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Weak Solutions for a Well-Posed Hele-Shaw Problem.

S. N. ANTONTSEV - A. M. MEIRMANOV - V. V. YURINSKY

Sunto. – *Analizziamo l'esistenza e l'unicità di soluzioni deboli del problema ben posto di Hele-Shaw con condizioni generali sul contorno assegnato, equazione governante non-omogenea nel dominio incognito e condizione dinamica non-omogenea a contorno libero. Il nostro approccio permette anche di indebolire le restrizioni sui dati iniziali e di contorno. Otteniamo infine alcune stime per la soluzione negli spazi BV, proviamo un teorema di comparazione, e mostriamo che la soluzione dipende in modo continuo dai dati iniziali e di contorno.*

Summary. – *We analyze existence and uniqueness of weak solutions to the well-posed Hele-Shaw problem under general conditions on the fixed boundaries and non-homogeneous governing equation in the unknown domain and non-homogeneous dynamic condition on the free boundary. Our approach allows us also to minimize the restrictions on the boundary and initial data. We derive several estimates on the solutions in BV spaces, prove a comparison theorem, and show that the solution depends continuously on the initial and boundary data.*

1. – Introduction.

The aim of this article is to generalize the approaches of Kamin, Oleinik, and Kruzhkov to the study of well-posed Hele–Shaw problem. We establish existence and uniqueness of weak solutions under a non-homogeneous condition on the free boundary and general boundary conditions on the fixed boundary.

The Hele-Shaw problem is a well-known model of liquid filtration in a porous medium. In this model, the evolution of liquid pressure is considered in the variable flow region $\Omega(t)$, $0 \leqslant t \leqslant T$, which is a part of the larger fixed domain $Q \subset \mathbb{R}^n$.

In $\Omega(t)$ the pressure $p(x, t)$ satisfies the field equation, which is below the Poisson equation (here and in the sequel all variables are dimensionless)

$$-\Delta p = \operatorname{div} F(x)\,. \tag{1.1}$$

This equation follows from Darcy's law for the liquid velocity, $-v = \nabla p + F(x)$, and the continuity equation $\operatorname{div} v = 0$. The nature of the body force $F(x)$ may vary, it being caused, e.g., by gravity or rotation. A recent example of a free boundary problem arising in physics with a source term of this kind is contained in [13].

The boundary of the unknown flow region $\Omega(t) \subset Q$ is

$$\partial\Omega(t) = S \cup \Gamma(t),$$

where both the fixed boundary $S = \partial Q$ and the free (moving) boundary $\Gamma(t)$ may each include a finite number of disjoint bounded connected components.

At the initial moment, the free boundary is known:

$$\Gamma(0) \equiv \Gamma^0, \qquad \Omega(0) = \Omega^0. \tag{1.2}$$

Its evolution is determined by the boundary conditions

$$p\,|_{\Gamma(t)} = 0 \tag{1.3}$$

and

$$-V_\nu(x, t) = \nabla p(x, t)\cdot\nu(x, t) + F(x)\cdot\nu, \tag{1.4}$$

where $V_\nu(x, t)$ is the velocity of the free boundary in the direction of the normal $\nu(x)$ measured at the point $x \in \Gamma(t)$. (When $\nabla p \neq 0$, the normal is $\nu = |\nabla p|^{-1}\nabla p$, and the normal velocity is $V_\nu = -|\nabla p|^{-1} p_t$.)

On the fixed boundary S, we assume the third type boundary condition

$$a(x)\,\frac{\partial p}{\partial \nu} + b(x)\,p = p_s(x, t), \qquad x \in S, \tag{1.5}$$

where $\partial/\partial\nu$ is the derivative in the outward normal direction.

The above problem is *well-posed* if its solution is nonnegative, and *ill-posed* otherwise. We use the abbreviation WHSP to refer to the well-posed Hele-Shaw problem.

An alternative description of a solution to the Hele-Shaw problem is in terms of the indicator function $\chi : Q \times (0, T) \to \{0, 1\}$ of the flow domain and the extended pressure $\tilde{p}$:

$$\Omega(t) = \{x \in Q : \chi(x, t) = 1\}, \qquad \tilde{p}(x, t) = \begin{cases} p(x, t), & x \in \Omega(t), \\ 0, & x \in Q \setminus \Omega(t). \end{cases} \tag{1.6}$$

The above equations imply the following integral identity:

$$\int_Q \int_0^T (\chi\varphi_t - \chi F\cdot\nabla\varphi + \tilde{p}\Delta\varphi)\,dx\,dt = -\int_Q \chi_0\varphi(x, 0)\,dx + \int_{S'}\int_0^T p_s\,\frac{\partial\varphi}{\partial\nu}\,ds\,dt - \int_{S\setminus S'}\int_0^T \left(\frac{\varphi p_s}{a} + \varphi\chi F\cdot\nu\right) ds\,dt, \tag{1.7}$$

where S' is the part of the fixed boundary that carries the Dirichlet boundary condition ($a = 0,\ b = 1$), the test function φ belongs to the space

$$K \stackrel{\text{def}}{=} \left\{ \varphi \in W_2^{2,1}(Q \times [0, T]),\ \varphi(x, T) = 0,\ \left(a \frac{\partial \varphi}{\partial \nu} + b\varphi \right) \Big|_{x \in S} = 0 \right\},$$

and

$$\chi_0(x) \stackrel{\text{def}}{=} \begin{cases} 1, & x \in \Omega^0, \\ 0, & x \in Q \backslash \Omega^0, \end{cases} \tag{1.8}$$

is the indicator function of the known initial flow domain.

In a loose sense, the definition of χ and the integral identity (1.7) correspond to the non-linear equation

$$\frac{\partial}{\partial t}\chi - \Delta \tilde{p} = \operatorname{div}(\chi F), \qquad \chi \in H(\tilde{p}), \tag{1.9}$$

where H is the Heaviside graph,

$$H(\xi) = \{0\} \text{ for } \xi < 0, \qquad H(0) = [0, 1], \qquad H(\xi) = \{1\} \text{ for } \xi > 0,$$

and the inclusion reflects the fact that $\tilde{p}$ vanishes outside the flow region.

We define weak solutions to problem (1.1)-(1.5) following [14].

DEFINITION 1.1. – *A pair* $(\tilde{p}, \chi)$ *of nonnegative functions from the space* $L^\infty(Q_T)$ *with* $Q_T = Q \times (0, T)$ *defines a* weak solution *to problem* (1.1)-(1.5) *if* χ *satisfies the integral equality* (1.7)-(1.8) *and the inclusion*

$$\forall (x, t) \qquad \chi(x, t) \in H(\tilde{p}(x, t)). \tag{1.10}$$

Note that problem (1.1)-(1.5) corresponds to the one-phase Stefan model problem of a phase transition with vanishing heat capacity. The Stefan problem has been intensively studied over past decades. Its analysis was initiated by Rubinstein [16].

The concept of the weak solution has been introduced by Kamin [6] and Oleinik [14] who proved the existence and uniqueness of global-in-time weak solutions to the multidimensional two-phase Stefan problem. Using the Kamin-Oleinik approach, several authors have obtained stability estimates for weak solutions (see, e.g., [1]). In early nineties, Götz and Zaltzman have used Kruzhkov's method of BV estimates [8] to analyze qualitative properties of weak solutions to the multidimensional two-phase Stefan problem [3]. A review of work on the Stefan problem was given by Meirmanov in [12], and most of subsequent development is covered by Visintin's book [17].

The vanishing heat capacity does not allow us to extend the cited results for the Stefan problem to WHSP in a straightforward manner. The main difficulty lies in obtaining *a priori* estimates for weak solutions. In particular, the maximum principle fails for zero heat capacity. In the general case, a weak solution to WHSP has no regularity in the time variable. This motivated the independent study of WHSP undertaken by several authors (see [5] and references therein), who mainly used the variational inequality methods.

Using the approach through variational inequalities, problem (1.1)-(1.5) was studied for a one-component fixed boundary under strong restrictions on the functions in condition (1.5) on the given boundary. Namely, for the Laplace equation ($f(x)=0$) or the Cauchy problem with no fixed boundary and a specific source term in the Poisson equation, it was assumed that either the pressure p_s is strictly positive in the Dirichlet condition ($a=0$) or the flux p_s is constant in the Neumann condition ($a=1$). Elliot and Janovsky [2] have proved the existence theorem for WHSP with a constant flux on the fixed boundary. The Cauchy problem with the source term $f(x)=C\log|x+iy|$ in the Poisson equation was investigated by Gustafsson [4]. Louro and Rodrigues [11] studied WHSP with strictly positive Dirichlet data on the fixed boundary. WHSP with non-local boundary conditions was considered by Primicerio and Rodrigues [15].

It is well-known that $\chi(x,t)$is an increasing function of time if the free boundary condition (1.4) is homogeneous (i.e., $F(x)$ has compact support in Ω^0). This fact allows one to apply the Duvaut transformation and reformulate problem (1.1)-(1.5) as a variational inequality.

However, if condition (1.4) is non-homogeneous, then the inclusion

$$\Omega(t_1)\subset\Omega(t_2), \qquad t_1<t_2$$

may fail, and one cannot reduce the problem to a variational inequality. In fact, if $p_s=0$ and $\operatorname{div}F=0$, then $p(x,t)\equiv 0$ in $\Omega(t)$, and $\chi(x,t)$ satisfies the equation

$$\frac{\partial\chi}{\partial t}-F\cdot\nabla\chi=0\,. \tag{1.11}$$

Thus, the behavior of $\chi(x,t)$ depends only on the source term F, and in the general case the flow region $\Omega(t)=\{\chi=1\}$ is not necessarily increasing.

