delta_{AB} D\Gamma^B = \sum_{|B| \leqslant |A|-1} \tilde{\delta}_{AB}\Gamma^B D$$

with numerical coefficients γ_{ABC}, δ_{AB}, $\tilde{\delta}_{AB}$. We first observe that the commutation rule with the wave operator and an energy argument give for every multi-index A

$$(51)\qquad \|\partial_t \Gamma^A u(t)\|^2 + \|\nabla \Gamma^A u(t)\|^2 = \|\partial_t \Gamma^A u(0)\|^2 + \|\nabla\Gamma^A u(0)\|^2 \qquad \forall t \geqslant 0,$$

where in the right side the norms are evaluated at time $t = 0$. It readily follows that

$$(52)\qquad \sum_{1 \leqslant |\alpha| \leqslant 3} |||\partial^\alpha u(t)|||_2 \leqslant C|||\nabla u(t)|||_4 = C\max_{|A| \leqslant 4}\|\Gamma^A \nabla u(t)\| \leqslant C\max_{|A|\leqslant 4}\Big(\|\nabla\Gamma^A u(t)\| + \sum_{|B| \leqslant |A|-1}|\delta_{AB}|\|D\Gamma^B u(t)\|\Big) \leqslant C\max_{|A|\leqslant 4}(\|\partial_t \Gamma^A u(0)\| + \|\nabla\Gamma^A u(0)\|) \leqslant C\|\nabla f, g\|_{\widehat{H}^4(R^2)}.$$

We proceed with the estimate of $|||u(t)|||_2$. Let us define $v(t, x) =$

$\int_0^t u(s, x)\, ds + \phi(x)$, where ϕ is such that $\Delta\phi = g$ in $\boldsymbol{R}^2$. Then v solves

$$\begin{aligned} &(\partial_{tt}^2 - \Delta)\, v = 0 \quad \text{in } [0, \infty) \times \boldsymbol{R}^2, \\ &v(0, \cdot) = \phi, \\ &\partial_t v(0, \cdot) = f \qquad \text{in } \boldsymbol{R}^2. \end{aligned}$$

As in (51) we have

$$\|\partial_t \Gamma^A v(t)\|^2 + \|\nabla \Gamma^A v(t)\|^2 = \|\partial_t \Gamma^A v(0)\|^2 + \|\nabla \Gamma^A v(0)\|^2 \quad \forall t \geq 0. \tag{53}$$

Since $\partial_t v = u$, from (53) we obtain

$$|||u(t)|||_2 = |||\partial_t v(t)|||_2 \leq C|||Dv(0)|||_2. \tag{54}$$

Substituting the initial values of v yields

$$\begin{aligned} |||\partial_t v(0)|||_2 \leq C\,(&\max_{|B| \leq 2} \|\Lambda^B f\| + \sum_i (\|x_i \nabla f\| + \|x_i \Delta f\|) + \\ &\sum_{i,j} \|x_i x_j \Delta f\| + \max_{|B| \leq 1} (\|\Lambda^B \Delta\phi\| + \sum_i \|\Lambda^B (x_i \Delta\phi)\|), \\ |||\nabla v(0)|||_2 \leq C(&\max_{|B| \leq 2} \|\Lambda^B \nabla\phi\| + \sum_i (\|x_i \partial\nabla\phi\| + \|x_i \Delta\nabla\phi\|) + \\ &\sum_{i,j} \|x_i x_j \Delta\nabla\phi\| + \max_{|B| \leq 1} (\|\Lambda^B \nabla f\| + \sum_i \|\Lambda^B (x_i \nabla f)\|). \end{aligned} \tag{55}$$

The terms in (55) containing f are easily estimated by $C\|f\|_{\widehat{H}^2(R^2)}$. The last step consists in estimating $\nabla\phi$ by g. As in the proof of Lemma 2.4 we can take the solution ϕ of $\Delta\phi = g$ such that

$$\|\nabla\phi\| \leq C\|g\|_{L^2_{\log}(R^2)}. \tag{56}$$

By application to $\Delta\phi = g$ of the operators Λ^B, commutation of the operators and (56) we show that the terms in (55) containing ϕ are estimated by $C\|g\|_{H^2_{\log}(R^2)}$; in particular we use the estimates

$$\begin{aligned} &\|x_i \Delta\nabla\phi\| = \|x_i \nabla g\| \leq C\|\nabla g\|_{L^2_{\log}(R^2)} \leq C\|g\|_{H^1_{\log}(R^2)}, \\ &\|x_i x_j \Delta\nabla\phi\| = \|x_i x_j \nabla g\| \leq C\|x_i \nabla g\|_{L^2_{\log}(R^2)} \leq C\|g\|_{H^1_{\log}(R^2)}. \end{aligned}$$

From (54) we then obtain

$$|||u(t)|||_2 \leq C(\|f\|_{\widehat{H}^2(R^2)} + \|g\|_{H^2_{\log}(R^2)}). \tag{57}$$

In the end, write again $u_1, \tilde{f}, \tilde{g}$ instead of u, f, g. (52) and (57) give

$$\sum_{|\alpha| \leq 3} ||| \partial^\alpha u_1(t) |||_2 \leq C(\|\tilde{f}\|_{\widehat{H}^5(R^2)} + \|\tilde{g}\|_{\widehat{H}^4(R^2)} + \|\tilde{g}\|_{H^2_{\log}(R^2)}) \leq \tag{58}$$

$$M_2 := C(\|f\|_{\widehat{H}^5} + \|g\|_{\widehat{H}^4} + \|g\|_{H^2_{\log}}). \qquad \blacksquare$$

END OF THE PROOF OF THEOREM 1.1. – From $M_1 = \|f\|_{W^{5,1}} + \|g\|_{W^{4,1}}$ and (58) we have

$$M_1 + M_2 \leq C(\|f\|_{\widehat{H}^5} + \|f\|_{W^{5,1}} + \|g\|_{\widehat{H}^4} + \|g\|_{H^2_{\log}} + \|g\|_{W^{4,1}}).$$

Substituting in (50) gives the thesis.

4. – Appendix.

We report some elementary estimates used above.

LEMMA 4.1. – *There exists a constant $C > 0$ such that for all $t \geq 0$*

$$\int_0^t (1+t-s)^{-1}(1+s)^{-1}\, ds \leq C(1+t)^{-1}\log(1+t), \tag{59}$$

$$\int_0^t (1+t-s)^{-1}(1+s)^{-1/2}\, ds \leq C(1+t)^{-1/2}\log(1+t), \tag{60}$$

$$\int_0^t (1+t-s)^{-1/2}(1+s)^{-1}\log(e+s)\, ds \leq C(1+t)^{-1/2}\log^2(1+t). \tag{61}$$

PROOF. – Estimates (59) and (60) are proven in [6], see formula (5.49), p. 43; (61) may be proved following the lines of the proof of (5.49).

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