Bollettino Unione Matematica Italiana

Alessandro Tancredi, Alberto Tognoli

A note on global Nash subvarieties and Artin-Mazur theorem

Bollettino dell'Unione Matematica Italiana, Serie 8, Vol. **7-B** (2004), n.2, p. 425–431.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2004_8_7B_2_425_0>

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2004.

Bollettino U. M. I.
(8) **7**-B (2004), 425-431

A Note on Global Nash Subvarieties and Artin-Mazur Theorem.

ALESSANDRO TANCREDI - ALBERTO TOGNOLI (*)

Sunto. – *Si prova che ogni sottospazio di Nash connesso di* $\mathbb{R}^n$ *che abbia equazioni globali è Nash isomorfo ad una componente connessa di una varietà algebrica che, nel caso compatto, può essere scelta con due sole componenti connesse arbitrariamente vicine. Alcuni esempi illustrano i limiti dei risultati ottenuti e degli strumenti utilizzati.*

Summary. – *It is shown that every connected global Nash subvariety of* $\mathbb{R}^n$ *is Nash isomorphic to a connected component of an algebraic variety that, in the compact case, can be chosen with only two connected components arbitrarily near each other. Some examples which state the limits of the given results and of the used tools are provided.*

1. – Introduction.

It is well known (cf. [BCR]), by a simple application of Artin-Mazur Theorem, that for every connected semi-algebraic Nash submanifold M of $\mathbb{R}^n$ there exist an algebraic variety Z in $\mathbb{R}^{n+s}$ and an open Nash embedding $\sigma : M \to Z$ such that $\sigma(M)$ is a connected component of Z.

We show that this result can be easily extended to global Nash subvarieties S of M, by also getting that the inverse of σ and some arbitrarily chosen Nash functions on S are induced by regular functions on Z. However we remark by a counterexample that this result can not be a characterization of the global Nash subvarieties. Moreover, in the compact case, we show that Z can be chosen with only two connected components, both Nash isomorphic to S and arbitrarily near each other.

The Artin-Mazur Theorem is a powerful tool in the study of Nash functions that we have used several times with some extensions (cf. [TT1], [TT2], [TT3]). In particular in [TT3], in order to get some algebraic approximations, we considered an algebraic subvariety W in $\mathbb{R}^n$ contained in M and we gave very strong conditions for the algebraicity of $\sigma(W)$, which were quite natural in the

(*) The authors are members of G.N.S.A.G.A. of I.N.D.A.M. This work is partially supported by M.I.U.R. and by European Contract HPRN-CT-2001-00271.

complex Nash setting. Here we show that, in general, $\sigma(W)$ can not be algebraic and then that stronger tools than Artin-Mazur theorem are necessary in order to establish deeper results about the existence of algebraic structures on Nash subvarieties (cf. [AK], [BCR], [To]).

2. – Preliminary remarks and notations.

We recall some results and definitions that we shall use in the following without further references (cf. [BCR], [TT1]).

By an *algebraic variety* we mean an affine algebraic subvariety of some space $\mathbb{R}^n$ and we consider the strong topology on it.

Let Ω be an open subset of $\mathbb{R}^n$; a *Nash function* on Ω is a real analytic function which verifies on Ω an algebraic relation over the polynomial ring $\mathbb{R}[X_1, \dots, X_n]$.

A *Nash subvariety* X of $\mathbb{R}^n$ is a locally closed subset which is, locally, the zero set of a finite number of Nash functions. *Nash functions* on open sets of X are locally restrictions of Nash functions on open sets of $\mathbb{R}^n$.

We denote by $\mathcal{N}_X$ the sheaf of Nash functions on X.

A *Nash submanifold* is a Nash subvariety which is smooth as an analytic subvariety. A Nash subvariety X of a Nash submanifold M is, locally, the zero set of a finite number of Nash functions on open subsets of M.

A *Nash map* between Nash subvarieties is locally defined by Nash functions. A *Nash isomorphism* of Nash subvarieties is a Nash map whose inverse is also a Nash map.

