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Rania Wazir

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A Bound for the Average Rank of a Family of Abelian Varieties.

Rania Wazir

Sunto. – *Si considera una famiglia di varietà abeliane $A/\mathbb{Q}(T)$ e si determina un estremo superiore per il rango di Mordell-Weil medio, in termini del rango di Mordell-Weil della fibra generica. Questo risultato è basato su stime di Michel per il rango medio di una famiglia di varietà abeliane, ed estende un lavoro precedente di Silverman sulle superficie ellittiche.*

Summary. – *In this note, we consider a one-parameter family of Abelian varieties $A/\mathbb{Q}(T)$, and find an upper bound for the average rank in terms of the generic rank. This bound is based on Michel's estimates for the average rank in a one-parameter family of Abelian varieties, and extends previous work of Silverman for elliptic surfaces.*

1. – Introduction.

Let $\pi : \mathcal{A} \to \mathbb{P}^1$ be a proper flat morphism of smooth projective varieties defined over $\mathbb{Q}$, with generic fiber an Abelian variety A defined over $\mathbb{Q}(T)$. This can be thought of as a family of Abelian varieties over $\mathbb{Q}$, parametrized by $\mathbb{Z}$. By the Mordell-Weil Theorem, for any Abelian variety defined over a field k finitely generated over $\mathbb{Q}$, the set of rational points is a finitely generated Abelian group; its rank is called the Mordell-Weil rank. In particular, in the case of the fibration $\pi : \mathcal{A} \to \mathbb{P}^1$, this holds both for the generic fiber $A/\mathbb{Q}(T)$, and for the special fibers $\mathcal{A}_t/\mathbb{Q}$.

Very little is known about the Mordell-Weil rank, and there are many open questions and conjectures, even in the «simplest» case of elliptic curves over $\mathbb{Q}$. One area of much recent interest, is to find bounds on the average rank of a one parameter family of Abelian varieties $\mathcal{A}_t$ as t varies in $\mathbb{Z}$, (see, for example, the work on families of elliptic curves of Fouvry-Pomykala [FP93], Michel [Mic95], and Silverman [Sil98], and the work of Michel [Mic97] on families of A belian varieties). In this paper, we use some estimates of Michel to obtain an upper bound for the average rank of the special fibers in terms of the generic rank.

THEOREM 1.1. – *Assume $\mathcal{A}$ satisfies the «standard conjectures» as described below. Furthermore, fix a base point z_0, and suppose that the monodromy representation μ_{z_0} is irreducible. Then, as $T \to \infty$, we have*:

$$\frac{1}{2X} \sum_{|t| \leq X} \text{rank } \mathcal{A}_t(\mathbb{Q}) \leq \left(\frac{\mathcal{L}_X}{2X \log X} + \text{rank } \mathcal{A}(\mathbb{Q}(T)) + \frac{g}{2} \right) (1 + o(1)).$$

This paper was inspired by Professor Joseph Silverman's article [Sil98] on elliptic surfaces, and I would like to thank him for many interesting discussions on elliptic curves and Mordell-Weil ranks. I am also indebted to Professors Alberto Conte and Siman Wong for their help and advice.

1.1. *Definitions.*

For the remainder of this article, we will write simply $\mathcal{A}$ to indicate the fibration $\pi : \mathcal{A} \to \mathbb{P}^1$. Its generic fiber A is an Abelian variety of dimension g, defined over $\mathbb{Q}(T)$. Furthermore, we assume always A has trivial Chow trace, and that $A(\mathbb{Q}(T)) \neq \emptyset$.

Let $D := \{ t \in \mathbb{C} \mid \mathcal{A}_t \text{ is singular} \} \cup \{ \infty \}$, let $P(T) \in \mathbb{Z}[T]$ be the separable polynomial that vanishes on D, and set $d := |D| = \deg(P) + 1$. Henceforth, all sums $\sum_{|t| \leq X}$ will be taken over $t \notin D$.

Let $U := \mathbb{P}^1_{\mathbb{C}} - D$, and fix a base-point $z_0 \in U$; we thus obtain a monodromy representation

$$\mu_{z_0}\colon \pi_1(U, z_0) \to H^1_{\text{ét}}(\mathcal{A}_{z_0}/\overline{\mathbb{Q}}; \mathbb{Q}_l) \cong \mathbb{Q}_l^{2g},$$

which we denote by μ_{z_0}.

