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Grassmann Defective Surfaces.

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Sunto. – *Una varietà proiettiva si dice* $(1, h)$*-difettiva se la dimensione della Grassmanniana delle rette contenute negli h-piani* $(h+1)$*-secanti la varietà ha dimensione minore dell'ordinario. Nel presente lavoro, ispirato a una classica nota di Alessandro Terracini, si dimostra un criterio di* $(1, h)$*-difettività per superficie algebriche e si presentano alcune sue conseguenze: in particolare, si deduce che l'immersione di Veronese di grado abbastanza alto di una superficie liscia con gruppo di Picard isomorfo a* $\mathbb{Z}$ *non è* $(1, h)$*-difettiva, estendendo così il risultato ottenuto per* $\mathbb{P}^2$ *dallo stesso Terracini, e si esibiscono nuovi esempi di superficie rigate* $(1, h)$*-difettive.*

Summary. – *A projective variety V is* $(1, h)$*-defective if the Grassmannian of lines contained in the span of* $h+1$ *independent points on V has dimension less than the expected one. In the present paper, which is inspired by classical work of Alessandro Terracini, we prove a criterion of* $(1, h)$*-defectivity for algebraic surfaces and we discuss its applications to Veronese embeddings and to rational normal scrolls.*

1. – Introduction.

Here we study complex projective surfaces from the point of view of Grassmann defectivity. Roughly speaking (but a precise definition is stated at the beginning of section 2), given an algebraic surface S embedded in some projective space we are interested in the dimension of the Grassmannian of lines contained in the span of $h+1$ independent points on S. Of course there is a naïve expectation suggested by an easy parameter count, but already in the nineteen century it was clear that the expected dimension is not necessarily attained. The exceptional surfaces are said to be $(1, h)$-defective. For instance, in the paper [9] published by London in 1890, the 3-Veronese embedding of $\mathbb{P}^2$ is claimed to be $(1, 4)$-defective (see Remark 2.2 in [8] for a modern proof of this fact). Indeed, the study of Grassmann defectivity for Veronese embeddings of $\mathbb{P}^2$ historically arose as a variation of the so-called Waring problem (see Problem 7.6 in [5]) and it was tackled from this point of view by various authors, among whom we wish to mention at least Alessandro Terracini. In his beautiful paper [10], going back to 1915, he was able to prove that

London's example is the unique $(1, h)$-defective Veronese embedding of $\mathbb{P}^2$. The interested reader can find in [8] a modern revisitation of this classical result. Terracini's method turns out to be quite powerful and it suggests more general applications. Indeed, the present paper is entirely devoted to working out Terracini's ideas in the case of arbitrary algebraic surfaces.

In particular, after recalling some basic facts from [8], we present in section 2 a compact characterization of $(1, h)$-defective surfaces. As a consequence, it follows that Terracini's theorem holds for any smooth algebraic surface S with $\mathrm{Pic}(S) \cong \mathbb{Z}$: namely, in section 3 we prove that high degree Veronese embeddings of such surfaces are never $(1, h)$-defective. Furthermore, our characterization can be also applied as an effective tool for producing examples: for instance, in section 4 we exhibit a new series of examples of $(1, h)$-defective surfaces, which are rational normal scrolls satisfying simple numerical conditions.

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2. – Preliminaries and the main result.

Let $V \subset \mathbb{P}^r$ be an integral nondegenerate projective variety of dimension n defined over the complex field $\mathbb{C}$.

DEFINITION 1. – *The h-*secant variety $\mathrm{Sec}_h(V)$ *of V is the Zariski closure of the set $\{p \in \mathbb{P}^r$: p lies in the span of $h+1$ independent points of $V\}$. V is h-*defective *with h-defect $\delta_h(V)$ if*

$$\delta_h(V) = \min\{(n+1)(h+1) - 1, r\} - \dim \mathrm{Sec}_h(V) > 0 .$$

DEFINITION 2. – *The (k, h)-*Grassmann secant variety $\mathrm{Sec}_{k,h}(V)$ *of V is the Zariski closure of the set $\{l \in \mathbb{G}(k, r)$: l lies in the span of $h+1$ independent points of $V\}$. V is (k, h)-*defective *with (k, h)-defect $\delta_{k,h}(V)$ if*

$$\delta_{k,h}(V) = \min\{(h+1)\,n + (k+1)(h-k), (k+1)(r-k)\} - \dim \mathrm{Sec}_{k,h}(V) > 0 .$$

We recall the following results, taken from [8] but essentially already contained in [10].