We say that a Nash subvariety is *coherent* if it is coherent as an analytic subvariety.

A Nash subvariety X of a Nash submanifold M is a *global* Nash subvariety if it is the zero set of finitely many Nash functions on M. It is well known that a Nash subvariety may fail to be a global Nash subvariety (cf. e.g. Remark 3). However coherent semi-algebraic Nash subvarieties are global Nash subvarieties (cf. [CRS], [CS]).

We will use the Artin-Mazur Theorem in the following form (cf. [BCR]):

THEOREM 1. – *Let M be a connected semi-algebraic Nash submanifold of dimension m of $\mathbb{R}^n$ and $f : M \to \mathbb{R}^q$ a Nash map.*

Then there exist a nonsingular irreducible affine algebraic subvariety $N \subset \mathbb{R}^{n+q+s}$ of dimension m, an open Nash embedding $\sigma : M \to N$ and a regular map $h : N \to \mathbb{R}^q$ such that $\pi\sigma = id_M$ and $h\sigma = f$, where $\pi : N \to \mathbb{R}^n$ is induced by the canonical projection $\mathbb{R}^{n+q+s} \to \mathbb{R}^n$ and $h : N \to \mathbb{R}^q$ is induced by the canonical projection $\mathbb{R}^{n+q+s} \to \mathbb{R}^q$.

Moreover $\sigma(M)$ is a connected component of $\pi^{-1}(M)$.

3. – Global Nash subvarieties

LEMMA 1. – *Let M be a semi-algebraic Nash submanifold of $\mathbb{R}^n$ and let S be a closed Nash subvariety of M which is global in an open semi-algebraic neighborhood of S in M. Then S is global in M.*

Moreover every Nash function on S that extends to a Nash function on a neighborhood of S in M extends to M.

PROOF. – There exist an open semi-algebraic neighborhood U of S in M and a coherent sheaf $\mathcal{C}$ of $(\mathcal{N}_M|_U)$-modules such that $S = Supp(\mathcal{C})$, where $\mathcal{C}$ is a quotient of $\mathcal{N}_M|_U$ by an ideal generated by finitely many Nash functions on U. There exist a coherent sheaf $\mathcal{E}$ of $\mathcal{N}_M$-modules and a semi-algebraic open neighborhood V of S in U such that $\mathcal{C}|_V \cong \mathcal{E}|_V$ and $\mathcal{E}|_{M-S} = 0$. Let $\mathfrak{J}$ be the annihilator of $\mathcal{E}$: since $\mathcal{C}$ and $\mathcal{E}$ coincide on an open semi-algebraic neighborhood of S, the coherent ideal $\mathfrak{J}$ is a finite ideal of $\mathcal{N}_M$, and then (cf. [CS]) it is generated by finitely many Nash functions $f_1, \dots, f_p \in \mathcal{N}_M(M)$ and so $S = \{x \in M | f_1(x) = \dots = f_p(x) = 0\}$.

If ϕ is a Nash function defined on some open neighborhood U of S in M let us consider its canonical image in $\Gamma(U, (\mathcal{N}_M/\mathfrak{J})|_U)$. Since $\Gamma(U, (\mathcal{N}_M/\mathfrak{J})|_U) \cong \Gamma(M, \mathcal{N}_M/\mathfrak{J})$, by the results of [CS] it follows that there exists a Nash function $\tilde{\phi}$ on M such that $\tilde{\phi}|_S = \phi|_S$.