If B is an Abelian variety of dimension g, then the conductor of B, N_B, is defined by

$$N_B := \prod_p p^{\delta_p},$$

where the exponent δ_p satisfies (see, for example, [BK94, Theorem 6.2]):

1. $\delta_p = O(g \log g)$ for all p
2. $\delta_p = 0$ if and only if B has good reduction at p
3. $0 < \delta_p \leq g$ if B has semistable reduction at P
4. $\delta_p \leq 2g$ for all $p > 2g + 1$.

For the family of varieties $\mathcal{A}$, we then let

$$\mathcal{L}_X := \sum_{|t| \leq X} \log N_{\mathcal{A}_t}.$$

1.2. *The conjectures.*

The «standard conjectures» relate to the L-series $L(\mathcal{V}, s)$ and $L_2(\mathcal{V}, s)$ that can be attached to $H^1_{\text{ét}}(\mathcal{V}/\overline{k}; \mathbb{Q}_l)$ and $H^2_{\text{ét}}(\mathcal{V}/\overline{k}; \mathbb{Q}_l)$, respectively, of a smooth projective variety $\mathcal{V}$ defined over a field k finitely generated over $\mathbb{Q}$. In the case of varieties $\mathcal{V}/\mathbb{Q}$, these are defined as:

$$L_i(\mathcal{V}, s) := \prod_{\mathfrak{p}} \det\left(1 - \mathrm{Frob}_{\mathfrak{p}} p^{-s} \,|\, H^i_{\text{ét}}(\mathcal{V}/\overline{\mathbb{Q}}; \mathbb{Q}_l)\right)^{-1}.$$

We refer to the articles of Serre [Ser65] and Tate [Tat65] for more general definitions.

CONJECTURE 1.1 (Taniyama-Weil). – *Let B be an Abelian variety defined over $\mathbb{Q}$. Then the L-series $L(B, s)$ can be analytically continued to all of $\mathbb{C}$, and satisfies a functional equation:*

$$\zeta(B, s) = w(B)\, \zeta(B, 1 - s),$$

where

$$w(B) = \pm 1 \qquad \text{and} \qquad \zeta(B, s) := N_B^{s/2}((2\pi)^{-s}\,\Gamma(s))^g\, L(B, s).$$

In the case of an elliptic curve $E/\mathbb{Q}$, this conjecture is a consequence of the Modularity conjecture, which, thanks to the work of Wiles, Taylor-Wiles, Diamond et al, is now a theorem.

CONJECTURE 1.2 (Birch and Swinnerton-Dyer)

$$\operatorname*{ord}_{s=1} L(B, s) = \text{rank } B(\mathbb{Q}).$$

Unfortunately, very little is known for this conjecture. In the case of elliptic curves $E/\mathbb{Q}$, we have the following partial result:

$$\text{If } \operatorname*{ord}_{s=1} L(E, s) \leq 1 \text{ then } \operatorname*{ord}_{s=1} L(E, s) = \text{rank } E(\mathbb{Q}).$$

CONJECTURE 1.3 (Generalized Riemann Hypothesis). – *Assume $B/\mathbb{Q}$ is an Abelian variety for which Conjecture* 1.1 *holds. Then every zero of $L(B, s)$ lies on the line* $\mathrm{Re}\,(s) = 1$.

CONJECTURE 1.4 (Tate's Conjecture). – *Let k be a field finitely generated over $\mathbb{Q}$, and let $\mathcal{V}$ be a smooth projective variety defined over k. Then $L_2(\mathcal{V}, s)$ has a meromorphic continuation to $\mathbb{C}$, and satisfies*

$$-\operatorname*{ord}_{s=2} L_2(\mathcal{V}, s) = \text{rank } \mathrm{NS}(\mathcal{V}/k),$$

where $\mathrm{NS}(\mathcal{V}/k)$ *is the k-rational part of the Neron-Severi group of $\mathcal{V}$. Furthermore, we assume that $L_2(\mathcal{V}, s)$ does not vanish on the line* $\mathrm{Re}\,(s) = 1$; *strictly speaking, this is not traditionally part of Tate's conjecture. However, it is known to hold true in all cases where Tate's Conjecture is known.*

2. – The average rank.

The goal of this section is to prove Theorem 1.1. In order to obtain a bound on the average rank in terms of the generic rank, we start first with some analytic estimates obtained by Michel.