In what follows, we study the existence and uniqueness of weak solutions to WHSP with the Poisson field equation (1.1) under the non-homogeneous condition (1.4) and general boundary conditions (1.5). We construct its solution as a limit of solutions to a family of regularized problems.

To avoid complications in constructing solutions to the regularized problems in an unbounded domain Q, we first restrict the equations to the bounded domain Q_{mod} (see (2.1) below). We establish the existence and uniqueness of the solution and its properties for the «modified problem», i.e., for the auxiliary problem in Q_{mod}, under the additional assumption that the pressure vanishes on the supplementary spherical boundary.

The key ingredient of the method used to return to the unbounded domain are estimates for the maximum of the solution to the modified WHSP which have explicit dependence on the radius R_* of (2.1). These estimates permit us to show that for an arbitrary time interval $[0, T]$ there exists a common radius $R_* = R_*(T)$ such that the modified problem is equivalent to the original one:

$$\tilde{p}(x, t) = 0, \qquad |x| \geqslant \frac{1}{2} R_*, \ t \in [0, T].$$

To arrive at this last conclusion, we use explicit solutions of the Hele-Shaw problem and the comparison theorems for generalized solutions.

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2. – The main results.

2.1. *Notation.*

Below, we use the standard notation $|x|$ for the Euclidean norm in $\mathbb{R}^n$, and $\text{dist}(a, G) = \inf\{|a-y| : y \in G\}$ is the corresponding distance from a point to a set. The δ-neighborhood of a set $G \subset \mathbb{R}^n$ is

$$\mathcal{U}_\delta(G) = \{x : \text{dist}(x, G) < \delta\}, \qquad \delta > 0.$$

A ball of radius r centered at a is $B(r, a)$; we abbreviate $B(r) = B(r; 0)$. The characteristic function of a set is $\mathbf{1}_G(x) = \begin{cases} 1, & x \in G, \\ 0, & x \notin G. \end{cases}$

As usual, for $T > 0$

$$Q_T = Q \times (0, T), \quad S_T = S \times (0, T), \quad \text{etc.}$$

Notation of function spaces is that of [9, 10]. The definition of functions of bounded variation and the space BV is standard (see, e.g., § IV.7 in [7]).

We use several kinds of constants in our calculations. Below C (with or without indices) denotes a constant that depends only on the number of dimensions n and the shape of S. A constant whose value is determined by the above and the constants in the boundary and initial conditions is denoted M (with or without indices). We write K for a constant that depends on arguments introduced in intermediate calculations.

In the proof below, we first consider an auxiliary Hele-Shaw problem in the bounded «modified domain» defined using a large enough radius R_* as

$$Q_{\text{mod}} = Q \cap B(R_*)\,. \tag{2.1}$$

We use a cutoff function of special form to derive the a priori estimate for pressure on Q_{mod}:

$$\zeta_\delta(x) = Z\left(\frac{1}{\delta}\,\text{dist}\,(x,\,S)\right) Z\left(\frac{1}{\delta}\,\text{dist}\,(x,\,\partial B(R_*))\right), \tag{2.2}$$

where $Z : \mathbb{R} \to [0,\,1]$ is an infinitely differentiable function such that $Z(0)=0$, $Z(t) = 1$ for $t \geqslant 1$, and for some $c > 0$

$$Z'(t) \geqslant 0,\; cZ(t) \geqslant tZ'(t),\; c\sqrt{Z(t)} \geqslant Z'(t) + |Z''(t)|\,, \quad t \geqslant 0\,. \tag{2.3}$$

It follows from the regularity of S that if δ is sufficiently small, then for each point $x \in \mathcal{U}_\delta(S)$ there is a unique point $\xi(x) \in S$ such that

$$\text{dist}\,(x,\,S) = |x - \xi(x)|\,,\; \nabla\,\text{dist}\,(x,\,S) = \frac{x - \xi(x)}{|x - \xi(x)|}\,, \tag{2.4}$$

and

$$x - \xi(x) = -\nu(\xi(x))\,|x - \xi(x)|\,,$$

where $\nu(x)$ is the unit vector of the outward normal at the point $x \in \partial Q_{\text{mod}}$.

2.2. *Existence and uniqueness theorems.*

Below, we consider problem (1.1)-(1.5) under the following restrictions on the given data:

CONDITION 2.1. – *The fixed boundary is twice continuously differentiable, and $a, b \geqslant 0$, $a^2 + b^2 = 1$. The initial flow region is bounded and regular:*

$$\chi_0(x) \in BV(Q), \quad \chi_0(x) = 0,\; x \in \mathbb{R}^n \backslash B(R_0),\; R_0 < \infty\,. \tag{2.5}$$

CONDITION 2.2. – *On the given boundary*

$$0 \leqslant p_s \leqslant M_0. \tag{2.6}$$

CONDITION 2.3. – *The source term is smooth, $F \in C^2(Q)$, and $|F|^{(2)} \leqslant M_0$. It is solenoidal outside a bounded domain,* supp (div F) $\subset B(R_0)$ *for some $R_0 > 0$, and on the fixed boundary*

$$F(x) \cdot \nu(x) \leqslant 0\,,\; x \in S, \tag{2.7}$$

where $\nu(x)$ is the normal vector. Moreover, outside a larger sphere $B(R_1)$,

$R_0 < R_1$, *the source* F *satisfies the condition*

$$F(x)\cdot x \leqslant 0, \quad |x| > R_1. \tag{2.8}$$

For the cutoff function (2.2), it follows from (2.4) that

$$\nabla \zeta_\delta(x) = -\frac{1}{\delta} Z'\left(\frac{1}{\delta}\operatorname{dist}(x, S)\right) \nu(\xi(x)).$$

Consequently, (2.3) and Conditions 2.2-2.3 imply the inequality

$$\begin{aligned} F(x)\cdot\nabla\zeta_\delta(x) &= -\frac{1}{\delta} Z'\left(\frac{1}{\delta}\operatorname{dist}(x, S)\right) F(x)\cdot\nu(\xi(x)) \\ &\geqslant -\frac{1}{\delta} Z'\left(\frac{1}{\delta}\operatorname{dist}(x, S)\right) F(\xi(x))\cdot\nu(\xi(x)) - \\ &\frac{1}{\delta} M_0 \,|x - \xi(x)|\, Z'\left(\frac{1}{\delta}\operatorname{dist}(x, S)\right) \geqslant -cM_0\,\zeta_\delta(x) \end{aligned} \tag{2.9}$$

for all points in the δ neighborhood of the boundary of the domain $Q \cap B(R)$ provided that $R \geqslant R_1 > R_0$ (here $\nu(x)$ is the unit vector of the outward normal to $\partial(Q \cap B(R))$). The constant c does not depend on the constants in the cited restrictions.

We now state our principal result on existence, stability, and comparison of solutions to the Hele-Shaw problem.

THEOREM 2.1. – *The following assertions hold true under Conditions* 2.1-2.3:

(a) *Problem* (1.1)-(1.5) *has at least one weak solution* $(\tilde{p}(x, t), \chi(x, t))$ *with a compact support.*

(b) *The first component of the solution* $\tilde{p}(x, t)$ *is bounded, and the second one* $\chi(x, t)$ *has bounded variation:* $\chi \in BV(Q'_T)$ *if* $Q' \subset Q$.

(c) *Under the assumption* $a \geqslant a_0 = \text{const} > 0$, *the weak solutions to problem* (1.1)-(1.5) *enjoy the following property of stability and monotonicity:*

If pairs $(\tilde{p}^1(x, t), \chi^1(x, t))$ *and* $(\tilde{p}^2(x, t), \chi^2(x, t))$ *are weak solutions of* (1.1)-(1.5) *that correspond, respectively, to the given data* $p_s = p_{s,i}$, $\chi_0 = \chi_{0,i}$, *and* $F = F_i$, $i = 1, 2$, *then*

$$\max_{0 \leqslant t \leqslant T} \int_Q |\chi_1(x, t) - \chi_2(x, t)|\, dx \leqslant \int_Q |\chi_{0,1}(x) - \chi_{0,2}(x)|\, dx + \frac{1}{a_0}\int_{S_T} |p_{s,1} - p_{s,2}|\, ds\, dt\,. \tag{2.10}$$

Inequality (2.10) *remains true if* $a \geqslant 0$ *and* $p_{s,1} = p_{s,2}$.