PROPOSITION 1. – *Let $V \subset \mathbb{P}^r$ be an integral nondegenerate projective variety of dimension n. Let $\sigma : \mathbb{P}^k \times V \to \mathbb{P}^{r(k+1)+k}$ be the Segre embedding of $\mathbb{P}^k \times V$. Then V is (k, h)-defective with defect $\delta_{k,h}(V) = \delta$ if and only if $\sigma(\mathbb{P}^k \times V)$ is h-defective with defect $\delta_h(\sigma(\mathbb{P}^k \times V)) = \delta$.*

LEMMA 1. – *Let $V \subset \mathbb{P}^r$ be an integral nondegenerate projective variety of dimension n. For $k \leqslant r$, let $\sigma : \mathbb{P}^k \times V \to \mathbb{P}^{r(k+1)+k}$ be the Segre embedding of $\mathbb{P}^k \times V$. Fix $p^{(0)}, \dots, p^{(h)}$ general points on V and $\lambda^{(0)}, \dots, \lambda^{(h)}$ general points in $\mathbb{P}^k$, so that $P^{(j)} := (\lambda_0^{(j)} p^{(j)}, \dots, \lambda_k^{(j)} p^{(j)})$ is a general point on $\sigma(\mathbb{P}^k \times V) \subset \mathbb{P}^{r(k+1)+k}$ for $j = 0 \dots h$; finally, take a general point $P \in \langle P^{(0)}, \dots, P^{(h)} \rangle$. Then there is a natural identification between*:

- *hyperplanes $H \subset \mathbb{P}^{r(k+1)+k}$ such that $T_P(\mathrm{Sec}_h(\sigma(\mathbb{P}^k \times V))) \subset H$;*

- *k-dimensional linear systems $\mathcal{H}$ of hyperplane sections of $V \subset \mathbb{P}^r$ with a projectivity $\omega : \mathcal{H} \to \mathbb{P}^k$ such that all the elements of the linear system pass through the points $p^{(j)} \in V$ and for every j the hyperplane section of the linear system corresponding to $\lambda^{(j)}$ is tangent to V at $p^{(j)}$.*

Let $S \subset \mathbb{P}^r$ be a nondegenerate integral algebraic surface. If H is a hyperplane section of S, then $\mathbb{P}^r = \mathbb{P}V$, where $V \subseteq H^0(S, \mathcal{O}_S(H))$ and equality holds if and only if $S \subset \mathbb{P}^r$ is linearly normal. Moreover, $\mathbb{P}V$ naturally identifies to the hyperplane linear system of S and for any effective divisor D on S the hyperplane sections of S passing through D form a linear system $\mathbb{P}V(-D)$.

THEOREM 1. – *The surface S is $(1, h)$-defective with defect δ if and only if*

- *either the linear system $\mathcal{L}(2^{h+1})$ of hyperplane sections of S with $h+1$ assigned general double points contains exactly $2r - 4h + \delta - 2$ independent pencils. In this case S sits in a $(s+2)$-dimensional cone over a curve with vertex a linear space of dimension $s \leqslant h-1$, and $r \geqslant 2h + s + 3$;*

- *or there exists an $(h+1)$-dimensional involution $\mathcal{D}$ on S such that $2 \dim \mathbb{P}V(-D) = h + 2r + 1 - \min\{4(h+1) - 1, 2r+1\} + \delta - 1$, where D is a general divisor in $\mathcal{D}$.*

PROOF. – By Proposition 1, S is $(1, h)$-defective with defect δ if and only if $\sigma(\mathbb{P}^1 \times S) \subset \mathbb{P}^{2r+1}$ is h-defective with defect δ, i. e.

$$\dim \mathrm{Sec}_h(\sigma(\mathbb{P}^1 \times S)) = 2r + 1 - c < 4(h+1) - 1 \tag{1}$$

where $c := 2r + 1 - \min\{4(h+1) - 1, 2r+1\} + \delta \geqslant 1$.