LEMMA 2.1 (Michel [Mic97, Proposition 5.1]). – *If $\pi : \mathcal{A} \to \mathbb{P}^1$ is a family of Abelian varieties with irreducible monodromy representation μ_{z_0}, then*

$$\left| \sum_{t \in \mathbb{F}_p} \operatorname{Trace}(\operatorname{Frob}_\mathfrak{p} \,|H^1_{\text{ét}}(\mathcal{A}_t/\overline{\mathbb{Q}};\, \mathbb{Q}_l)) \right| \leqslant 2g(d-2)\, p$$

Furthermore, μ_{z_0} is irreducible for any family $\mathcal{A}$ arising as the Jacobian fibration attached to a Lefschetz fibration $f : \mathcal{Y} \to \mathbb{P}^1$ with $\dim H^1_{\text{ét}}(\mathcal{Y}/\overline{\mathbb{Q}};\, \mathbb{Q}_l) = 0$.

As we will be using the Frobenius trace above frequently, we define, for any smooth, projective variety $\mathcal{V}/\mathbb{Q}$:

$$a_p(\mathcal{V}) := \operatorname{Trace}(\operatorname{Frob}_\mathfrak{p} \,|H^1_{\text{ét}}(\mathcal{V}/\overline{\mathbb{Q}};\, \mathbb{Q}_l)),$$

$$b_p(\mathcal{V}) := \operatorname{Trace}(\operatorname{Frob}_\mathfrak{p} \,|H^2_{\text{ét}}(\mathcal{V}/\overline{\mathbb{Q}};\, \mathbb{Q}_l)).$$

For singular varieties, we set $a_p(\mathcal{V}) = b_p(\mathcal{V}) = 0$. Furthermore, we will be most interested in averaging the Frobenius traces over the special fibers, and so we let:

$$\mathfrak{A}_p(\mathcal{A}) := \frac{1}{p^m} \sum_{x \in \mathbb{F}_p} a_p(\mathcal{A}_x),$$

$$\mathfrak{B}_p(\mathcal{A}) := \frac{1}{p^m} \sum_{x \in \mathbb{F}_p} b_p(\mathcal{A}_x).$$

It is important to note that, from Lemma 2.1 above, it follows that $\mathfrak{A}_p(\mathcal{A})$ is bounded independently of p.

The following estimate is based on a generalization to Abelian varieties of Mestre's explicit formula for elliptic curves. Define first the test function

$$F_\lambda(x) := \max\left\{0,\, 1 - \left|\, \frac{x}{\lambda} \,\right|\right\}.$$

LEMMA 2.2 (Michel [Mic97, Lemma 6.1 and 7.1 ff]). – *Suppose $\pi : \mathcal{A} \to \mathbb{P}^1$ is a family of Abelian varieties such that Conjectures* 1.1, 1.2, *and* 1.3 *hold for all special fibers $\mathcal{A}_t$. Then, for any $\lambda > 1$, and the function F_λ as defined* *above, we have:*

$$\lambda \sum_{|t| \leqslant X} \operatorname{rank} \mathcal{A}_t(\mathbb{Q}) \leqslant \mathcal{L}_X - 2 \sum_{\mathfrak{p}} F_\lambda(\log p) \log p \mathfrak{A}_p(\mathcal{A}) + $$

$$+ k_{\mathcal{A}} \frac{\lambda}{2}(1 + o(1)) 2X + O(X) + O(e^{\lambda/2}),$$

where $|k_{\mathcal{A}}| \leqslant g$, *and in the case that* $\mathcal{A}$ *arises from a Lefschetz fibration,* $k_{\mathcal{A}} = 1$.

To simplify notation, we let

$$S(X) := \sum_{\mathfrak{p}} F_\lambda(\log p) \log p \mathfrak{A}_p(\mathcal{A}).$$

In order to bound the term $S(X)$, we will need the following result:

THEOREM 2.1 (Wazir [Waz03, Corollary 6.1]). – *If Conjecture* 1.4 *holds for* $\mathcal{A}$, *then*

$$\lim_{X \to \infty} \frac{1}{X} \sum_{p \leqslant X} -\mathfrak{A}_p(\mathcal{A}) \log p + \lim_{X \to \infty} \frac{1}{X} \sum_{p \leqslant X} \mathfrak{B}_p(\mathcal{A}) \frac{\log p}{p} =$$

$$= \operatorname{rank} A(\mathbb{Q}(T)) + \operatorname{rank} \operatorname{NS} A(\mathbb{Q}(T)).$$