(d) *If* $a \geqslant 0$ *and the given data satisfy the inequalities*

$$p_{s,1} \leqslant p_{s,2}, \qquad F_1 = F_2, \qquad \chi_{0,1} \leqslant \chi_{0,2},$$

then

$$\chi_1(x, t) \leqslant \chi_2(x, t), \qquad \tilde{p}_1(x, t) \leqslant \tilde{p}_2(x, t). \tag{2.11}$$

(e) *If* $\operatorname{supp} F \subset \Omega^0$ *and* $\operatorname{div} F \geqslant 0$, *then* $\chi(x, t)$ *is an increasing function of time* t, *and the integral equality* (1.7) *can be rewritten as*

$$\int_{Q_T} (\chi \varphi_t + \tilde{p} \Delta \varphi)\, dx\, dt = -\int_Q \chi_0 \varphi(x, 0)\, dx + \int_{S_T^+} p_s \frac{\partial \varphi}{\partial \nu}\, ds\, dt - \int_{S_T \setminus S_T^+} \frac{\varphi p_s}{a}\, ds\, dt - \int_{Q_T} \varphi \operatorname{div} F\, dx\, dt. \tag{2.12}$$

Moreover, if the given data satisfy the inequalities

$$p_{s,1} \leqslant p_{s,2}, \ \operatorname{div} F_1 \leqslant \operatorname{div} F_2, \ \chi_{0,1} \leqslant \chi_{0,2},$$

then the corresponding solutions $(\tilde{p}_1, \chi_1)$ *and* $(\tilde{p}_2, \chi_2)$ *satisfy inequalities* (2.11).

Theorem 2.1 can be complemented by the following uniqueness result.

THEOREM 2.2 (Uniqueness). – *If the closures of* S' *and* $S'' = S \setminus S'$ *are disjoint and* $b = 0$, $F \cdot \nu = 0$ *on* S'', *then each bounded weak solution* (p, χ) *to problem* (1.1)-(1.5) *is unique.*

3. – Proof of Theorem 2.1.

We consider only the case of unbounded domain Q because all the arguments apply to that of bounded one with minor modifications. The proof consists of several steps, which are exposed in separate subsections below.

First, we replace Q with the bounded modified domain Q_{mod} of (2.1) and regularize the original problem for Q_{mod} to obtain smooth approximations to its solution. The regularized problems that are solved in Q_{mod} depend on an auxiliary large parameter $R_* > 0$ (see (2.1)). We suppose that $R_* > \max\{R_1, 4R_0\}$ (see Conditions 2.1 and 2.3), so Q_{mod} includes the initial flow region. We solve the auxiliary problem under the assumption that the pressure vanishes on the additional spherical boundary:

$$p^\varepsilon |_{\partial B(R_*)} = 0. \tag{3.1}$$

We derive uniform estimates for the solutions of the regularized problem.

Next, we establish the convergence of the solutions of the regularized problems to a solution of the original problem in the modified domain. We use the term «modified problem» to refer to the original WHSP restricted to Q_{mod} with the additional condition (3.1).

Finally, it is shown that for each $T > 0$ and $t \in [0, T]$ the flow domain $\Omega(t)$ does not reach the exterior boundary of Q_{mod} if R_* is chosen large enough, so the solution of the modified problem also solves the original one for the unbounded domain. The argument is based on comparing the solution of WHSP on Q_{mod} with a spherically symmetric solution of WHSP that has known speed of propagation of the free boundary.

3.3 *Regularization.* – We construct a family of regularizations of system (1.7), (1.8), (1.10) depending on a small parameter $\varepsilon > 0$.

First, we approximate the Heaviside function by a smooth function $H_\varepsilon(p)$ in such a way that

$$\forall p \in \mathbb{R} H_\varepsilon'(p) > 0, \quad \forall p \geq 0 \lim_{\varepsilon \to 0} H_\varepsilon(p) = H(p),$$
(3.2)
$$H_\varepsilon(p) = \frac{1}{\varepsilon} p \text{ for } p \leq \varepsilon, \quad H_\varepsilon(p) = \varepsilon(p - 2\varepsilon) + 1 \text{ for } p > 2\varepsilon\,.$$

In this case H_ε has a well defined inverse function $\Phi_\varepsilon(\chi) \stackrel{\mathrm{def}}{=} H_\varepsilon^{-1}(\chi)$. We regularize the initial conditions so that

$$\chi(x, 0) = \chi_0^\varepsilon(x) \in [0, 1], \quad \chi_0^\varepsilon(x) = 1 \text{ in } \Omega^0, \quad \|\chi_0^\varepsilon\|_{BV} \leq M_0, \tag{3.3}$$

where the value of M_0 is determined by the BV norm of the initial function χ_0, and

$$\chi_0^\varepsilon(x) \searrow \chi_0(x) \text{ as } \varepsilon \searrow 0\,.$$

We use the boundary conditions (3.1) and

$$a_\varepsilon(x) \frac{\partial p}{\partial \nu} + b(x)\, p = p_s(x, t), \quad x \in S, \quad a_\varepsilon = a + \varepsilon\,. \tag{3.4}$$

Below the term «regularized solution» refers to the solution of the regularized analogue of the integral equality (1.7) with the inclusion of (1.10) replaced by equality. In other terms, the regularized solution satisfies the equations that correspond to (1.9),

$$\frac{\partial}{\partial t} \chi^\varepsilon(p) - \Delta p^\varepsilon = \operatorname{div}(\chi^\varepsilon(p)\, F^\varepsilon), \qquad p^\varepsilon = \Phi_\varepsilon(\chi^\varepsilon), \tag{3.5}$$

as well as the pertinent boundary and initial conditions (3.3) and (3.1).

The existence of a unique smooth regularized solution $p^{\varepsilon}(x, t)$ is well known [9].

3.4. *Uniform* a priori *estimates.*

Throughout this subsection, the time $T > 0$ is supposed to have the same arbitrarily chosen fixed value, and notation Q refers to the modified domain (2.1), Q_T to $Q_{\text{mod}} \times (0, T)$, etc. Notation of constants is as described in subsection 2.1. Dealing with regularized solutions (3.5), we omit the superscript «ε» wherever possible.

The first estimate for the modified problem is given by Lemma 3.2 below.

To derive it, we construct the function $u(x)$ as a solution of the boundary value problem

$$-\Delta u - \varepsilon F \cdot \nabla u = \Phi, \qquad x \in Q, \tag{3.6}$$

$$u\big|_{|x| = R_*} = 0, \tag{3.7}$$

$$\frac{\partial u}{\partial \nu}\bigg|_S = 1 \tag{3.8}$$

where the source term $\Phi(x) = M_0 \mathbf{1}_{B(R_0)}(x)$ is constant in $B(R_0)$ and vanishes in $Q \backslash B(R_0)$. To avoid trivial complications we suppose that the constant M_0 in (2.6), (2.9), and Condition 2.2 is the same, and $\varepsilon > 0$ is much smaller than $\min\{M_0, 1\}$.

LEMMA 3.1. – *If* $0 < \varepsilon \leqslant \varepsilon_1 = \varepsilon_1(R_*)$, *then the solution of problem* (3.6), (3.7), (3.8) *admits the following estimates*:

$$\begin{aligned} 0 \leqslant u(x) &\leqslant K_1(M_0) \ln R_* \quad &&\text{for } n = 2, \\ 0 \leqslant u(x) &\leqslant K_1(M_0) &&\text{for } n > 2. \end{aligned} \tag{3.9}$$

Proof of this lemma is placed in Appendix A.1.

REMARK 3.1. – *The estimates of Lemma* 3.1 *are sharp. To check this, it suffices to consider the solution of the problem*

$$\Delta u = 0, \; R_0 < |x| < R_*, \quad u\big|_{\partial B(R_*)} = 0, \; \frac{\partial u}{\partial \nu}\bigg|_{\partial B(R_0)} = 1 .$$

LEMMA 3.2 (The first *a priori* estimate). – *The following estimate holds uniformly in* $\varepsilon \in]0, \varepsilon_*]$, $\varepsilon_* = \varepsilon_*(M_0, R_*)$: *for* $(x, t) \in Q_T$

$$0 \leqslant p^{\varepsilon}(x, t) \leqslant K(M_0, R_*), \tag{3.10}$$

where $K(M_0, R_*) = C(1 + K_2(M_0) \ln R_*)$ *for* $n = 2$ *and* $K(M_0, R_*) = C(1 + K_2(M_0))$ *for* $n > 2$.

PROOF. – The maximum principle shows that the solution $\chi(x, t)$ of problem (3.1), (3.3), (3.4), (3.5) is nonnegative, so $p(x, t) = \Phi_\varepsilon(\chi^\varepsilon(x, t)) \geqslant 0$ by (3.2). We define the function

$$v(x, t) = \frac{p(x, t)}{u(x) + 1} \geqslant 0 ,$$

where u is the solution to problem (3.6)-(3.8). By the choice of u and (3.5), the new function v solves the problem

$$(1 + u)\, H'(p)\, v_t = (1 + u)\, \Delta v + \Lambda \cdot \nabla v - cv, \; x \in Q, \; t \geqslant 0 ,$$

$$v\big|_{\partial B(R_*)} = 0, \quad \left((1 + u)\, a \frac{\partial v}{\partial \nu} + (a + b(1 + u))\, v\right)\Big|_S = p_s, \quad v\big|_{t=0} = \frac{p_0}{1 + u} ,$$

where $\Lambda = (1 + u)\, H'(p)\, F + 2\nabla u$ and

$$c = \Phi(x) + (\varepsilon - H'(p)\, F \cdot \nabla u - (1 + u) \frac{H(p)}{p} \operatorname{div} F . \tag{3.11}$$

If the maximum of $v(x, t)$ is positive and attained on the fixed boundary S, then the estimate follows directly from the boundary conditions for v and Lemma 3.1:

$$\max_S v \leqslant \frac{p_s}{a + b(1 + u)} \leqslant \frac{p_s}{a + b} \leqslant p_s \leqslant M_0 .$$

Similarly,

$$\max v\big|_{t=0} \leqslant \frac{p_0}{u + 1} \leqslant p_0 \leqslant M_0 .$$

Suppose that the maximum of v is positive and attained in the interior of Q. In this case, it can only be attained at a point where $c < 0$. We show now that this implies an upper bound on p.