LEMMA 2. – *Let X' be a compact connected component of a real algebraic variety X of $\mathbb{R}^n$. There exist a compact algebraic variety Z of $\mathbb{R}^{n+2}$, which has only two connected components, Z_1 and Z_2, arbitrarily near each other, and a regular map $\tau: Z \to X$, induced by the canonical projection of $\mathbb{R}^{n+2}$ onto $\mathbb{R}^n$, such that $\tau(Z) = X'$ and $\tau|_{Z_i}: Z_i \to X'$ is a Nash isomorphism, for $i = 1, 2$.*

PROOF. – As in [Wa], by assuming $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ canonically embedded into $\mathbb{R}^{n+2}$, we can find an open disk B of $\mathbb{R}^{n+1}$ centered at the origin of radius b such that $X' \subset B$. Let ε be the distance from X' to the boundary of B and let $a > b + \varepsilon$ be a real number. Let D be the open disk of $\mathbb{R}^{n+1}$ centered at the origin of radius a and let $X'' = X - X'$. Then there exists a continuous function f on $\overline{D} \cap \mathbb{R}^n$ such that $f|_{X'} = 0$ and $f|_{X'' \cap \overline{D}} = a + \varepsilon$. By the Weierstrass approximation theorem there exists a polynomial $F \in \mathbb{R}[X_1, \dots, X_n]$ such that $\|f - F\|_{\overline{D} \cap \mathbb{R}^n} < \varepsilon$. Let $\widetilde{X}$ be the graph of the regular function $F|_X$ and let $\widetilde{X'}$ be the graph of the Nash function $F|_{X'}$. Of course $\widetilde{X'}$ is a Nash subvariety which is a compact connected component of the algebraic variety $\widetilde{X}$ and the canonical projection $\mathbb{R}^{n+1} \to \mathbb{R}^n$ induces a Nash isomorphism $\widetilde{X'} \to X'$.

By the construction made $\widetilde{X'}$ is contained in D and $\widetilde{X} - \widetilde{X'}$ is contained in $\mathbb{R}^{n+1} - \overline{D}$. Let Q be the sphere of $\mathbb{R}^{n+2}$ centered at the origin of radius a and let us consider in $\mathbb{R}^{n+2}$ the cylinders $\widehat{X}$ over $\widetilde{X}$ and $\widehat{X'}$ over $\widetilde{X'}$. We have

$Q \cap \widehat{X} = Q \cap \widehat{X'}$ and then $Z = Q \cap \widehat{X'}$ is an algebraic variety which has only two connected components Z_1 and Z_2. Moreover the canonical projection $\mathbb{R}^{n+2} \to \mathbb{R}^{n+1}$ induces a Nash isomorphism $Z_i \to \widetilde{X}'$, $i = 1, 2$. We can conclude that there exists a regular map $\tau: Z \to X$, induced by the canonical projection $\mathbb{R}^{n+2} \to \mathbb{R}^n$, such that $\tau|_{Z_i}: Z_i \to X'$ is a Nash isomorphism, for $i = 1, 2$.

By replacing the sphere Q with a suitable ellipsoid we can get that Z_1 and Z_2 are arbitrarily near each other.

THEOREM 2. – *Let S be a closed connected Nash subvariety of $\mathbb{R}^n$ which is a global Nash subvariety in some open semi-algebraic neighborhood of S in $\mathbb{R}^n$.*

Let Y be an algebraic subvariety of $\mathbb{R}^q$ and $\phi : S \to Y$ a Nash map that extends, as a map into $\mathbb{R}^q$, to an open neighborhood of S.

Then there exist an algebraic variety Z of some $\mathbb{R}^m$, a connected component Z' of Z, regular maps $\chi : Z \to \mathbb{R}^q$ and $\varrho : Z \to \mathbb{R}^n$ such that $\varrho(Z) = S$, $\chi = \phi\varrho$ and $\varrho|_{Z'}: Z' \to S$ is a Nash isomorphism.

Moreover if S is compact, the variety Z can be determined with only two connected components, Z_1 and Z_2, arbitrarily near each other, and $\varrho|_{Z_i}$: $Z_i \to S$ is a Nash isomorphism, for $i = 1, 2$.

PROOF. – If S is compact, by using the stereographic projection we may assume that S is a Nash subvariety of a compact semi-algebraic submanifold M; otherwise we can assume $M = \mathbb{R}^n$. In any case, by Lemma 1, we may assume that $S = \{x \in M \mid f_1(x) = \ldots = f_p(x) = 0\}$, where $f_i \in \mathcal{N}_M(M)$, $i = 1, \ldots, p$, and that ϕ extends to a Nash map $\widetilde{\phi} : M \to \mathbb{R}^q$.