Assuming further that Conjecture 1.4 is also true for the generic fiber A, we get

$$\lim_{X \to \infty} \frac{1}{X} \sum_{p \leqslant X} \mathfrak{B}_p(\mathcal{A}) \frac{\log p}{p} = \operatorname{rank} \operatorname{NS} A(\mathbb{Q}(T)) \tag{1}$$

and therefore

$$\lim_{X \to \infty} \frac{1}{X} \sum_{p \leqslant X} -\mathfrak{A}_p(\mathcal{A}) \log p = \operatorname{rank} A(\mathbb{Q}(T)). \tag{2}$$

We are finally ready to estimate $S(X)$. Proceeding as in Michel, Fouvry-Pomykala, and Silverman, we split the interval $|t| \leqslant X$ into $\left[\frac{2X+1}{p}\right]$ intervals of length p, and one interval, I_p, of length at most $p-1$. Noting also that $F_\lambda(\log p) = 0$ when $p > e^\lambda$, we rewrite the sum $S(X)$ as:

$$S(X) = \sum_{p \leqslant e^\lambda} F_\lambda(\log p)(\log p) \left[\frac{2X+1}{p}\right] \mathfrak{A}_p(\mathcal{A}) + \tag{3}$$

$$+ \sum_{p \leqslant e^\lambda} F_\lambda(\log p) \frac{\log p}{p} \sum_{t \in I_p} a_p(\mathcal{A}_t).$$

Using additive characters and Lemma 2.1, we find

$$\sum_{p \leqslant e^{\lambda}} F_{\lambda}(\log p) \frac{\log p}{p} \sum_{t \in I_p} a_p(\mathcal{A}_t) = O(\lambda e^{\lambda}). \tag{4}$$

Furthermore, recall that $\mathfrak{A}_p(\mathcal{A})$ is bounded independently of p; therefore, removing the greatest integer brackets in the first term introduces an error of the order

$$\sum_p F_{\lambda}(\log p)(\log p) = O(e^{\lambda}).$$

Thus, it remains to estimate

$$\sum_{p \leqslant e^{\lambda}} -F_{\lambda}(\log p) \frac{\log p}{p} \mathfrak{A}_p(\mathcal{A}) = \sum_{p \leqslant e^{\lambda}} -\frac{\log p}{p} \mathfrak{A}_p(\mathcal{A}) + \frac{1}{\lambda} \sum_{p \leqslant e^{\lambda}} \frac{(\log p)^2}{p} \mathfrak{A}_p(\mathcal{A}).$$

Using Abel's identity [Apo76, Theorem 4.2], we have:

$$\sum_{p \leqslant e^{\lambda}} -\frac{\log p}{p} \mathfrak{A}_p(\mathcal{A}) = \frac{1}{e^{\lambda}} \sum_{p \leqslant e^{\lambda}} -\mathfrak{A}_p(\mathcal{A}) \log p + \int_1^{e^{\lambda}} \left(\frac{1}{x} \sum_{p \leqslant x} -\mathfrak{A}_p(\mathcal{A}) \log p \right) \frac{1}{x}\, dx$$

$$\frac{1}{\lambda} \sum_{p \leqslant e^{\lambda}} \frac{(\log p)^2}{p} \mathfrak{A}_p(\mathcal{A}) =$$

$$= -\frac{1}{e^{\lambda}} \sum_{p \leqslant e^{\lambda}} -\mathfrak{A}_p(\mathcal{A}) \log p + \frac{1}{\lambda} \int_1^{e^{\lambda}} \left(\frac{1}{x} \sum_{p \leqslant x} -\mathfrak{A}_p(\mathcal{A}) \log p \right) \frac{1 - \log x}{x}\, dx.$$

Summing this, we get:

$$\sum_p -F_{\lambda}(\log p)\, \mathfrak{A}_p(\mathcal{A}) \log p = \tag{5}$$

$$= \left(\frac{\lambda + 1}{\lambda} \int_1^{e^{\lambda}} \frac{1}{x}\, dx - \frac{1}{\lambda} \int_1^{e^{\lambda}} \frac{\log x}{x}\, dx \right) (1 + o(1)) \operatorname{rank} A(\mathbb{Q}(T))$$

$$= \left(\frac{\lambda}{2} + 1 \right) (1 + o(1)) \operatorname{rank} A(\mathbb{Q}(T))$$

The equations (4) and (5) give the following estimate for $S(X)$:

$$S(X) = -(2X + 1) \operatorname{rank} A(\mathbb{Q}(T)) (1 + o(1)) \frac{\lambda}{2} + O(\lambda e^{\lambda}). \tag{6}$$