Hence if we take a general point P as in the statement of Lemma 1, then $T_P(\mathrm{Sec}_h(\sigma(\mathbb{P}^1 \times S)))$ is contained in exactly c independent hyperplanes H_t $(1 \leqslant t \leqslant c)$. By Lemma 1, each H_t gives rise to a pencil $\mathcal{F}_{Ht}$ of hyperplane sections of S all passing through $h+1$ general points $p^{(0)}, \dots, p^{(h)}$ of S in such a way that for every $p^{(j)}$ at least one curve of the pencil passes doubly through $p^{(j)}$. By an infinitesimal Lemma already known to Terracini and reproved in modern times by Ciliberto and Hirschowitz in [6], we have that every H_t is tangent to $\sigma(\mathbb{P}^1 \times S)$ along a positive dimensional variety Σ passing through

$P^{(0)}, \dots, P^{(h)}$ (notice that by Bertini's theorem Σ does not depend on t). The points of Σ are indeed Segre images of pairs (λ, p) such that for every t the curve of $\mathcal{F}_{Ht}$ corresponding to λ has a singular point in p. If Σ is one-dimensional we have *a priori* two cases:

(i) $\sigma^{-1}(\Sigma) = \bigcup_{j=0}^{h} \mathbb{P}^1 \times \{p^{(j)}\}$, so that all the curves of $\mathcal{F}_{H_t}$ pass doubly through $p^{(0)}, \dots, p^{(h)}$;

(ii) $\sigma^{-1}(\Sigma)$ surjects on $\mathbb{P}^1$ and projects to S over a curve D passing through $p^{(0)}, \dots, p^{(h)}$, so that $\mathcal{F}_{H_t}$ has D as a base curve.

Notice that if Σ is higher dimensional we fall *a fortiori* in case (ii).

In case (i) we have $\mathcal{F}_{H_t} \subseteq \mathcal{L}(2^{h+1})$ for every $1 \leqslant t \leqslant c$. We claim that

$$\dim\langle \mathcal{F}_{H_1} \dots \mathcal{F}_{H_c} > \geqslant \frac{c}{2}.$$

To check the claim, let $\langle \mathcal{F}_{H_1} \dots \mathcal{F}_{H_c} \rangle$ be spanned by the columns of a matrix

$$\begin{pmatrix} f_{01} & \cdots & f_{0c} \\ f_{11} & \cdots & f_{1c} \end{pmatrix}$$

whose entries are hyperplane sections of S. Just making elementary operations on columns, we obtain

$$\begin{pmatrix} f_{01} & \cdots & f_{0x} & 0 & \cdots & 0 \\ f_{11} & \cdots & f_{1x} & g_{1x+1} & \cdots & g_{1c} \end{pmatrix}$$

where

$$x = \dim \langle f_{01} \dots f_{0c} \rangle.$$

Since both $f_{01} \dots f_{0x}$ and $g_{1x+1} \dots g_{1c}$ are linearly independent, we deduce

$$\dim\langle \mathcal{F}_{H_1} \dots \mathcal{F}_{H_c} \rangle \geqslant \max\{x, c-x\} \geqslant \frac{c}{2}$$

and the claim is checked.