If at the point of maximum $p \leqslant 2\varepsilon$, then

$$\max v = \frac{p}{u + 1} \leqslant 2\varepsilon \leqslant M_0 .$$

If at the point of maximum $p > 2\varepsilon$, then we can use (3.2) to conclude that

$$c < 0, \; H'(p) = \varepsilon, \; \frac{H(p)}{p} = \varepsilon + \frac{1 - 2\varepsilon^2}{p},$$

so (3.11) implies an inequality for p:

$$0 > c = \Phi(x) - (1 + u)\left(\varepsilon + \frac{1 - 2\varepsilon^2}{p}\right) \operatorname{div} F. \tag{3.12}$$

The maximum of v cannot be attained on $Q \backslash B(R_0)$: on this part of the domain $\Phi = \operatorname{div} F = 0$ and $F = 0$, so $c = 0$, which contradicts (3.12).

If the maximum is attained at a point $x \in B(R_0)$, then $\Phi(x) = M_0$, so inequality (3.12) reads

$$0 > c = M_0 - (1 + u)\left(\varepsilon + \frac{1 - 2\varepsilon^2}{p}\right) \operatorname{div} F. \tag{3.13}$$

Therefore it follows from Condition 2.3 that there exists a constant C' such that

$$M_0(1 + u)\frac{1 - 2\varepsilon^2}{p} \geqslant M_0 - \varepsilon(1 + u)\, C' M_0.$$

Combining this estimate with the inequality of Lemma 3.1, we see that if $\varepsilon < \varepsilon_1(R_*)$ is chosen small enough, then at the point of maximum

$$p \leqslant \frac{(1 - 2\varepsilon^2)(1 + u)}{1 - \varepsilon C_2(1 + u)} \leqslant 2(1 + u) \leqslant 2(1 + \widehat{K}(M_0, R_*)),$$

where

$$\widehat{K}(M_0, R_*) = \begin{cases} K_1(M_0) \ln R_*, & n = 2, \\ K_1(M_0), & n > 2. \end{cases}$$

This proves Lemma 3.2. ■

We derive now some additional uniform estimates for the regularized solution. Below ζ_δ is the cutoff function (2.2).

LEMMA 3.3 (The second *a priori* estimate). – *For all* $0 < \varepsilon < \varepsilon_*(M_0, R_*)$ *and* $0 < \delta \leqslant \delta_0$

$$\int_{Q_T} \zeta_\delta \, |\nabla p^\varepsilon|^2 \, dx\, dt \leqslant K(M_0, R_*, \delta). \tag{3.14}$$

PROOF. – Multiplying equation (3.5) by $\zeta_\delta(x)\, p^\varepsilon(x, t)$ and integrating over Q we obtain

$$\frac{d}{dt}\int_Q \zeta_\delta(x)\, V(p^\varepsilon)\, dx + \int_Q \zeta_\delta(x)\, |\nabla p^\varepsilon|^2 dx = -\int_Q \chi\zeta_\delta F^\varepsilon \cdot \nabla p^\varepsilon dx + \sum_{k=1}^{3} I_k,$$

where $V(p) \overset{\text{def}}{=} \int_0^p qH_\varepsilon'(q)\, dq$ for $p \geq 0$ and

$$I_1 = \frac{1}{2}\int_Q |p^\varepsilon|^2 \Delta\zeta_\delta dx,\; I_2 = -\frac{1}{2}\int_S |p^\varepsilon|^2 \frac{\partial\zeta_\delta}{\partial\nu} dS,\; I_3 = -\int_Q \chi p^\varepsilon F^\varepsilon \cdot \nabla\zeta_\delta dx\,.$$

We estimate the terms on the right-hand side using the uniform estimate (3.10):

$$|I_1| + |I_2| + |I_3| \leq C\left(\left(\frac{1}{\delta}K(M_0, R_*)\right)^2 + \frac{1}{\delta}K(M_0, R_*)\right) \leq K_1(M_1, \delta)\,.$$

We complete the proof applying the estimate $|F^\varepsilon(x)| \leq CM_0$ and the Cauchy inequality:

$$\left|\int_Q \chi\zeta_\delta F^\varepsilon \cdot \nabla p^\varepsilon dx\right| \leq \frac{1}{2}\int_Q \zeta_\delta(x)\, |\nabla p^\varepsilon|^2 dx + \frac{1}{2}\int_Q \zeta_\delta\, |F^\varepsilon|^2 dx\,. \qquad \blacksquare$$

3.5. *Stability, monotonicity, and BV estimates.*

The regularized solutions enjoy the following property of stability and monotonicity. Consider the regularized solutions $(p_i^\varepsilon(x, t), \chi_i^\varepsilon(x, t))$, $i = 1, 2$, that correspond to the given data $p_s = p_{s,i}$, $\chi_0 = \chi_{0,i}$, and the source terms $F_i = F$. Recall that we always suppose $a \geq 0$ (see Condition 2.1).

LEMMA 3.4. – *If the given data satisfy the inequalities*

$$p_{s,1} \leq p_{s,2},\; F_1 = F_2,\; \chi_{0,1} \leq \chi_{0,2},$$

then

$$\chi_1^\varepsilon(x, t) \leq \chi_2^\varepsilon(x, t) \;\text{ and }\; p_1^\varepsilon(x, t) \leq p_2^\varepsilon(x, t)\,. \tag{3.15}$$

If $a \geq a_0 = const > 0$ *in Condition* 2.1, *then the regularized solutions satisfy inequality* (2.10). *Inequality* (2.10) *remains true without the assumption that* $a \geq 0$ *is separated from zero in the special case* $p_{s,1} = p_{s,2}$.

PROOF. – The main step in the proof of estimate (3.15) is an application of the maximum principle. The rest of this argument is an adaptation of results obtained by several authors [1, 12].

To derive the stability estimate (2.10), we multiply the equation for the difference $\overline{\chi} = \chi_1^\varepsilon(x, t) - \chi_2^\varepsilon(x, t)$ by $\overline{\chi}/\sqrt{\overline{\chi}^2 + \lambda^2}$, integrate over Q, and pass to the limit as $\lambda \to 0$ using condition (2.7) on S and the fact that $\bar{p}\,\bar{\chi} \equiv (p_1^\varepsilon - p_2^\varepsilon)\,\bar{\chi} \geqslant 0$. ■

The following lemma provides the necessary *BV* estimates.

LEMMA 3.5 (BV estimates). – *For each* $\varepsilon \in]0, < \varepsilon_*]$ *and* $t \in [0, T]$

$$\int_Q \zeta_\delta(x)\,|\chi^\varepsilon(x+h, t) - \chi^\varepsilon(x, t)|\,dx \leqslant K_2\,|h| \tag{3.16}$$

and

$$\int_Q \zeta_\delta(x)\,|\chi^\varepsilon(x, t+\tau) - \chi^\varepsilon(x, t)|\,dx \leqslant K_2\sqrt{|\tau|}\,, \tag{3.17}$$

where ζ_δ *is the same as in Lemma* 3.3, *and the constant* K_2 *can be described in terms of those in Conditions* 2.1-2.3 *and Lemma* 3.3: $K_2(M_0, \delta, R_*) = C\max\left\{M_0, \frac{1}{\delta^2}\sqrt{K(M_0, R_*, \delta)}\right\}$.

In the statement of the lemma $T > 0$ is arbitrary; the constants $\varepsilon_* = \varepsilon_*(M_0, R_*)$ and $K(M_0, R_*, \delta)$ are those of Lemma 3.3.

PROOF of Lemma 3.5 is a modification of the argument used by Götz and Zaltzman [3]. Its main idea is to derive integral estimates for the functions $q_i = \partial\chi^\varepsilon/\partial x_i$ and $\boldsymbol{q} = \sum_{k=1}^n |q_i|$ through calculations with an appropriately regularized sign function.

Differentiating equation (3.5) with respect to x_i, we arrive at the equation

$$\frac{\partial q_i}{\partial t} - \frac{\partial}{\partial x_i}(\nabla\chi\cdot F + \chi\,\mathrm{div}\,F) - \mathrm{div}\,(\nabla(\Phi'(\chi)\,q_i)) = 0\,, \tag{3.18}$$

where $\Phi'(\chi)\,q_i = \partial p/\partial x_i$. We multiply (3.18) by $\zeta_\delta q_i/\sqrt{q_i^2 + \lambda^2}$ and integrate the result over the cylinder Q_t. After integration by parts, we derive for $t \in [0, T]$ the equality

$$\int_Q \zeta_\delta\sqrt{q_i^2 + \lambda^2}\,dx\,\Big|_0^t + I_1 + I_2 + I_3 + I_4 = J_1 + J_2 + J_3, \tag{3.19}$$

where the summands on the right-hand side are

$$J_1 = \int_{Q_t} \frac{-(F\cdot\nabla\zeta_\delta)\,q_i^2}{\sqrt{q_i^2 + \lambda^2}}\,dx\,dt\,, \qquad J_2 = \int_{Q_t} \Delta\zeta_\delta\,\frac{\partial p}{\partial x_i}\,\frac{q_i}{\sqrt{q_i^2 + \lambda^2}}\,dx\,dt\,,$$

$$J_3 = \int_{Q_t} \zeta_\delta\,\frac{q_i}{\sqrt{q_i^2 + \lambda^2}}\left(\chi\,\frac{\partial}{\partial x_i}(\mathrm{div}\,F) + \sum_{j=1}^n q_j\,\frac{\partial F_j}{\partial x_i}\right)dx\,dt\,.$$