Combining this with Lemma 2.2, we obtain:

$$\frac{1}{2X}\sum_{|t|\leq X}\text{rank } \mathcal{A}_t(\mathbb{Q}) \leq \frac{\mathcal{L}_X}{2X\lambda} + (1+o(1)) \text{ rank } A(\mathbb{Q}(T)) + $$
$$+O\left(\frac{e^{\lambda}}{2X}\right) + \frac{g}{2}(1+o(1)) + O\left(\frac{1}{\lambda}\right) + O\left(\frac{e^{\lambda}}{\lambda X}\right).$$

Finally, by setting $\lambda = \log X - \log\log X$, we have

$$\frac{1}{2X}\sum_{|t|\leq X}\text{rank } \mathcal{A}_t(\mathbb{Q}) \leq \frac{\mathcal{L}_X}{2X(\log X - \log\log X)} +$$
$$+(1+o(1)) \text{ rank } A(\mathbb{Q}(T)) + \frac{g}{2} + o(1),$$

and the theorem follows, letting $X \to \infty$.

We would also like to obtain a bound on $\dfrac{\mathcal{L}_X}{\log X}$. For this purpose, let

$$S := \{s \in D \mid \mathcal{A}_s \text{ has additive reduction}\},$$

and define a «conductor polynomial»

$$N(T) := P(T) \prod_{s\in S}(T-s) = \prod_{s\in D}(T-s)\prod_{s\in S}(T-s).$$

By considering the properties of δ_p, we obtain:

$$\begin{aligned}\log N_{\mathcal{A}_t} &= \sum_p \delta_p \log p\\ &\leq \sum_{p|P(t)} g\log p + \sum_{p|P(t)} 2g\log p + \sum_{p<2g+1} g\log g\log p\\ &\leq g\log N(t) + O(g\log g\log Z),\end{aligned}$$

where $Z := \prod_{p\leq 2g+1} p$. Therefore

$$\sum_{|t|\leq X}\log N_{\mathcal{A}_t} \leq g\deg N(T)(1+o(1))2X \leq 2g(d-1)(1+o(1))2X. \tag{7}$$

Putting this estimate together with Theorem 1.1, we obtain the following bounds for average rank:

$$\frac{1}{2X}\sum_{|t|\leq X}\text{rank } \mathcal{A}_t(\mathbb{Q}) \leq \left(g\deg N(T) + \text{rank } \mathcal{A}(\mathbb{Q}(T)) + \frac{g}{2}\right)(1+o(1))$$
$$\leq \left(2g(d-1) + \text{rank } \mathcal{A}(\mathbb{Q}(T)) + \frac{g}{2}\right)(1+o(1)).$$

In the case of a Lefschetz fibration, this estimate can be improved, as will be shown below.

3. – Lefschetz fibrations.

We now consider a special family of Abelian varieties: a family of the form $\mathrm{Jac}\,(Y)/\mathbb{Q}(T)$, where Y is the generic fiber of a Lefschetz fibration, and $\mathrm{Jac}\,(Y)$ has trivial Chow trace.

To be more precise, let $\mathcal{Y}$ be a smooth, projective surface defined over $\mathbb{Q}$, with Lefschetz fibration $f : \mathcal{Y} \to \mathbb{P}^1$. Then f is a semi-stable fibration, with section $\sigma : \mathbb{P}^1 \to \mathcal{Y}$. Let $\mathrm{Jac}\,(Y)$ be the Jacobian Variety of the generic fiber Y, and set $\mathcal{A} \to \mathbb{P}^1$ the Néron model of $\mathrm{Jac}\,(Y)$. Then $A(\mathbb{Q}(T)) \neq \emptyset$, and we have, for all smooth fibers $\mathcal{Y}_t$, $\mathrm{Jac}\,(\mathcal{Y}_t) = \mathcal{A}_t$. If furthermore $\mathrm{Jac}\,(Y)$ has trivial Chow trace, it follows that the first Betti number

$$b_1(\mathcal{Y}, l) := \dim\,(H^1_{\text{ét}}(\mathcal{Y}/\overline{\mathbb{Q}};\, \mathbb{Q}_l)) = \dim\,(H^1_{\text{ét}}(\mathbb{P}^1/\overline{\mathbb{Q}};\, \mathbb{Q}_l)) = 0,$$

and, in fact, the two conditions are equivalent. See, for example, [Shi99, Theorem 3].