By the claim, to prove that all surfaces falling in this case are indeed h-defective, it will be sufficient to show that $\operatorname{expdim} \mathcal{L}(2^{h+1}) < \frac{c}{2}$. Using (1) we compute:

$$\operatorname{expdim} \mathcal{L}(2^{h+1}) = r - 3(h+1) \leqslant r - \frac{3}{4}(2r + 3 - c) =$$

$$= \frac{-2r - 9 + 3c}{4} < \frac{c}{2} - 1$$

(notice that by definition of δ we have $c \leqslant 2r+1$). Hence S is h-defective with defect strictly greater than 1 and from [2], Classification Theorem 1.3, it follows that S sits in a $(s+2)$-dimensional cone over a curve with vertex a linear space of dimension $s \leqslant h-1$, and that $r \geqslant 2h+s+3$. In particular, we have $r \geqslant 2h+1$, so $c = 2r - 4h + \delta - 2$.

In case (ii), as $p^{(0)}, \ldots, p^{(h)}$ vary on S, the curve D moves in an $(h+1)$-dimensional involution $\mathcal{D}$ (for the concept of an involution, see [2], § 5). If D pass doubly through $p^{(0)}, \ldots, p^{(h)}$, we may argue exactly as in case (i); otherwise, the moving parts of the c independent pencils $\mathcal{F}_{H_t}$ are contained in $\mathbb{P}V(-D)$ and have to pass through $p^{(0)}, \ldots, p^{(h)}$ in correspondence with prescribed general coefficients. Hence the generic fiber of the natural map

$$\mathbb{G}(\mathbb{P}^1, \mathbb{P}V(-D)) \to (\mathbb{P}^1)^{h+1}/\mathrm{Aut}(\mathbb{P}^1)$$

which associates to a one-dimensional linear system the $(h+1)$-ple of coefficients corresponding to its curves through $p^{(0)}, \ldots, p^{(h)}$, must have dimension $c-1$. It follows that

$$\dim \mathbb{G}(\mathbb{P}^1, \mathbb{P}V(-D)) - (h+1) + \#\mathrm{Aut}(\mathbb{P}^1) = c - 1\,,$$

i.e.

$$2 \dim \mathbb{P}V(-D) = h + c - 1\,.$$

Conversely, from each of the two conditions described in the statement we obtain exactly c independent pencils of hyperplane sections of S passing doubly through any fixed $h+1$ general points of S in correspondence of prescribed general coefficients in $\mathbb{P}^1$. Applying Lemma 1 we deduce that $T_P(\mathrm{Sec}_h(\sigma(\mathbb{P}^1 \times S)))$ is contained in exactly c independent hyperplanes of $\mathbb{P}^{2r+1}$; hence $\dim \mathrm{Sec}_h(\sigma(\mathbb{P}^1 \times S)) = 2r+1-c$ and $\sigma(\mathbb{P}^1 \times S)$ is h-defective with defect δ. Now the thesis directly follows from Proposition 1. ■

3. – Application to Veronese embeddings.

We recall the following standard definition:

DEFINITION 3. – *Let V be an algebraic variety and fix an ample divisor H on V. Then for any $n > > 0$ the n-Veronese of V (with respect to H) is the embedding of V into $\mathbb{P}H^0(V, nH)$ by the complete linear series $|nH|$.*

COROLLARY 1. – *Let S be an integral algebraic surface with* $\mathrm{Pic}(S) = \mathbb{Z}$ *and let H be the ample divisor which generates* $\mathrm{Pic}(S)$ *over $\mathbb{Z}$. If $n \gg 0$ and $h \geqslant 2$, then the n-Veronese of S is $(1, h)$-defective if and only if for some integer m with $1 \leqslant m \leqslant n$*

either

$$\begin{aligned} h &\leqslant h^0(S, mH) - 2 \\ 2h &\geqslant h^0(S, nH) - 2 \\ h &\leqslant 2h^0(S, (n-m)H) - 2 \end{aligned}$$

or

$$\begin{aligned} h &\leqslant h^0(S, mH) - 2 \\ 2h &\leqslant h^0(S, nH) - 2 \\ 3h &\geqslant 2h^0(S, nH) - 2h^0(S, (n-m)H) - 2 \,. \end{aligned}$$