The numbered summands on the left hand side of (3.19) are

$$I_1 = \int\limits_{Q_t} \zeta_\delta \Phi'(\chi) |\nabla q_i|^2 \frac{\lambda^2}{(q_i^2+\lambda^2)^{3/2}} dx\,dt \geqslant 0$$

and several «small» summands which converge to zero as $\lambda \to 0$ by the dominated convergence theorem,

$$I_2 = \int\limits_{Q_t} \frac{\zeta_\delta \lambda^2 q_i}{(q_i^2+\lambda^2)^{3/2}} (F\cdot\nabla q_i)\, dx\,dt, \qquad \lim_{\lambda\to 0} I_2 = 0\,,$$

$$I_3 = \int\limits_{Q_t} \zeta_\delta \Phi''(\nabla\chi\cdot\nabla q_i) \frac{q_i\lambda^2}{(q_i^2+\lambda^2)^{3/2}} dx\,dt, \qquad \lim_{\lambda\to 0} I_3 = 0\,,$$

$$I_4 = -\int\limits_{Q_t} (\nabla\zeta_\delta\cdot\nabla q_i)\, \Phi'(\chi) \frac{q_i\lambda^2}{(q_i^2+\lambda^2)^{3/2}} dx\,dt, \qquad \lim_{\lambda\to 0} I_4 = 0\,.$$

Obviously, $\sqrt{q_i^2+\lambda^2} \searrow |q_i|$ as $\lambda \searrow 0$, so it suffices to evaluate the limit of the first summand on the left hand side of (3.19) in terms of $\int\limits_{Q_t} \zeta_\delta \boldsymbol{q} dx\,dt$ and eventually use the Gronwall inequality to get the desired estimate for instant values of $\int \zeta_\delta \boldsymbol{q}$. By the above, the terms I_k, $k \geqslant 2$, are no obstacles to this approach.

The summands on the right hand side need more attention, and it is at this stage that the special choice of the cutoff ζ_δ is essential.

On the neighborhood of the fixed boundary where the integrand does not vanish $-F(x)\cdot\nabla\zeta_\delta(x) \leqslant cM_0\zeta_\delta(x)$ by (2.9), so

$$J_1 \leqslant cM_0 \int\limits_{Q_t} \zeta_\delta |q_i| dx\,dt \leqslant cM_0 \int\limits_{Q_t} \zeta_\delta \boldsymbol{q}\, dx\,dt\,.$$

Assumption (2.3) and smoothness of the fixed boundary permit us to use estimate (3.14) to evaluate J_2 using the Cauchy-Buniakovsky inequality:

$$|J_2| \leqslant \left(\int\limits_{Q_t} \zeta_\delta |\nabla p|^2 dx\,dt\right)^{1/2} \left(\int\limits_{Q_t} \frac{|\Delta\zeta_\delta|^2}{\zeta_\delta} dx\,dt\right)^{1/2} \leqslant$$

$$\frac{C}{\delta^2}\left(\int\limits_{Q_T} \zeta_\delta |\nabla p^\varepsilon|^2 dx\,dt\right)^{1/2} \leqslant \frac{C}{\delta^2}\sqrt{K(M_0, R_*, \delta)}\,.$$

Since $\chi = H_\varepsilon(p) \leqslant 1 + \varepsilon p$, the estimate of Lemma 3.2 for the pressure and the assumption $|F|^{(2)} \leqslant M_0$ yield the following estimate for the integral J_3:

$$|J_3| \leqslant CM_0 \int\limits_{Q_t} (1 + \zeta_\delta \boldsymbol{q})\, dx\,dt\,.$$

Next, we pass to the limit in $\lambda \to 0$ and sum the inequalities that result from (3.19) with $i = 1, 2, \ldots, n$ to obtain for $\boldsymbol{q}$ the estimate

$$\int_Q \zeta_\delta(x)\, \boldsymbol{q}(x, t)\, dx \leq K_2 \left(1 + \int_0^t \left(\int_Q \zeta_\delta \boldsymbol{q}\, dx \right) dt \right) \tag{3.20}$$

with $K_2 = C \max \left\{ M_0, \frac{1}{\delta^2} \sqrt{K(M_0, R_*, \delta)} \right\}$. Combined with Gronwall's inequality, inequality (3.20) guarantees estimate (3.16).

The rest of the proof is the same as in [3]. ■

3.6. *Convergence of regularized solutions.*

Estimate (3.10) allows us to choose sequences $\varepsilon = \varepsilon_j \searrow 0$ for which p^{ε_j} and χ^{ε_j} converge in the following sense:

$$p^\varepsilon(x, t) \xrightarrow{L_2(Q_T)\text{-weak}} \tilde{p}(x, t), \; \chi^\varepsilon(x, t) \xrightarrow{L_2(Q_T)\text{-weak}} \chi(x, t) \tag{3.21}$$

(here and below we omit the index «j»). It is clear that the limit pressure inherits the estimate of Lemma 3.2

$$0 \leq \tilde{p}(x, t) \leq M_1. \tag{3.22}$$

Making use of the inequality

$$\chi^\varepsilon(x, t) = H_\varepsilon(p^\varepsilon(x, t)) \leq H_\varepsilon(\max_\varepsilon p^\varepsilon) \leq 1 + \varepsilon M_1$$

we find that

$$0 \leq \chi^\varepsilon(x, t) \leq 1 + \varepsilon M_1 \text{ and } 0 \leq \chi(x, t) \leq 1 . \tag{3.23}$$

Weak convergence in (3.21) allows us to conclude that the functions χ and $\tilde{p}$ satisfy the integral equality (1.7).

Using (3.16), (3.17), and a standard diagonalization procedure in δ, we obtain sequences of solutions that converge strongly in the modified region (2.1), which proves (1.10) and assertions (a)-(d) of Theorem 2.1 — the latter follow from Lemma 3.4 and Lemma 3.5.

The last statement (e) of Theorem 2.1 does not follow immediately from the assumption $\operatorname{supp}(\operatorname{div} F) \subset \Omega^0$. Yet, if we prove that there exists at least one solution $(\chi, \tilde{p})$ to problem (2.12), (1.8), (1.10) such that

$$\Omega^0 \subset \operatorname{supp}(\chi(t)), \; 0 \leq t \leq T ,$$

then this solution can only be the unique solution to the original problem (1.7), (1.8), (1.10). To show that problem (2.12), (1.8), (1.10) does have a solution, we repeat the preceding steps in the proof of (a)-(d) with minor modifications.

The monotonicity of $\chi(x, t)$ follows from the comparison result (2.11).

To establish it, we consider the solution (p^1, χ^1) that corresponds to the data

$$p_s^1(x, t) = \begin{cases} p_s, & 0 \leqslant t \leqslant t_0, \\ 0, & t > t_0, \end{cases} \qquad f^1 = \begin{cases} f, & 0 \leqslant t \leqslant t_0, \\ 0, & t > t_0, \end{cases} \qquad \chi_0^1 = \chi_0.$$

For this solution, its second component stops to evolve after t_0:

$$\chi^1(x, t) = \begin{cases} \chi(x, t), & 0 \leqslant t \leqslant t_0, \\ \chi(x, t_0), & t \geqslant t_0. \end{cases}$$

Let us compare $\chi^1(x, t)$ with the corresponding component of the solution (p^2, χ^2) obtained from the original data $p_s^2 = p_s, f^2 = f$, and $\chi_0^2 = \chi_0$. Conditions of part (b) of the theorem being satisfied, it follows from (2.11) that

$$\chi(x, t_0) = \chi^1(x, t) \leqslant \chi^2(x, t) = \chi(x, t).$$

3.7. *Equivalence of modified and original problems.*

To complete the proof of Theorem 2.1 in the case of unbounded Q, we now prove that the solutions of the modified and original problems coincide over each finite time interval $[0, T]$ for sufficiently large $R_* = R_*(T)$.

Below, the time $T > 0$ is an arbitrary fixed number. Let us consider the weak solution $(\tilde{p}, \chi)$ of the modified problem obtained above. To show that it solves the original problem with unbounded domain Q, it suffices to prove the inclusion

$$\operatorname{supp} \chi(x, t) \subset B\left(\frac{1}{2} R_*\right), \qquad t \in [0, T]. \tag{3.24}$$

We prove (3.24) applying our comparison results for bounded domains to the solutions of the modified problem and a spherically symmetric auxiliary problem in the spherical layer $B(R_*) \backslash B(R_0)$ for large R_*.

Let us define the upper barrier functions $(\tilde{p}_1, \chi_1)$ as the solution of (1.7) and (1.10) in the layer $R_0 < |x| < R_*$ with the source term

$$F_1 = -\frac{M_0 R_*^{n-1}}{r^{n-1}} \frac{x}{r}, \qquad r = |x|, \tag{3.25}$$

the boundary conditions

$$\tilde{p}_1 \big|_{|x| = R_0} = K, \qquad \tilde{p}_1 \big|_{|x| = R_*} = 0, \tag{3.26}$$

with the constant K defined using the constant in Lemma 3.1 as

$$K = \begin{cases} K_1(M_0)\ln R_*, & n=2, \\ K_1(M_0), & n>2, \end{cases}$$

and the initial condition

$$\chi_1(x, 0) = \begin{cases} 1, & R_0 < |x| \leqslant R_1, \\ 0, & R_1 < |x| \leqslant R_*, \end{cases}$$

in which R_1 is one more parameter chosen so that $R_1/R_0 > e$.