THEOREM 3.1. – *Let $f : \mathcal{Y} \to \mathbb{P}^1$, $\mathcal{A} \to \mathbb{P}^1$ be as defined above, and assume also that Conjecture* 1.4 *holds for $\mathcal{Y}$; then, as $X \to \infty$, we have:*

$$\frac{1}{2X}\sum_{|t|\leqslant X}\mathrm{rank}\;\mathcal{A}_t(\mathbb{Q}) \leqslant \left(\frac{\mathcal{L}_X}{2X\log X} + \mathrm{rank}\;\mathcal{A}(\mathbb{Q}(T)) + \frac{1}{2}\right)(1+o(1)).$$

Since we are assuming that Conjecture 1.4 holds for $\mathcal{Y}$, [Won02, Theorem 5] gives:

$$\lim_{X\to\infty}\frac{1}{X}\sum_{p\leqslant X} -\mathfrak{A}_p(\mathcal{Y})\log p = \mathrm{rank}\;A(\mathbb{Q}(T)).$$

The fiber $\mathcal{Y}_t$ has good reduction at p exactly when $\mathrm{Jac}\,(\mathcal{Y}_t)$ has good reduction at p, so $a_p(\mathcal{Y}_t) = a_p(\mathcal{A}_t)$ by [Mil86, Corollary 9.6], and therefore $\mathfrak{A}_p(\mathcal{Y}) = \mathfrak{A}_p(\mathcal{A})$. The theorem follows as before.

In order to improve the bound on $\frac{\mathcal{L}_X}{\log X}$ given in Equation 2.7, we note that, for a Lefschetz fibration, we have:

i. $\delta_p(\mathcal{A}_t) = 0$ when $\mathcal{Y}_t$ (and hence also $\mathcal{A}_t$) has good reduction at p

ii. $\delta_p(\mathcal{A}_t) = 1$ when $\mathcal{Y}_t$ has bad reduction at p, and $p > 2g+1$.

From this it follows, by the same calculation as before, that

$$\frac{\mathcal{L}_X}{\log X} \leqslant (d-1)2X(1+o(1))$$

and therefore

$$\frac{1}{2X}\sum_{|t|\leqslant X} \text{rank } \mathcal{A}_t(\mathbb{Q}) \leqslant \left(d - \frac{1}{2} + \text{rank } \mathcal{A}(\mathbb{Q}(T))\right)(1+o(1)). \tag{8}$$

4. – Examples.

We describe here some examples of families of Abelian varieties for which Tate's Conjecture is known, and use the formulas of this paper to compute their average rank. Since very little is known about Tate's Conjecture in varieties of dimension three or more, we restrict ourselves to considering elliptic surfaces, and families of Jacobians arising from a Lefschetz fibration of a surface over $\mathbb{P}^1$.

For the next two examples, let $\mathcal{E}$ be a non-split elliptic surface, with section, defined over $\mathbb{Q}$. For elliptic surfaces, non-split is equivalent to non-constant j-invariant, and this in turn implies that the monodromy representation is irreducible. Furthermore, Conjecture 1 holds for all the fibers.

EXAMPLE 4.1. – *Suppose $\mathcal{E}/\mathbb{Q}$ is a $\mathbb{Q}$-rational elliptic surface (i.e. birational, over $\mathbb{Q}$, to $\mathbb{P}^2$). Then Conjecture* 1.4 *holds for $\mathcal{E}$, and assuming that Conjectures* 1.2 *and* 1.3 *hold for all nonsingular fibers, we have:*

$$\frac{1}{2X}\sum_{|t|\leqslant X} \text{rank } \mathcal{E}_t(\mathbb{Q}) \leqslant \left(\frac{\mathcal{L}_X}{2X\log X} + \text{rank } \mathcal{E}(\mathbb{Q}(T)) + \frac{1}{2}\right)(1+o(1)).$$

Furthermore, rational elliptic surfaces over $\mathbb{Q}$ have no additive reduction (i.e. $S=\emptyset$, and $\deg N(T) = \deg P(T)$), and

$$\text{rank } \mathcal{E}(\mathbb{Q}(T))) \leqslant \text{rank } \mathcal{E}(\overline{\mathbb{Q}}(T)) \leqslant 8 \quad \text{[Shi92, p. 110]}.$$

Therefore,

$$\frac{1}{2X}\sum_{|t|\leqslant X} \text{rank } \mathcal{E}_t(\mathbb{Q}) \leqslant \left(d + 7 + \frac{1}{2}\right)(1+o(1)).$$

We consider next a concrete example, shown by Rosen and Silverman [Remark 4.1.2] to satisfy Conjecture 1.4, and to have generic rank 2.