PROOF. – We are going to apply Theorem 1. To begin with, notice that the first possibility cannot occur. Indeed, if S sits in a cone over a curve C, then there is a natural map $f\colon S \to C$ and $f^*(\mathcal{O}_C(1)).\ F = 0$ for any fiber F of f, contradicting the assumption $\mathrm{Pic}(S) = \mathbb{Z}$. Next, in this case we may rephrase the second possibility in terms of linear systems. Indeed, by definition (see [2], § 5), an involution $\mathcal{D}$ is in particular an algebraic family of divisors parametrized by a reduced variety. Hence the hypothesis $\mathrm{Pic}(S) = \mathbb{Z}$ implies that all divisors in $\mathcal{D}$ are linearly equivalent and $\mathcal{D}$ turns out to be a linear system. Now the thesis follows from Theorem 1 via a straightforward computation. ■

THEOREM 2. – *Let S be a smooth algebraic surface with* $\mathrm{Pic}(S) = \mathbb{Z}$. *Then there exists an integer $n_0 = n_0(S)$ such that for any $n \geqslant n_0$ and $h \geqslant 2$ the n-Veronese of S is* not *$(1, h)$-defective.*

REMARK 1. – *The classical case $S = \mathbb{P}^2$ (where $n_0 := 4$) is stated in* [10] *on p.* 100 *and in* [8] *as Theorem* 1.4. *However, there are many other interesting examples of algebraic surfaces S with* $\mathrm{Pic}(S) = \mathbb{Z}$. *Here we wish to recall at least a couple of facts. If $S(d)$ is the quasi-projective variety parametrizing smooth surfaces of degree d in $\mathbb{P}^3$, by the Noether-Lefschetz theorem there is a countable set of proper irreducible closed subvarieties of $S(d)$ such that for any point s outside the union of these subvarieties the corresponding surface S has* $\mathrm{Pic}(S) \cong \mathbb{Z}$ *generated by $\mathcal{O}_S(1)$ (see* [7], *p.* 341). *Moreover, for every $g \geqslant 2$ there is a unique family of $K3$ surfaces of degree $2g - 2$ in $\mathbb{P}^g$ depending on* 19 *moduli, whose generic surface has Picard group generated by the hyperplane section (see* [4], *p.* 108).

PROOF OF THEOREM 2. – Let H be as in the statement of Corollary 1. If K is the canonical divisor on S, we have $K = kH$ with $k \in \mathbb{Z}$. The Riemann-Roch theorem gives:

$$h^0(S,\,aH) - h^1(S,(k-a)H) + h^0(S,(k-a)H) = \chi + \frac{1}{2}(a^2 - ak)\,d$$

where $\chi := \chi(\mathcal{O}_S)$ and $d := H^2$. Notice that if $a > k$ we have $H^1(S,(k-a)\,H) = 0$ by the Kodaira vanishing theorem. Hence we may introduce the following (biregular) invariant of S:

$$M := \max_{1 \leqslant a \leqslant k} h^1((k-a)\,H)$$

so that for any $b \in \mathbb{Z}$

$$h^0(S,\,bH) \leqslant \chi + \frac{1}{2}(b^2 - bk)\,d + M\,.$$

Now, if we take $n > k$, by Corollary 1 the n-Veronese of S is $(1,\,h)$-defective if and only if for some m with $1 \leqslant m \leqslant n$ either

$$h \leqslant \chi + \frac{1}{2}(m^2 - mk)\,d + M - 2 \tag{2}$$

$$2h \geqslant \chi + \frac{1}{2}(n^2 - nk)\,d - 2 \tag{3}$$

$$h \leqslant 2\chi + \big((n-m)^2 - (n-m)\,k\big)\,d + 2M - 2 \tag{4}$$

or

$$h \leqslant \chi + \frac{1}{2}(m^2 - mk)\,d + M - 2 \tag{5}$$

$$2h \leqslant \chi + \frac{1}{2}(n^2 - nk)\,d - 2 \tag{6}$$

$$3h \geqslant (2nm - m^2 - mk)\,d - 2M - 2 \tag{7}$$

In the sequel, we will denote by C_i, $i \in \mathbb{N}$, nonnegative constants depending only on the numerical invariants k, χ, d and M. We start by excluding the first possibility. Indeed, from (2) and (3) we get