The solution $(\tilde{p}_1, \chi_1)$ is spherically symmetrical and can be found explicitly. In particular

$$\chi_1(x, t) = \begin{cases} 1, & R_0 < |x| < R(t), \\ 0, & |x| > R(t), \end{cases}$$

where $R(t)$ is determined by the following equations for the ratio $z = (R(t)/R_0)^n$ (which proves non decreasing): for $n=2$

$$\frac{dz}{dt} = \left(\frac{4K_1(M_0)\ln R_*}{R_0^2 \ln z} + 2\,\frac{M_0 R_*}{R_0^2} \right) \geqslant 0\,,$$

while for $n>2$ and $\alpha = 1 - \dfrac{2}{n}$

$$\frac{dz}{dt} = \left(\frac{n(n-2)\,K_1(M_0)}{R_0^2}\,\frac{z^{\alpha}}{z^{\alpha}-1} + \frac{nM_0R_*^{n-1}}{R_0^n} \right) \geqslant 0\,.$$

In both cases $z(t) > e$ for all $t > 0$ and

$$\frac{dz}{dt} = \frac{d}{dt}\left(\frac{R(t)}{R_0} \right)^n \leqslant C_1 K + C_2 R_*^{n-1} M_0, \tag{3.27}$$

where

$$C_1 = \max\left\{ 1,\ \frac{e^{\alpha}}{e^{\alpha}-1}\,\frac{n^2}{R_0^2} \right\}, \qquad C_2 = \frac{n}{R_0^n}\,.$$

Integrating (3.27) we conclude that

$$\frac{R(t)}{R_0} \leqslant (R_1^n + (C_1 K + C_2 R_*^{n-1} M_0)\,t)^{1/n}. \tag{3.28}$$

Below, we consider very large values of the size R_* of domain (2.1). The above inequality implies that the size $R(t)$ of the flow region for the auxiliary problem grows slower than R_*: for each dimension n and fixed $T > 0$

$$\lim_{R_* \to \infty} \max_{0 \leqslant t \leqslant T} \frac{R(t)}{R_*} = 0 . \tag{3.29}$$

Note that the regularized solutions $\chi_1^\varepsilon(x)$ of the auxiliary problem are also spherically symmetrical, and if we choose $\chi_{1,0}^\varepsilon(r)$ as a decreasing function, then the radial derivative of χ_1 remains non positive,

$$\frac{\partial \chi_1^\varepsilon}{\partial r} \leqslant 0 . \tag{3.30}$$

We show now that $\chi^\varepsilon(x, t) \leqslant \chi_1^\varepsilon(|x|, t)$ in $(Q_{\text{mod}} \backslash B(R_0)) \times (0, T)$.

The differences $\overline{\chi} = \chi_1^\varepsilon - \chi^\varepsilon$ and $\overline{p} = p_1^\varepsilon - p^\varepsilon$ solve the equation

$$\frac{\partial \overline{\chi}}{\partial t} - F \cdot \nabla \overline{\chi} - \Delta \overline{p} = I \equiv (F_1 - F) \cdot \nabla \chi_1^\varepsilon ,$$

with boundary and initial data that satisfy the conditions $\overline{\chi}(x, 0) \geqslant 0$, $\overline{\chi}\,|_S \geqslant 0$, and $\overline{\chi}\,|_{\partial B(R_*)} = 0$. Hence, if we show that

$$I \geqslant 0 , \tag{3.31}$$

then it would follow that $\overline{\chi}(x, t) \geqslant 0$ for $(x, t) \in Q_T$.

Inequality (3.31) is a simple consequence of the choice of F_1 and (3.30). Namely, for $r \leqslant R_*$

$$I \equiv (F_1 - F) \cdot \nabla \chi_1^\varepsilon = -\left(\left(\frac{M_0 R_*^{n-1}}{r^{n-1}} \frac{x}{r} + F \right) \cdot \frac{x}{r} \right) \frac{\partial \chi_1^\varepsilon}{\partial r} =$$

$$- \frac{\partial \chi_1^\varepsilon}{\partial r} \left(\frac{M_0 R_*^{n-1}}{r^{n-1}} + \left(F \cdot \frac{x}{r} \right) \right) \geqslant \left| \frac{\partial \chi_1^\varepsilon}{\partial r} \right| (M_0 - |F|) \geqslant 0 .$$

Passing to the limit in $\varepsilon \to 0$, we get the estimate

$$\chi(x, t) \leqslant \chi_1(x, t) \qquad \text{for} \qquad |x| \leqslant R(t) ,$$

so $\chi(x, t)$ also vanishes outside $B(R(t))$. This estimate and (3.29) show that for each $T > 0$ there exists a value of $R_* = R_*(T) > 0$ such that

$$\forall t \in [0, T] \qquad \operatorname{supp} \chi(t) \subset B\left(\frac{1}{2} R_* \right) .$$

Hence the solution to the modified problem solves the original problem in unbounded domain as well. ∎

4. – Proof of Theorem 2.2.

Let (p, χ) be a solution of problem (1.1)-(1.5) which corresponds to the data (p_s, χ_0), and denote by $(\tilde{p}, \tilde{\chi})$ a solution to the same problem with the same data (p_s, χ_0) which is obtained by the regularization method. As shown above, the support of the latter solution is compact in Q_T. By Definition 1.1, the differences $\overline{\chi} = \chi - \tilde{\chi}$ and $\overline{p} = p - \tilde{p}$ satisfy the integral identity

$$I \equiv \int_0^T \int_Q (\overline{\chi}\varphi_t - \overline{\chi}F\cdot\nabla\varphi + \overline{p}\Delta\varphi)\,dx\,dt = 0\,. \tag{4.1}$$

We introduce the domains $Q_T^+ = \{(x, t) \in Q_T,\ |\overline{\chi}(t, x)| > 0\}$ and $Q_T = Q \times (0, T)$. Using these, we rewrite (4.1) in the form

$$I = \int_{Q_T^+} \overline{\chi}(\varphi_t - F\cdot\nabla\varphi + \mu\Delta\varphi)\,dx\,dt = 0, \qquad \mu = \overline{p}/\overline{\chi}\,, \tag{4.2}$$

where $\mu \geq 0$ by (1.10).

We use μ of (4.2) to introduce the auxiliary functions

$$\mu_\varepsilon(x, t) = \begin{cases} \mu(x, t), & 0 \leq \mu(x, t) \leq \dfrac{1}{\varepsilon}\,, \\[2ex] \dfrac{1}{\varepsilon}\,, & \mu(x, t) > \dfrac{1}{\varepsilon}\,, \end{cases}\,, \qquad \nu_\varepsilon(x) = \varepsilon e^{-|x|} < 1\,, \tag{4.3}$$

and

$$\lambda(x, t) = \lambda_\varepsilon(x, t) \stackrel{\text{def}}{=} \mu_\varepsilon(x, t) + \nu_\varepsilon(x)\,. \tag{4.4}$$

Note that $\nu_\varepsilon \leq \lambda_\varepsilon \leq \dfrac{1}{\varepsilon} + \nu_\varepsilon$ and $\lambda_\varepsilon = \mu + \nu_\varepsilon$ if $0 \leq \mu \leq \dfrac{1}{\varepsilon}$, while $\lambda_\varepsilon = \dfrac{1}{\varepsilon} + \nu_\varepsilon$ if $\mu > \dfrac{1}{\varepsilon}$.

The above functions are used for ε so small that $\varepsilon \max p(t, x) < 1$. We apply the following auxiliary proposition proved in Appendix A.2.

LEMMA 4.1. – *Suppose that $F \in C^1$ and the continuously differentiable function $h(x, t)$ vanishes for x outside a compact subset of Q. If the source term satisfies Condition 2.3 and*

$$\varepsilon e^{-|x|} \leq \lambda(x, t) \leq \frac{1}{\varepsilon} + \varepsilon e^{-|x|}, \qquad \varepsilon > 0\,, \tag{4.5}$$

then the problem

$$\frac{\partial\varphi}{\partial t} - F\cdot\nabla\varphi + \lambda\Delta\varphi = h(t, x), \ (x, t) \in Q_T = Q \times (0, T), \tag{4.6}$$

$$\varphi|_{S'} = 0, \ \frac{\partial\varphi}{\partial\nu}\Big|_{S''} = 0, \ t \in (0, T), \qquad \varphi|_{t=T} = 0, \ x \in Q, \tag{4.7}$$

has a solution $\varphi \in W^{2,1}(Q_T)$. *Moreover, this solution admits the estimates*

$$\max_{t\in[0,T]} \int_Q |\nabla\varphi(x, t)|^2 dx + \int_0^T \int_Q \lambda|\Delta\varphi|^2 dx\,dt \leqslant C_0, \tag{4.8}$$

where $C_0 = C_0\left(T, \int_0^T \int_Q |\nabla h|^2 dx\,dt, \|F\|^{(1)}_{Q_T}\right)$, *and*

$$\int_0^T \int_Q \frac{1}{\lambda}\left|\frac{\partial\varphi}{\partial t}\right|^2 dx\,dt \leqslant 3\int_0^T \int_Q \left(\lambda|\Delta\varphi|^2 + \frac{h^2 + |F|^2|\nabla\varphi|^2}{\lambda}\right) dx\,dt. \tag{4.9}$$