EXAMPLE 4.2. – *Let $\mathcal{E} \to \mathbb{P}^1$ be the elliptic surface defined by the Weierstrass Equation*

$$y^2 = x^3 - 33s^4x + s^4(8s^2 + 1).$$

Then $\mathcal{E}$ has j-invariant given by:

$$j(\mathcal{E}) = \frac{9199872s^4}{5260s^4 - 16s^2 - 1},$$

which is clearly not constant over $\mathbb{Q}$. Therefore, assuming the fibers of $\mathcal{E}$ satisfy Conjectures 1.2 *and* 1.3, *we obtain the following estimate on the average rank:*

$$\frac{1}{2X}\sum_{|t|\leqslant X} \text{rank } \mathcal{E}_t(\mathbb{Q}) \leqslant \left(\deg N(T) + 2 + \frac{1}{2}\right)(1 + o(1)).$$

Our next examples will be families of Abelian varieties arising as Jacobians of a Lefschetz fibration $f : \mathcal{Y} \to \mathbb{P}^1$, where $\mathcal{Y}$ is a smooth projective surface with $H^1_{\text{ét}}(\mathcal{Y}/\overline{\mathbb{Q}}; \mathbb{Q}_l) = 0$, and for which Conjecture 1.4 is known to hold. For this purpose, we provide a list of surfaces where the conjecture is known to be true, and whose first Betti number is trivial:

1. rational surfaces ($\kappa(\mathcal{Y}) = -\infty$).
2. $K3$ surfaces of type CM. ($\kappa(\mathcal{Y}) = 0$).
3. Fermat surfaces. ($\kappa(\mathcal{Y}) = 2$ for surfaces of degree 4 or higher).

The first item is proven in [RS98, Theorem 1.8]; the rest of the list is extracted from [Ram89], where also other examples of surfaces for which Conjecture 1.4 holds can be found.

In the case of a Lefschetz fibration $f : \mathcal{Y} \to \mathbb{P}^1$, [Kat74, Proposition 3.2.10] shows that d does not depend on the choice of fibration, but only on the surface $\mathcal{Y}$, and on the choice of embedding $f : \mathcal{Y} \hookrightarrow \mathbb{P}^N$; in particular,

$$d = b_2(\mathcal{Y}, l) + 2 + \delta + 4(g - 1). \tag{9}$$

where δ is the cardinality of the intersection between $\mathcal{Y}$ and the axis of the fibration.

EXAMPLE 4.3. – *Let $\mathcal{Y}_m$ be the Fermat Surface of degree m in $\mathbb{P}^3$, given by the equation:*

$$X^m + Y^m + Z^m + U^m = 0.$$

In this case, we have $b_2(\mathcal{Y}, l) = 1$, hence $d = 3 + \delta + 4(g - 1)$. If we assume *also* $\gcd(m, 6) = 1$, *then by* [Shi82, *Example* 4.3],

$$\text{rank NS}(\mathcal{Y}_m/\overline{\mathbb{Q}}) = 3(m-1)(m-2)+1.$$

Furthermore, by the Shioda-Tate formula [Shi99, *Theorem* 3],

$$\text{rank }(A(\mathbb{Q}(T))) \leqslant \text{rank }(A(\overline{\mathbb{Q}}(T)) \leqslant \text{rank } NS(\mathcal{Y}_m/\overline{\mathbb{Q}}) - 2.$$

We thus obtain the following estimate on average rank:

$$\frac{1}{2X}\sum_{|t|\leqslant X}\text{rank } \mathcal{A}_t(\mathbb{Q}) \leqslant \left(4g+\delta+3(m-1)(m-2)-\frac{5}{2}\right)(1+o(1)).$$

EXAMPLE 4.4. – *For this final example, we consider a Lefschetz fibration on the elliptic modular surface with level M structure* $\mathcal{E}[M]$ *over* $\mathbb{Q}$ (*note, we are not giving* $\mathcal{E}[M]$ *its elliptic surface fibration structure!*) *For* $M \geqslant 4$, *Wong* [Won02, *Theorem* 4] *shows that* $\mathcal{E}[M]$ *satisfies the Tate Conjecture. Furthermore, when* $M=4$ *or* 5, *the first Betti number* $b_1(\mathcal{E}[M], l) = 0$. *Fix* $M=4$, *and let* $\mathcal{Y} = \mathcal{E}[4]$. *We use* [Shi72, *p.*38] *to determine the geometric invariants of* $\mathcal{E}$:

$$\begin{aligned}\text{rank NS}(\mathcal{Y}/\overline{\mathbb{Q}}) &= 20\\ p_g(\mathcal{Y}) &= 1\\ b_2(\mathcal{Y}) &= \text{rank NS}(\mathcal{Y}/\overline{q}) + 2p_g(\mathcal{Y}) = 22\,.\end{aligned}$$