$$n^2 - kn \leqslant 2(m^2 - km) + C_1, \tag{8}$$

while (3) and (4) together give us

$$(2(n-m))^2 - 4kn + 4km \geqslant n^2 - kn - C_2. \tag{9}$$

From (9) we deduce

$$(2(n-m))^2 \geqslant n^2 + 3kn + \min\{0,\ -4kn\} - C_2,$$

hence

$$(2(n-m))^2 \geqslant (n - 2\,|k|\,)^2$$

for $n \geqslant n_0(k,\, C_2)$. It follows that

$$m \leqslant \frac{n}{2} + |k|$$

and from (8) we obtain

$$n^2 - kn \leqslant \frac{1}{2} n^2 + 2\,|k|^2 + 2\,|k|n + \max\{0,\ -2kn\} + C_1.$$

Therefore

$$\frac{1}{2} n^2 \leqslant C_3 n + C_4$$

and if we take $n_0 \gg 0$ this is clearly impossible for any $n \geqslant n_0$. So the first case cannot occur.

Next we turn to the second possibility. From (5) and (7) we get

$$\left(2n - \frac{5}{2} m + \frac{1}{2} k\right) m \leqslant C_5, \tag{10}$$

while (6) and (7) together give us

$$2(2n - m - k)\, m \leqslant \frac{3}{2}(n^2 - kn) + C_6. \tag{11}$$

From (10) we deduce

$$n \leqslant \frac{5}{4} m - \frac{1}{4} k + \frac{C_5}{2m} \leqslant \frac{5}{4} m - \frac{1}{4} k + \frac{C_5}{2},$$

hence

$$m \geqslant \frac{4}{5} n + \frac{1}{5} k - \frac{2}{5} C_5.$$

So using (11) we obtain

$$\begin{aligned} \frac{3}{2} n^2 - \frac{3}{2} kn + C_6 &\geqslant 2(2n - m - k)\, m \geqslant 2(n-k)\, m \\ &\geqslant 2(n-k)\left(\frac{4}{5} n + \frac{1}{5} k - \frac{2}{5} C_5\right). \end{aligned}$$

Therefore

$$-\frac{1}{10}n^2 + C_7 n + C_8 \geqslant 0$$

and if we take $n_0 \gg 0$ this is clearly impossible for any $n \geqslant n_0$. So the proof is over. ■

4. – Application to rational normal scrolls.

Turning to examples of $(1, h)$-defective surface, the following holds.

THEOREM 3. – *Let* $0 < a_0 \leqslant a_1$ *and* $h \geqslant 2$ *be integers. Assume that* $a_0 + a_1 \geqslant 2h$ *and* $2a_0 \leqslant h - 2$, *or* $a_0 + a_1 \leqslant 2h$ *and* $2a_1 \geqslant 3h + 2$, *or* $a_0 + a_1 = 2h$ *and* h *even. Then the rational normal scroll* $S(a_0, a_1) \subset \mathbb{P}^{a_0 + a_1 + 1}$ *is* $(1, h)$*-defective.*

REMARK 2. – *If we specialize to the case* $h = 2$, *we obtain that* $S(1, 3)$ *and* $S(2, 2)$ *are the unique rational normal scrolls satisfying the numerical hypotheses of Theorem* 3. *We should expect this result, since it is classically known that* $S(1, 3)$ *and* $S(2, 2)$ *are* $(1, 2)$*-defective and by the classification of* $(1, 2)$*-defective surfaces due to Chiantini and Coppens* (*see* [3]) *these two surfaces are indeed the unique smooth* $(1, 2)$*-defective surfaces.*

PROOF OF THEOREM 3. – By Theorem 1, in order to prove that a scroll $S = S(a_0, a_1)$ is $(1, h)$-defective it is sufficient to exhibit a divisor D on S such that

$$\begin{aligned} &h^0(S, \mathcal{O}_S(D)) \geqslant h + 2 \\ (12) \qquad &2h^0(S, \mathcal{O}_S(H - D)) \geqslant h + 2(a_0 + a_1 + 1) + 1 \\ &\qquad - \min\{4(h+1) - 1, 2(a_0 + a_1 + 1) + 1\} + 1 + 1 . \end{aligned}$$