By Theorem 1 there exist a nonsingular algebraic variety N of $\mathbb{R}^{n+p+q+s}$ and an open Nash embedding $\sigma : M \to N$ such that, if π_i is the i-th canonical projection of $\mathbb{R}^{n+p+q+s}$, $\pi_i \sigma(x) = x_i$, for $i = 1, \ldots, n$, $\pi_{n+j}\sigma = f_j$, for $j=1,\ldots,p$, and $\pi_{n+p+l}\sigma = \widetilde{\phi}_l$, for $l=1,\ldots,q$. Let us consider the regular functions on N defined by $h_j = \pi_{n+j}|_N$, $j=1,\ldots,p$ and $\psi_l = \pi_{n+p+l}|_N$, $l=1,\ldots,q$.

Let $X = \{z \in N \mid h_1(z) = \ldots = h_q(z) = 0\}$; since, in any case, $\sigma(M)$ is a connected component of N and $\sigma(S) = X \cap \sigma(M)$, then $\sigma(S)$ is a connected component X' of X. Let π be the restriction to X of the canonical projection $(\pi_1, \ldots, \pi_n)$ and ψ the restriction to X of the regular map $(\psi_1, \ldots, \psi_q)$; $\pi : X \to \mathbb{R}^n$ is a regular map that induces a Nash isomorphism $X' \to S$, whose inverse is $\sigma|_S$ and ψ is a regular map $X \to \mathbb{R}^q$ such that $\psi|_{X'} = \phi\pi|_{X'}$.

In the noncompact case we get the conclusion by setting $Z = X$, $Z' = X'$, $\varrho = \pi$ and $\chi = \psi$.

When S is compact the conclusion follows from Lemma 2 by setting $\varrho = \pi\tau$ and $\chi = \psi\tau$.

REMARK 1. – *If* S *is coherent, by the results of* [CRS] *and* [CS] *any Nash map* $\phi : S \to Y$ *extends to a Nash map* $\tilde{\phi} : M \to \mathbb{R}^q$.

REMARK 2. – *Let* S *be a circle and* X *a regular algebraic curve which has only two ovals* X' *and* X''. *By Theorem* 1, *for any Nash isomorphism* $\phi : S \to X'$ *there exist an algebraic variety* N, *an open Nash embedding* $\sigma : S \to N$ *and a regular map* $\chi : N \to X$ *such that* $\chi\sigma = \phi$. *Two cases are possible* (*cf.* [To]): *either there is a second oval above* S *and* χ *is a biregular isomorphism or* χ *has degree* 2. *The Theorem* 2 *says that is always possible to reduce to the second case.*

4. – Some counterexamples.

REMARK 3. – *The converse of the previous theorem does not hold; in fact there exist Nash subvarieties which are not global and which are Nash isomorphic to algebraic varieties. By little alterations of the arguments given in* [NT] *we can produce such examples for, compact or noncompact, semi-algebraic Nash subvarieties too. Let* $X = \{x \in \mathbb{R}^3 \mid x_1^4 + x_2^4 + x_1^2(1 - x_3)(2 - x_3) = 0\}$, $Z = \{x \in \mathbb{R}^3 \mid x_1^4 + x_2^4 x_3^4 + x_1^2(1 - x_3)(2 - x_3) = 0\}$ *and* $Y = \{x \in Z \mid x_3 \neq 0\} \cup \{(0, 0, 0)\}$. *Let us consider the Nash map* $\phi : X \to \mathbb{R}^3$ *defined by* $\phi(x_1, x_2, x_3) = (x_1, x_2/x_3, x_3)$ *for* $x_3 \neq 0$ *and* $\phi(0, 0, 0) = (0, 0, 0)$, *and the Nash map* $\phi : Y \to \mathbb{R}^3$ *defined by* $\psi(x_1, x_2, x_3) = (x_1, x_2 x_3, x_3)$ *for* $x_3 \neq 0$ *and* $\psi(0, 0, 0) = (0, 0, 0)$. *It is trivial to see that* ϕ *induces a Nash isomorphism* $X \to Y$ *whose inverse is* ψ. *As in* [NT] *we can prove that* Y *is not a global Nash subvariety and that* ϕ *does not extend to* $\mathbb{R}^3$; *of course* X *and* Y *are not coherent Nash subvarieties.*