As in the previous example, we have the bound $\text{rank }(A(\mathbb{Q}(T))) \leqslant \text{rank } NS(\mathcal{Y}/\overline{\mathbb{Q}}) - 2 = 18$, *and therefore*

$$\frac{1}{2X}\sum_{|t|\leqslant X}\text{rank } \mathcal{A}_t(\mathbb{Q}) \leqslant 4g+\delta+35+\frac{1}{2}\,.$$

REFERENCES

[Apo76] T. APOSTOL, *Introduction to analytic number theory*, Undergrad. Texts Math., Springer-Verlag, Berlin, 1976.

[BK94] A. BRUMER - K. KRAMER, *The conductor of an abelian variety*, Comp. Math, **92** (1994), 227-248.

[BLR90] S. BOSCH - W. LUTKEBOHMERT - M. RAYNAUD, *Neron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Bd. 21, Springer Verlag, Berlin, 1990.

[Del81] P. DELIGNE, *La conjecture de Weil II*, Publ. Math. IHES, **52** (1981), 313-428.

[FP93] E. FOUVRY - J. POMYKALA, *Rang des courbes elliptiques et sommes d'exponentielles*, Monatsh. Math., **116** (1993), 115-125.

[Kat74] N. M. KATZ, *Etude cohomologique des pinceaux de Lefschetz*, SGA 7 II, LNM 340, Springer Verlag, 1974, pp. 254-327.

[Mic95] P. MICHEL, *Rang moyen de familles de courbes elliptiques et lois de Sato-Tate*, Monatsh. Math., **120** (1995), 127-136.

[Mic97] P. MICHEL, *Le rang de familles de variétés abéliennes*, J. Alg. Geom., **6** (1997), 201-234.

[Mil86] J. S. MILNE, *Jacobian varieties*, Arithmetic Geometry, Springer-Verlag, Berlin (1986), 167-212.

[Ram89] D. RAMAKRISHNAN, *Regulators, algebraic cycles, and values of L-functions*, Algebraic K-theory and Algebraic Number Theory (M. R. Stein and R. K. Dennis, eds.), Contemp. Math., vol. 83, AMS, Providence, 1989, pp. 183-307.

[RS98] M. ROSEN - J. SILVERMAN, *On the rank of an elliptic surface*, Invent. Math., **133** (1998), 43-67.

[Ser65] J.-P. SERRE, *Zeta and L functions*, Harper and Row, New York, 1965, pp. 82-92.

[Shi72] T. SHIODA, *On elliptic modular surfaces*, J. Math. Soc. Japan, **24** (1972), 20-59.

[Shi82] T. SHIODA, *On the Picard number of a Fermat surface*, J. Fac. Sci. Univ. Tokyo, Sec. IA, **28** (1982), 725-734.

[Shi92] T. SHIODA, *Some remarks on elliptic curves over function fields*, Astérisque, **209** (1992), 99-114.

[Shi99] T. SHIODA, *Mordell-Weil lattices for higher genus fibration over a curve*, New trends in algebraic geometry (Warwick, 1996), London Math. Soc. Lecture Note Ser., 264, Cambridge Univ. Press, Cambridge, 1999, pp. 359-373.

[Sil94] J. H. SILVERMAN, *Advanced topics in the arithmetic of elliptic curves*, Graduate Texts in Mathematics, vol. 151, Springer Verlag, 1994.

[Sil98] J. H. SILVERMAN, *The average rank of elliptic curves*, J. reine angew. Math., **504** (1998), 227-236.

[Tat65] J. TATE, *Algebraic cycles and poles of zeta functions*, Arithmetical Algebraic Geometry, Harper and Row, New York, 1965, pp. 93-110.

[Waz03] R. WAZIR, *A local-global summation formula for Abelian varieties*, Preprint, available on http://www.arxiv.org/abs/math.NT/0302266 (Feb. 2003).

[Won02] S. WONG, *On the Néron-Severi groups of fibered varieties*, To appear, J. reine Angew. Math. (2002)

Dipartimento di Matematica, Università degli Studi di Torino
Via Carlo Alberto, 10 - 10123 Torino (Italy).
E-mail: wazir@dm.unito.it

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