Let $\varphi(x, t)$ be a solution to problem (4.6)-(4.7) with the function λ given by (4.4) and an arbitrary function h satisfying conditions of Lemma 4.1. For this choice of φ, the relation (4.2) takes on the form

$$\int_{Q_T^+} \overline{\chi} h\,dx\,dt + I^+ = 0, \qquad I^+ = \int_{Q_T^+} \overline{\chi}(\mu - \mu_\varepsilon - \nu_\varepsilon)\,\Delta\varphi\,dx\,dt. \tag{4.10}$$

As shown in subsection 3.7, if x lies outside of some ball $B(R_0)$, then

$$\tilde{\chi}(x, t) = 0, \qquad \overline{\chi}(x, t) = \chi(x, t), \ \mu(x, t) = p(x, t). \tag{4.11}$$

We represent I^+ in the form

$$I^+ = I_1 + I_2, \tag{4.12}$$

where

$$I_1 = \int_{B_T^+} \overline{\chi}(\mu - \mu_\varepsilon - \nu_\varepsilon)\,\Delta\varphi\,dx\,dt, \qquad I_2 = \int_{Q_T^+\setminus B_T^+} \overline{\chi}(\mu - \mu_\varepsilon - \nu_\varepsilon)\,\Delta\varphi\,dx\,dt,$$

and $B_T^+ = \{(x, t) \in B(R_0) \times (0, T) \colon |\overline{\chi}(x, t)| > 0\}$.

We start with estimates over the set where $x \in B(R_0)$ (see (4.11)). Using estimate (4.8) of Lemma 4.1, we evaluate I_1 in the following way:

$$|I_1|^2 \leqslant \left(\int_{B_T^+} \overline{\chi}^2 \frac{(\mu - \mu_\varepsilon - \nu_\varepsilon)^2}{(\mu_\varepsilon + \nu_\varepsilon)} dx\,dt\right) \times \left(\int_{B_T^+} (\mu_\varepsilon + \nu_\varepsilon)|\Delta\varphi|^2 dx\,dt\right) \leqslant I_0 C_0, \tag{4.13}$$

where

$$I_0 = \int_{B_T^+} \overline{\chi}^2 \frac{(\mu - \mu_\varepsilon - \nu_\varepsilon)^2}{(\mu_\varepsilon + \nu_\varepsilon)} dx\,dt = \int_{B_T^+} \overline{p}^2 \left| \frac{\mu - \mu_\varepsilon - \nu_\varepsilon}{\mu} \right|^2 \frac{dx\,dt}{(\mu_\varepsilon + \nu_\varepsilon)} .$$

From the definition of the function μ_ε in (4.3), we conclude that at the points where $\mu > \frac{1}{\varepsilon}$

$$\left| \frac{\mu - \mu_\varepsilon - \nu_\varepsilon}{\mu} \right|^2 \frac{\overline{p}^2}{\mu_\varepsilon + \nu_\varepsilon} \leqslant \frac{\varepsilon}{1 + \varepsilon\nu_\varepsilon} \max \overline{p}^2, \qquad \lim_{\varepsilon \to 0} \frac{\varepsilon}{1 + \varepsilon\nu_\varepsilon} \max \overline{p}^2 = 0 .$$

At the points where $\mu < \frac{1}{\varepsilon}$ and $\mu = \mu_\varepsilon$ by (4.3)

$$\overline{\chi}^2 \frac{(\mu - \mu_\varepsilon - \nu_\varepsilon)^2}{\mu_\varepsilon + \nu_\varepsilon} = \overline{\chi}^2 \frac{\nu_\varepsilon^2}{\mu + \nu_\varepsilon} \leqslant \nu_\varepsilon, \qquad \lim_{\varepsilon \to 0} \nu_\varepsilon = 0 .$$

Thus $\lim_{\varepsilon \to 0} I_1 = 0$.

Let us now consider the integral I_2 over the complementary set $Q_T^+ \setminus B_T^+$, where we can make use of (4.11). In its integrand $\overline{\chi} = \chi$ and $\mu = p/\chi$. We decompose I_2 into integrals over the smaller domains

$$Q_0 = (Q_T^+ \setminus B_T^+) \cap \{0 < \chi(x,t) < 1\}, \qquad Q_1 = (Q_T^+ \setminus B_T^+) \cap \{\chi(x,t) = 1\} .$$

For $(x,t) \in Q_0$, it follows from the definition of solution that $p = 0$ and $\mu = \mu_\varepsilon = 0$. Thus

$$\left| \int_{Q_0} \overline{\chi}(\mu - \mu_\varepsilon - \nu_\varepsilon)\, \Delta\varphi\, dx\,dt \right| = \left| \int_{Q_0} \overline{\chi}\nu_\varepsilon \Delta\varphi\, dx\,dt \right| \leqslant$$

$$\left(\int_{Q_0} \nu_\varepsilon\, dx\,dt \right)^{1/2} \left(\int_{Q_0} (\mu_\varepsilon + \nu_\varepsilon)\, |\Delta\varphi|^2\, dx\,dt \right)^{1/2} \leqslant \sqrt{C_0\,\varepsilon} \left(\int_{Q_0} e^{-|x|}\, dx\,dt \right)^{1/2}$$

and consequently $\lim_{\varepsilon \to 0} \left| \int_{Q_0} \overline{\chi}(\mu - \mu_\varepsilon - \nu_\varepsilon)\, \Delta\varphi\, dx\,dt \right| = 0$.

For $(x,t) \in Q_1$, there are relations $\mu = p < \frac{1}{\varepsilon}$ and $\mu = \mu_\varepsilon$, so

$$\left| \int_{Q_1} \overline{\chi}(\mu - \mu_\varepsilon - \nu_\varepsilon)\, \Delta\varphi\, dx\,dt \right| = \left| \int_{Q_1} \overline{\chi}\nu_\varepsilon \Delta\varphi\, dx\,dt \right| \leqslant$$

$$\left(\int_{Q_1} (\mu_\varepsilon + \nu_\varepsilon)\, |\Delta\varphi|^2\, dx\,dt \right)^{1/2} \sqrt{\varepsilon} \left(\int_{Q_1} e^{-|x|}\, dx\,dt \right)^{1/2},$$

and consequently $\lim_{\varepsilon \to 0} \left| \int_{Q_1} \overline{\chi}(\mu - \mu_\varepsilon - \nu_\varepsilon)\, \Delta\varphi\, dx\,dt \right| = 0$.

Next, we pass to the limit in (4.10) for $\varepsilon \to 0$ and finally arrive at the equality

$$\int_{Q_T^+} \overline{\chi} h \, dx \, dt = 0 \tag{4.14}$$

for an arbitrary function h, which signifies that $\overline{\chi} = 0$ and, respectively, $\overline{p} = 0$. This proves the theorem. ∎

A. – Proofs of auxiliary propositions.

A.1. *Proof of Lemma* 3.1.

Let us define the function $\varphi(x)$ as the solution of (3.6), (3.7), (3.8) in the smaller domain $Q \cap B(3R_0)$ with the Dirichlet condition (3.7) satisfied on the sphere $\partial B(3R_0)$ instead of $\partial B(R_*)$. The solution $\varphi(x)$ is unique and bounded independently of R_*. Using the pertinent local estimates, we find that the gradient of φ is bounded at a finite distance from Ω^0:

$$|\nabla \varphi(x)| \leqslant C, \qquad R_0 \leqslant |x| \leqslant 3R_0.$$

Choose a function $\eta \in C^\infty(Q)$ such that

$$\eta(x) = \begin{cases} 1, & |x| \leqslant R_0, \\ 0, & |x| \geqslant 2R_0 \end{cases}$$

Conditions (3.7) and (3.8) hold for $\eta\varphi$ and

$$-\Delta(\eta\varphi) - \varepsilon F \cdot \nabla(\eta\varphi) = \Phi_1(x), \qquad x \in Q\,,$$

where

$$\Phi_1(x) = \begin{cases} \Phi(x), & |x| < R_0, \\ 0, & |x| > 2R_0, \end{cases} \qquad |\Phi_1(x)| \leqslant C \text{ if } R_0 \leqslant |x| \leqslant 2R_0.$$

The function $V = u - \eta\varphi$ solves the boundary value problem (3.6), (3.7) with the homogeneous boundary condition (3.8) and a bounded right hand side in (3.6), which vanishes outside of the spherical layer $B(2R_0) \backslash B(R_0)$.