In order to get an example in the compact case it is enough to take the projective closures of X *and* Z; *by embedding* $\mathbb{P}(\mathbb{R}^3)$ *into* $\mathbb{R}^9$ *by the Veronese map of degree* 2 *it is easy to produce the examples of a compact Nash subvariety* Y^*, *not global, which is Nash isomorphic to an algebraic variety* X^*.

By the example it also follows that there exist Nash maps on noncoherent Nash subvarieties which do not extend to any neighborhood. Of course this can happen only in the noncoherent case, by the cited results of [CRS] *and* [CS].

REMARK 4. – *In the Artin-Mazur Theorem* (*cf. Theorem* 1), *if the Nash map* f *is algebraic over some algebraic subvariety* W *contained in* M, *we can wonder whether* $\sigma(W)$ *is algebraic. In a previous paper* ([TT3]) *we gave very strong conditions for the algebraicity of* $\sigma(W)$ *in order to get some algebraic approximations. We shall show now that in general we can not get that the Nash subvariety* $\sigma(W)$ *is algebraic.*

Let X *be a compact regular plane algebraic curve with only two ovals and* *let S be one of these. We can suppose that X is a subvariety of a sphere Q of $\mathbb{R}^3$ and $S = \{x \in Q \mid g(x) = 0\}$, where g is a Nash function on Q (cf.* [CRS]). *Let Γ be a circle and t_0, t_1 be two diametrically opposite points of Γ; let α be a regular function on Γ, injective on the arcs $\overset{\frown}{t_0 t_1}$, $\overset{\frown}{t_1 t_0}$, and such that $\alpha(t_0) = 0$, $\alpha(t_1) = 1$. Let $M = Q \times \Gamma \subset \mathbb{R}^5$, $W = Q \times \{t_0\}$ and $f(x, t) = g(x)\,\alpha(t)$. It is easy to check that $\{(x, t) \in M \mid t \neq t_0,\ f(x, t) = 0\} = \{(x, t) \in S \times \Gamma \mid t \neq t_0\} \cong S \times (0, 1)$, where the bijection is a Nash isomorphism induced by a regular map, $\{(x, t) \in M \mid t = t_0, f(x, t) = 0\} = Q \times \{t_0\}$ and $\{(x, t) \in M \mid t = t_1, f(x, t) = 0\} = S \times \{t_1\}$. By Theorem* 1 *there exist a regular algebraic subvariety N of some $\mathbb{R}^n$, an open Nash embedding $\sigma : M \to N$ and a regular function $h : N \to \mathbb{R}$ such that $\pi\sigma = \mathrm{id}_M$, $h\sigma = f$, where $\pi : \mathbb{R}^n \to \mathbb{R}^5$ is the canonical projection. Let us suppose that $\sigma(W)$ is an algebraic subvariety of N: it is contained in the connected component $\sigma(M)$ of N and it has codimension one. Let F be the algebraic line bundle on N defined by $\sigma(W)$. By the construction made, the dual bundle F^ν of F is equivalent to the line bundle on N defined by the Nash submanifold $\sigma(Q \times \{t_1\})$. It follows that (cf.* [BCR]) *there exists an algebraic section γ of F^ν which approximates $\sigma(Q \times \{t_1\})$ and whose zero set is contained in $\sigma(M)$. Let $Z = \{x \in N \mid \gamma(x) = h(x) = 0\}$: Z is an algebraic subvariety which approximates $\sigma(S \times \{t_1\})$ and then $\pi(Z)$ approximates $S \times \{t_1\}$; if the approximation is strong enough $\pi(Z)$ can be identified to a section of the cylinder $S \times (0, 1)$ and then the canonical map $\pi(Z) \to S$ results a Nash isomorphism. It follows that there exists a bijective regular map $Z \to S$, but this is not possible by* [To] *since the Zariski closure of S contains another component of dimension one; we can conclude that $\sigma(W)$ is not an algebraic subvariety.*