On the other hand, the theory of divisors on a (nonsingular) rational normal scroll S is very well understood (see for instance [1] § 3.1 for a careful exposition of this subject). Namely, we have

$$\mathrm{Pic}(S) \cong \mathbb{Z}H \oplus \mathbb{Z}F \tag{13}$$

where H is a divisor associated to the tautological sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ of the corresponding projective bundle $\mathbb{P}(\mathcal{E})$ and F is a divisor associated to the fiber $\pi^* \mathcal{O}_{\mathbb{P}^1}(1)$ of the natural projection $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$. Moreover, since a global section

$$\sigma \in H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(a) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(b))$$

naturally identifies to a polynomial

$$\sigma = \sum_{i_0 + \ldots + i_k = a,\ i_j \geq 0} \sigma_{i_0 \ldots i_k} X_0^{i_0} \ldots X_k^{i_k}$$

(where $X_0 \ldots X_k$ are homogeneous coordinates on the fiber of $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$ and $\sigma_{i_0 \ldots i_k} \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i_0 a_0 + \ldots + i_k a_k + b)))$, we have

$$h^0(S, \mathcal{O}_S(aH + bF)) = \sum_{i_0 + \ldots + i_k = a,\ i_j \geq 0} \max\{i_0 a_0 + \ldots + i_k a_k + b + 1,\ 0\}. \tag{14}$$

It follows from (13) that the divisor D in (12) is of the form $D = aH + bF$; moreover, since (14) shows that $h^0(S, \mathcal{O}_S(aH + bF)) = 0$ for $a < 0$ and for $a = 0$ and $b < 0$, there are only two possibilities for D:

(i) $D = bF$

(ii) $D = H - bF$

with $b \geq 0$.

In case (i), from (12) and (14) we get:

$$\begin{aligned} &b + 1 \geq h + 2 \\ &2 \max\{a_0 - b + 1,\ 0\} + 2 \max\{a_1 - b + 1,\ 0\} \geq \\ &\quad h + 2(a_0 + a_1 + 1) + 1 - \min\{4(h+1) - 1,\ 2(a_0 + a_1 + 1) + 1\} + 2\,. \end{aligned} \tag{15}$$

In case (ii), from (12) and (14) we get:

$$\begin{aligned} &\max\{a_0 - b + 1,\ 0\} + \max\{a_1 - b + 1,\ 0\} \geq h + 2 \\ &2b + 2 \geq h + 2(a_0 + a_1 + 1) + 1 \\ &\qquad - \min\{4(h+1) - 1,\ 2(a_0 + a_1 + 1) + 1\} + 2\,. \end{aligned} \tag{16}$$

Let first $a_0 + a_1 \geq 2h$. In this case, we choose $D = bF$ with $b = h + 1$. Then the first inequality of (15) is satisfied; moreover, the assumption $2a_0 \leq h - 2$ implies that also the second inequality of (15) holds.

Let now $a_0 + a_1 \leq 2h$, so that in particular $a_0 \leq h$. As above, we choose $D = bF$ with $b = h + 1$. Then the first inequality of (15) is satisfied; moreover, the assumption $2a_1 \geq 3h + 2$ implies that also the second inequality of (15) holds.

Let finally $a_0 + a_1 = 2h$ and assume that h is even. Here we choose $D = H - bF$ with $b = \frac{h}{2}$, so that the second inequality of (16) is satisfied. If $a_0 \geq \frac{h}{2} - 1 = b - 1$, then the assumption $a_0 + a_1 \geq 2h$ implies that also the second inequality of (16) holds; if instead $a_0 < \frac{h}{2} - 1$, then the assumption $a_1 \geq \frac{3}{2}h + 1$ implies that also the second inequality of (16) holds.

Hence the proof is over. ∎

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