Let us rewrite (3.6) as an equation for V, multiply this equation by V, and integrate the result over the domain Q. Integration by parts shows that for $\varepsilon \leqslant \varepsilon_*(R_*)$

$$I_1^2 \equiv \int_Q |\nabla V|^2 \, dx \leqslant I_2 \equiv C \int_{|x| \leqslant 2R_0} |V| \, dx\,. \tag{A.1}$$

Now, we estimate the right hand side of this inequality. Put $I_0(r) \stackrel{\text{def}}{=}$

$\int\limits_{|x|=r} |V| dS$ and pass to the spherical coordinates (r, θ). Using the equality

$$|V(x)| = \left| \int\limits_{|x|}^{R_*} \frac{\partial \overline{V}}{\partial r}(r, \theta)\, dr \right|, \qquad \overline{V}(r, \theta) \stackrel{\text{def}}{=} (x(r, \theta)),$$

we obtain the estimates

$$\text{(A.2)} \qquad |V(x)| \leqslant \begin{cases} \sqrt{\ln (R_*/R_0)} \left(\int\limits_{|x|}^{R_*} r |\nabla V|^2 \, dr \right)^{1/2}, & n = 2, \\ \left(\dfrac{R_*^{n-2} - R_0^{n-2}}{R_0^{n-2} R_*^{n-2} (n-2)} \right)^{1/2} \left(\int\limits_{|x|}^{R_*} r |\nabla V|^2 \, dr \right)^{1/2}, & n > 2. \end{cases}$$

These estimates yield the inequalities

$$\text{(A.3)} \qquad I_0(r) \leqslant \begin{cases} C(\ln R_*)^{1/2} \left(\int\limits_Q |\nabla V|^2 \, dx \right)^{1/2} = C I_1 \sqrt{\ln R_*}, & n = 2, \\ C I_1, & n > 2. \end{cases}$$

Using (A.3) we find that

$$\text{(A.4)} \qquad I_2 = \int\limits_0^{2R_0} I_0(r)\, dr \leqslant \begin{cases} C I_1 \sqrt{\ln R_*}, & n = 2, \\ C I_1, & n > 2. \end{cases}$$

Combining (A.4) with (A.1) yields the estimate

$$\text{(A.5)} \qquad I_1 \leqslant \begin{cases} C \sqrt{\ln R_*}, & n = 2, \\ C, & n > 2. \end{cases}$$

Substituting (A.5) into (A.3) we see that

$$I_0 \leqslant \begin{cases} C \ln R_*, & n = 2, \\ C, & n > 2. \end{cases}$$

The inequality

$$\text{(A.6)} \qquad \max_{|x| = R_0} |V| \leqslant \begin{cases} C \ln R_*, & n = 2, \\ C, & n > 2, \end{cases}$$

now follows from the local estimate [9]. The maximum principle for the function $V(x)$ implies the boundedness of $|V(x)|$ and, consequently, that of $u(x)$ in $B(R_0)$:

$$\max_{|x| \leqslant R_0} u(x) \leqslant \begin{cases} C \ln R_*, & n = 2, \\ C, & n > 2. \end{cases}$$

Outside of $B(R_0)$, the function $u(x)$ satisfies the homogeneous equation (3.6). One more application of the maximum principle for $u(x)$ in $Q \cap B(R_0)$ completes the proof of estimates (3.9). ■

A.2. *Proof of Lemma* 4.1.

First, we consider problem (4.6)-(4.7) in the bounded domain $Q^R = Q \cap B(R)$ and suppose that $h(x, t)$ vanishes outside a compact subset of Q^R for each $t \in (0, T)$. The equation for φ is

$$\frac{\partial \varphi}{\partial t} - F \cdot \nabla \varphi + \lambda \Delta \varphi = h(x, t), \; x \in Q^R, \; t \in (0, T). \tag{A.7}$$

The boundary conditions are $\varphi|_{S''} = 0$, $\partial\varphi/\partial\nu|_{S'} = 0$, $\varphi|_{|x| = R} = 0$; moreover, $\varphi|_{t = T} = 0$.

We introduce the new time variable $\tau = T - t$. This transforms the problem stated above for equation (A.7) into that of finding $\varphi = \varphi(x, \tau)$ from the equation

$$-\frac{\partial \varphi}{\partial \tau} - F \cdot \nabla \varphi + \lambda \Delta \varphi = h, (x, \tau) \in Q_T^R \equiv Q^R \times (0, T), \tag{A.8}$$

under the boundary and initial conditions

$$\varphi|_{S'} = 0, \quad \left.\frac{\partial \varphi}{\partial \nu}\right|_{S''} = 0, \quad \varphi|_{|x| = R} = 0, \quad \varphi|_{\tau = 0} = 0\,. \tag{A.9}$$

In (A.8), we use the original notation of (A.7) for the function of the reversed time τ that corresponds to h.

It is well known [9] that problem (A.8)-(A.9) has a unique solution $\varphi \in W^{2,1}(Q_T^R)$. We seek estimates for this solution which do not depend either on R or on λ.

We multiply equation (A.8) by $\Delta\varphi$ and integrate over $Q^R \times (0, \tau)$. For each $\tau \in (0, T)$, this yields the equality

$$\frac{1}{2} \int_{Q^R} |\nabla \varphi(\tau, x)|^2 \, dx + \int_0^\tau \int_{Q^R} \lambda |\Delta \varphi|^2 \, dx \, d\tau' = \int_0^\tau I(\tau')\, d\tau' + J(\tau), \tag{A.10}$$

where

$$I = \int_{Q^R} (F \cdot \nabla \varphi)\, \Delta \varphi \, dx, \qquad J(\tau) = \int_0^\tau \int_{Q^R} h \Delta \varphi \, dx \, d\tau' . \tag{A.11}$$

It is evident that

$$|J(\tau)| = \left| \int_0^\tau \int_{Q^R} h \Delta \varphi \, dx \, d\tau' \right| = \left| \int_0^\tau \int_{Q^R} -\nabla h \nabla \varphi \, dx \, d\tau' \right| \leqslant \tag{A.12}$$

$$\frac{1}{2} \int_0^\tau \int_{Q^R} (|\nabla h|^2 + |\nabla \varphi|^2) \, dx \, d\tau' .$$

We represent the integrand of the first summand in (A.10) in the form

$$I = \int_{Q^R} \left(\frac{|\nabla \varphi|^2 \operatorname{div} F}{2} - \nabla \varphi \cdot [(\nabla \varphi \cdot \nabla) F] \right) dx + J_S, \tag{A.13}$$

where

$$J_S = \int_{\partial Q^R} \left((F \cdot \nabla \varphi) \frac{\partial \varphi}{\partial \nu} - \frac{1}{2} |\nabla \varphi|^2 (F \cdot \nu) \right) ds$$

is the boundary integral and (repeated indices imply summation)

$$(F \cdot \nabla \varphi)\, \Delta \varphi = F_j \frac{\partial \varphi}{\partial x_j} \frac{\partial^2 \varphi}{\partial x_i^2} , \qquad \nabla \varphi \cdot [(\nabla \varphi \cdot \nabla) F] = \frac{\partial F_j}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \frac{\partial \varphi}{\partial x_i} .$$

This identity is derived using integration by parts and the elementary formula

$$(F \cdot \nabla \varphi)\, \Delta \varphi = \operatorname{div} \left((F \cdot \nabla \varphi)\, \nabla \varphi - \frac{|\nabla \varphi|^2 F}{2} \right) + \frac{|\nabla \varphi|^2 \operatorname{div} F}{2} - \nabla \varphi \cdot [(\nabla \varphi \cdot \nabla) F] .$$

Using (A.9), (2.7), and (2.8) we conclude that the boundary integral in (A.13) is non-negative:

$$J_S = \int_{|x| = R} \left((F \cdot \nabla \varphi) \frac{\partial \varphi}{\partial \nu} - \frac{1}{2} |\nabla \varphi|^2 (F \cdot \nu) \right) ds = \frac{1}{2} \int_{|x| = R} |\nabla \varphi|^2 (F \cdot \nu) \, ds \leqslant 0 .$$

Hence in (A.10)

$$\int_0^\tau I(\tau)\,d\tau = \int_0^\tau \int_{B(R)} \left(\frac{1}{2} |\nabla\varphi|^2 \operatorname{div} F - \nabla\varphi \cdot [(\nabla\varphi\cdot\nabla)\,F] \right) dx\,dt \leqslant C(\|F\|^{(1)}) \int_0^\tau \int_{B(R)} |\nabla\varphi|^2\,dx\,dt\,. \tag{A.14}$$

We collect (A.10), (A.12), and (A.14) to obtain the following estimate, which is valid for $\tau \in (0, T)$:

$$\frac{1}{2} \int_{Q^R} |\nabla\varphi(\tau, x)|^2\,dx + \int_0^\tau \int_{Q^R} \lambda |\Delta\varphi|^2\,dx\,dt \leqslant C(\|F\|^{(1)}) \int_0^\tau \int_{Q^R} (|\nabla h|^2 + |\nabla\varphi|^2)\,dx\,dt\,. \tag{A.15}$$

The Gronwall inequality permits us to deduce from (A.15) an estimate for instant values of the L^2 norm of $\nabla\varphi$ that is uniform in $\tau \in (0, T)$. Combined with (A.15), this estimate yields the desired inequality (4.8), where the constant C_0 does not depend on R.

Denote by $\varphi_\varrho(t, x)$ the solution of (A.7) that corresponds to $R = \varrho$. For each fixed $R > 0$, the restrictions of functions from the family $\{\varphi_\varrho, \varrho > R\}$ constitute a weakly precompact class in $W^{2,1}(Q_T^R)$. Using a diagonal process, we can choose a sequence $\varrho_k \nearrow \infty$ so that for each $R > 0$ there is weak convergence $\varphi_{\varrho_k} \xrightarrow{\text{weak}} \varphi$ in $W^{2,1}(Q_T^R)$. The limit φ solves (4.6)-(4.7) and inherits the estimates of the lemma from φ_{ϱ_k}.

Inequality (4.9) is obvious from (4.6) and the Cauchy inequality. This proves the lemma. ■

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