It is known that in the literature (cf. [AK], [BCR], [To]) there exist several deeper results than Theorem 2 which state the existence of algebraic structures on Nash subvarieties even if, in general, they do not supply regular maps as ϱ and χ. In this framework we can state the following theorem, which may be interesting by itself, and which, together with the Remark 4, suggest that stronger results than Artin-Mazur Theorem are necessary in order to find such algebraic structures.

THEOREM 3. – *Let X' be a compact connected component of an irreducible algebraic variety X such that the set W' of singular points of X' is union of some irreducible components of the singular locus W of X.*

Then X' is homeomorphic to an algebraic variety Z.

PROOF. – Let Y be the desingularization (cf. [EV]) of X and let $\pi : Y \to X$ be the canonical map; $\pi^{-1}(W)$ is union of finitely many regular algebraic hypersurfaces in general position. Since π is a proper map $\pi^{-1}(X')$ is a compact connected component of Y and $\pi^{-1}(W')$ is an algebraic subvariety of Y contained in $\pi^{-1}(X')$. By the results of [To] there exists a regular algebraic subvariety $\tilde{X}'$ of Y Nash isomorphic, and arbitrarily near, to $\pi^{-1}(X')$, which contains $\pi^{-1}(W')$. Since there exists (cf. [AK]) a contraction of $\tilde{X}'$ along $\pi|_{\pi^{-1}(W')}$ we can get an algebraic variety Z homeomorphic to X'.

REFERENCES

[AK] S. AKBULUT - H. KING, *Topology of real algebraic sets*, Maht. Sci. Research Institute Publ. **25**, Springer, Berlin (1992).

[BCR] J. BOCHNACK - M. COSTE - M. F. ROY, *Real algebraic geometry*, Springer, Berlin, 1998.

[CRS] M. COSTE - R. RUIZ - M. SHIOTA, *Approximation in Nash manifolds*, Amer. J. Math., **117** (1995), 905-927.

[CS] M. COSTE - M. SHIOTA, *Nash functions on noncompact Nash manifolds*, Ann. Scient. Éc. Norm. Sup., **33** (2000), 139-149.

[EV] S. ENCINAS - O. VILLAMAYOR, *A new theorem of desingularization over fields of characteristic zero*, Preprint arXiv:math.AG/0101208

[NT] G. NARDELLI - A. TANCREDI, *A note on the extension of analytic functions off real analytic subsets*, Revista Matemática de la Universidad Complutense de Madrid, **9** (1996), 85-99.

[TT1] A. TANCREDI - A. TOGNOLI, *On the extension of Nash functions*, Math. Ann. **288** (1990), 595-604.

[TT2] A. TANCREDI - A. TOGNOLI, *Some remarks on the classification of complex Nash vector bundles*, Bull. Sci. math., **17** (1993), 177-183.

[TT3] A. TANCREDI - A. TOGNOLI, *On the algebraic approximation of Nash maps*, Ann. Univ. Ferrara, **38** (1992), 107-115.

[To] A. TOGNOLI, *Algebraic approximation of manifolds and spaces*, Sém. Bourbaki, **548** (1979/80).

[Wa] A. H. WALLACE, *Algebraic approximation of manifolds*, Proc. London Math. Soc. (3), **7** (1957), 196-210.

Alessandro Tancredi: Dipartimento di Matematica, Università di Perugia
Via Vanvitelli 1, 06123 Perugia, Italy; e-mail: altan@unipg.it

Alberto Tognoli: Dipartimento di Matematica, Università di Trento
Via Sommarive 14, I-38050 Povo (TN), Italy

Pervenuta in Redazione
il 19 aprile 2002